

OCCUPATION TIME PROBLEM FOR MULTIFRACTIONAL BROWNIAN MOTION

BY

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Abstract. In this paper, by using a Fourier analytic approach, we investigate sample path properties of the fractional derivatives of multifractional Brownian motion local times. We also show that those additive functionals satisfy a property of local asymptotic self-similarity. As a consequence, we derive some local limit theorems for the occupation time of multifractional Brownian motion in the space of continuous functions.

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1. INTRODUCTION

A fractional Brownian motion (fBm for short) $B^H = \{B^H(t), t \geq 0\}$ with Hurst index $H \in (0, 1)$ is a real-valued, centered Gaussian process with covariance function given by

$$\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}) \quad \text{for all } s, t \geq 0.$$

Several properties of fBm such as self-similarity and the stationarity of increments make it important in theory and also a useful model in various applications, such as in telecommunication, finance, image analysis among others. Note that in the case $H = \frac{1}{2}$, we retrieve the well-known Brownian motion. The fact that the Hurst parameter H is independent of time t makes the Hölder regularity of fBm constant along the paths. This property restricts its applications when modelling phenomena whose regularity evolves in time, such as Internet traffic and some highly textured images with strong global organization; see, for example, Lévy-Véhel [15] and Pesquet-Popescu and Lévy-Véhel [18].

The multifractional Brownian motion (mBm for short) was introduced as a natural extension of fBm to overcome this limitation. The basic idea is to replace the Hurst parameter H by a function $H(t) : [0, \infty) \rightarrow [a, b] \subset (0, 1)$ which is a Hölder continuous function of exponent $\beta > 0$, i.e., there exists a finite positive constant C such that

$$|H(t) - H(s)| \leq C|t - s|^\beta \quad \text{for all } s, t \geq 0.$$

1. Moving average representation of mBm (Lévy-Véhel and Peltier [16]):

$$\begin{aligned} \tilde{B}^{H(t)}(t) &= \frac{1}{\Gamma(H(t) + \frac{1}{2})} \left(\int_{-\infty}^0 [(t-u)^{H(t)-1/2} - (-u)^{H(t)-1/2}] W(du) \right. \\ &\quad \left. + \int_0^t (t-u)^{H(t)-1/2} W(du) \right), \quad t \geq 0, \end{aligned}$$

where W is a standard Brownian motion defined on \mathbb{R} .

2. Harmonizable representation of mBm (Benassi et al. [4]):

$$\widehat{B}^{H(t)}(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} \widehat{W}(d\xi), \quad t \geq 0,$$

where $\widehat{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$.

In the sequel, the notation $B := (B(t), t \geq 0)$ means that both previous representations of mBm may be chosen.

When $H(\cdot)$ varies with time, the mBm is no longer self-similar, even though it is seen as an intricacy; Lévy-Véhel and Peltier [16] proved that if $H(\cdot)$ satisfies the assumption

$$(\mathcal{H}_\beta) : \{H(\cdot) \text{ is } \beta\text{-Hölder continuous and } \sup_{t \in \mathbb{R}^+} H(t) < \beta\},$$

then the mBm has a property called the *local asymptotic self-similarity* (LASS in short) defined as follows:

$$\lim_{\rho \rightarrow 0^+} \text{law} \left\{ \frac{B(t + \rho u) - B(t)}{\rho^{H(t)}}, u \in \mathbb{R} \right\} = \text{law} \{B^{H(t)}(u), u \in \mathbb{R}\},$$

where $B^{H(t)}$ is an fBm with the Hurst parameter $H(t)$.

By using the concept of local nondeterminism and the assumption (\mathcal{H}_β) , Boufoussi et al. [6] proved the existence of a jointly continuous local time of mBm and studied its Hölder regularity in time and space. In Boufoussi et al. [7], the

authors proved that the local time of mBm has a kind of local asymptotic self-similarity (LASS) property. Through this result, they obtained some local limit theorems corresponding to the mBm (first order limit theorems).

The aim of this work is to establish some local limit theorems for the occupation time of mBm. To prove our main results we are led to study the regularities of the fractional derivative of local time of mBm, by using a representation based on Fourier analytic approach.

The rest of this paper is organized as follows. In Section 2, we give some basic facts about local times. In Section 3, we prove some Hölder regularities of the fractional derivative of local time of mBm. Section 4 is devoted to the asymptotic results: Subsection 4.1 is concerned with the LASS property of those additive functionals, and in Subsection 4.2, some local limit theorems for the occupation time of mBm are studied.

Most of the estimates in the sequel contain unspecified finite positive constants. We use the same symbol for these constants, even when they vary from one line to the next.

2. LOCAL TIMES

In this section we recall the definition and some properties of local times of Gaussian processes and a result on mBm that will be needed in the sequel.

Let $(X(t), t \in \mathbb{R}^+)$ be a separable random process with Borel sample function. The *occupation measure* of X is defined as follows:

$$\mu(A, B) = \lambda(\{s \in A, X(s) \in B\}) \quad \forall A \in \mathcal{B}(\mathbb{R}^+), \forall B \in \mathcal{B}(\mathbb{R}),$$

where λ is the Lebesgue measure on \mathbb{R}^+ . If $\mu(A, \cdot)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , we say that X has *local time* on A and define the local time $L(A, \cdot)$ as the Radon–Nikodym derivative of $\mu(A, \cdot)$. Sometimes, we write L_t^x instead of $L([0, t], x)$.

The following property of local time is called the *occupation density formula*: For every $t \in \mathbb{R}^+$ and every measurable function $f : \mathbb{R} \mapsto \mathbb{R}^+$,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(x) L_t^x dx.$$

The well-known Fourier analytic approach introduced by Berman [5] states that, for a fixed sample function at fixed t , the Fourier transform on x of L_t^x is the function

$$F(u) = \int_{\mathbb{R}} e^{iux} L_t^x dx.$$

Using the occupation density formula and the inverse Fourier transform of this

function, we have

$$(2.1) \quad L_t^x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_0^t e^{iu(X(s)-x)} ds \right) du.$$

The following result due to Boufoussi et al. ([6], Lemma 3.3) will extensively be used in the sequel.

LEMMA 2.1. *Under the assumption (\mathcal{H}_β) , for any even integer $m \geq 2$, there exist positive and finite constants $C_m > 0$ and $\delta > 0$ such that, for any $t, s \geq 0$ with $|t - s| < \delta$, any $x, y \in \mathbb{R}$ and any $0 < \xi < \min(1, (1 - \lambda)/(2\lambda))$, where $\lambda = \sup_{u \in [s, t]} H(u)$, we have*

$$(2.2) \quad \mathbb{E} [L_t^x - L_s^x]^m \leq C_m \frac{|t - s|^{m(1-\lambda)}}{\Gamma(1 + m(1 - \lambda))},$$

$$(2.3) \quad \mathbb{E} [L_t^y - L_s^y - L_t^x + L_s^x]^m \leq C_m |y - x|^{\xi m} \frac{|t - s|^{m(1-\lambda(1+\xi))}}{\Gamma(1 + m(1 - \lambda(1 + \xi)))}.$$

REMARK 2.1. *Estimates similar to (2.2) and (2.3) were proved for the multi-fractional Brownian sheet by Meerschaert et al. [17], Lemmas 3.5 and 3.7.*

3. FRACTIONAL DERIVATIVE OF LOCAL TIME OF mBm

The fractional derivatives have many uses, such as fractional integro-differentiation which has now become a significant topic in mathematical analysis. Fractional derivatives of local time have been discussed for physical purposes in the paper by Ezawa et al. [9]. For a complete survey on the fractional derivative we refer the reader to Hardy and Littlewood [13] and the book by Samko et al. [19].

DEFINITION 3.1. Let $0 < \theta < 1$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that belongs to $\mathcal{C}^\theta \cap L^1(\mathbb{R})$, where \mathcal{C}^θ is the space of locally θ -Hölder continuous functions on \mathbb{R} . For $0 < \gamma < \theta$, we define $D_\pm^\gamma f$ by

$$D_\pm^\gamma f(x) := \frac{\gamma}{\Gamma(1 - \gamma)} \int_0^{+\infty} \frac{f(x) - f(x \mp y)}{y^{1+\gamma}} dy.$$

The operators D_+^γ and D_-^γ are called, respectively, *right-sided* and *left-sided Marchaud fractional derivatives* of order γ . We put $D^\gamma := D_+^\gamma - D_-^\gamma$.

REMARK 3.1. 1. D_+^γ and D_-^γ satisfy the switching identity:

$$\int_{\mathbb{R}} f(x) D_-^\gamma g(x) dx = \int_{\mathbb{R}} g(x) D_+^\gamma f(x) dx$$

for any $f, g \in \mathcal{C}^\theta(\mathbb{R}) \cap L^1(\mathbb{R})$ and $0 < \gamma < \theta$.

2. $D_{\pm}^{\gamma} f$ is $(\theta - \gamma)$ -Hölder continuous whenever f is θ -Hölder continuous for any $0 < \gamma < \theta$.

3. For $h : \mathbb{R} \rightarrow \mathbb{R}$ and $a > 0$, we denote by h_a the function $x \rightarrow h(ax)$; then

$$D_{\pm}^{\gamma}(h_a) = a^{\gamma}(D_{\pm}^{\gamma}h)_a \quad \text{for all } \gamma > 0.$$

4. The fractional derivatives of the local time are particular continuous additive functionals of zero energy in the sense of Fukushima [11].

Lemma 2.1 allows us to define the fractional derivative of order γ of local time of mBm as follows:

$$D^{\gamma}L(t, \cdot)(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \frac{L_t^{x+a} - L_t^{x-a}}{a^{1+\gamma}} da \quad \text{for all } 0 < \gamma < \xi.$$

Using (2.1), we get

$$\begin{aligned} (3.1) \quad D^{\gamma}L(t, \cdot)(x) &:= \frac{\gamma}{2\pi\Gamma(1-\gamma)} \int_0^{+\infty} \int_{\mathbb{R}} \int_0^t \frac{e^{iu(B(s)-(x+a))} - e^{iu(B(s)-(x-a))}}{a^{1+\gamma}} dsduda \\ &= \frac{\gamma}{2\pi\Gamma(1-\gamma)} \int_0^{+\infty} \int_{\mathbb{R}} \int_0^t e^{iuB(s)} [e^{-iu(x+a)} - e^{-iu(x-a)}] \frac{1}{a^{1+\gamma}} dsduda. \end{aligned}$$

Here are the main results of this section.

THEOREM 3.1. *Let $0 < \gamma < \xi$ and $D \in \{D^{\gamma}, D_{\pm}^{\gamma}\}$. Under the assumption (\mathcal{H}_{β}) , there exist finite and positive constants $C_m > 0$ and $\delta > 0$ such that for every $t, s \geq 0$ with $|t - s| < \delta$, any $x \in \mathbb{R}$ and any $0 < \xi < \min(1, (1 - \lambda)/(2\lambda))$, where $\lambda = \sup_{u \in [s, t]} H(u)$, we have*

$$(3.2) \quad \|DL(t, \cdot)(x) - DL(s, \cdot)(x)\|_{2m} \leq C_m |t - s|^{1-\lambda(1+\gamma)},$$

where $\|\cdot\|_{2m} = [\mathbb{E}|\cdot|^{2m}]^{1/(2m)}$.

Proof. We will prove (3.2) for D^{γ} since the other cases may be treated in the same way. For all integers $m \geq 1$, we have for any $b > 0$

$$\begin{aligned} \|D^{\gamma}L_t(x) - D^{\gamma}L_s(x)\|_{2m} &= \left\| \int_0^{+\infty} \frac{\gamma}{\Gamma(1-\gamma)} \frac{L_t^{x+u} - L_t^{x-u} - L_s^{x+u} + L_s^{x-u}}{u^{1+\gamma}} du \right\|_{2m} \\ &\leq C(I_1(b) + I_2(b)), \end{aligned}$$

where

$$I_1(b) := \int_0^b \frac{\|L_t^{x+u} - L_t^{x-u} - L_s^{x+u} + L_s^{x-u}\|_{2m}}{u^{1+\gamma}} du$$

and

$$I_2(b) := \int_b^{+\infty} \frac{\|L_t^{x+u} - L_t^{x-u} - L_s^{x+u} + L_s^{x-u}\|_{2m}}{u^{1+\gamma}} du.$$

We estimate $I_1(b)$ and $I_2(b)$ separately.

In view of (2.3), we deduce

$$I_1(b) \leq C_m |t - s|^{1-\lambda(1+\xi)} \int_0^b u^{\xi-\gamma-1} du \leq C_m |t - s|^{1-\lambda(1+\xi)} b^{\xi-\gamma}.$$

Now, we deal with $I_2(b)$. We use (2.2) to conclude that

$$I_2(b) \leq C_m |t - s|^{1-\lambda} \int_b^{+\infty} \frac{1}{u^{1+\gamma}} du \leq C_m |t - s|^{1-\lambda} b^{-\gamma}.$$

Finally, by choosing $b = |t - s|^\lambda$, we obtain the desired result. ■

THEOREM 3.2. *Let $0 < \gamma < \xi$ and $D \in \{D^\gamma, D_\pm^\gamma\}$. Under the assumption (\mathcal{H}_β) , there exists a finite and positive constant $C_m > 0$ such that, for every $t, s \geq 0$ with $|t - s| < \delta$, any $(x, y) \in \mathbb{R}^2$ and any $0 < \xi < \min(1, (1 - \lambda)/(2\lambda))$, where $\lambda = \sup_{u \in [s, t]} H(u)$, we have*

$$(3.3) \quad \begin{aligned} & \|DL(t, \cdot)(x) - DL(t, \cdot)(y) - DL(s, \cdot)(x) + DL(s, \cdot)(y)\|_{2m} \\ & \leq C_m |t - s|^{1-\lambda(1+\xi)} |x - y|^{\xi-\gamma}. \end{aligned}$$

REMARK 3.2. *Similar results are obtained, respectively, by Ait Ouahra and Eddahbi [2] and Ait Ouahra [1] for the local time and the fractional derivative of local time of a symmetric stable process of index $1 < \alpha \leq 2$. The key ingredient in their proofs was the Markov property of the symmetric stable process, which is not satisfied by the mBm.*

Proof of Theorem 3.2. For any integer $m \geq 1$, we have

$$\begin{aligned} & \|DL(t, \cdot)(x) - DL(t, \cdot)(y) - DL(s, \cdot)(x) + DL(s, \cdot)(y)\|_{2m} \\ &= \left\| \int_{[0, \infty)} \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} du \right\|_{2m} \\ &\leq C \int_{[0, \infty)} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du \\ &\leq C(J_1(b) + J_2(b)), \end{aligned}$$

where

$$J_1(b) := \int_{[0,b]} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du,$$

$$J_2(b) := \int_{[b,+\infty[} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du$$

for some $b > 0$.

First, let us estimate $J_1(b)$. Combining the Minkowski inequality and (2.3), we obtain

$$\begin{aligned} & \|L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}\|_{2m} \\ & \leq \|L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u}\|_{2m} + \|L_t^{y+u} - L_s^{y+u} - L_t^{y-u} + L_s^{y-u}\|_{2m} \\ & \leq C_m \frac{|u|^\xi |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}. \end{aligned}$$

Therefore,

$$\begin{aligned} J_1(b) & \leq C_m \frac{|t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1+\lambda(1+\xi)))} \int_{[0,b]} \frac{u^\xi}{u^{1+\gamma}} du \\ & \leq C_m \frac{|t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))} b^{\xi-\gamma}. \end{aligned}$$

Now, we are going to estimate $J_2(b)$. Using the Minkowski inequality and decomposing otherwise, we obtain

$$\begin{aligned} & \|L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}\|_{2m} \\ & \leq \|L_t^{x+u} - L_s^{x+u} - L_t^{y+u} + L_s^{y+u}\|_{2m} + \|L_t^{x-u} - L_s^{x-u} - L_t^{y-u} + L_s^{y-u}\|_{2m} \\ & \leq C_m \frac{|x-y|^\xi |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}. \end{aligned}$$

Therefore, $J_2(b)$ is dominated by

$$b^{-\gamma} \cdot C_m \frac{|x-y|^\xi |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}.$$

By choosing $b = |x - y|$, we deduce that

$$\begin{aligned} & \|DL(t, \cdot)(x) - DL(t, \cdot)(y) - DL(s, \cdot)(x) + DL(s, \cdot)(y)\|_{2m} \\ & \leq C_m |t - s|^{(1-\lambda(1+\xi))} |x - y|^{\xi-\gamma}, \end{aligned}$$

which completes the proof of the theorem. ■

4. ASYMPTOTIC RESULTS

It is well known that techniques for proving limit theorems related to self-similar processes use the self-similarity of their local times. It is natural to expect the same when dealing with processes satisfying the LASS property. The answer to the preceding question is affirmative in the case of the mBm and the result is given by the following lemma. (See Theorem 5.1 in Boufoussi et al. [7].)

In the sequel, we will use the following notation: $B^{H(t)}(t) = B(t)$ for the sake of simplicity.

LEMMA 4.1. *Under the assumption (\mathcal{H}_β) , for any fixed t_0 , the local time of mBm is locally asymptotically self-similar with parameter $1 - H(t_0)$, in the sense that, for every $x \in \mathbb{R}$, the family of processes $\{Y_\rho(t, x), t \in [0, 1]\}_{\rho > 0}$ defined by*

$$(4.1) \quad Y_\rho(t, x) = \frac{L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0, \rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)}}$$

converges in law to the local time $\{l(t, x), t \in [0, 1]\}$ of the fBm $B^{H(t_0)}$ with Hurst parameter $H(t_0)$.

We will introduce the proof of this lemma in the Appendix, to clarify a passage in the estimate of $I_1^{\varepsilon, \rho}$ on page 864 in Boufoussi et al. [7].

REMARK 4.1. *The previous result can also be obtained by using propositions (4.1) and (4.2) in Jolis and Viles [14], which allows us to identify the limit law as local time.*

4.1. LASS for the fractional derivative of local time of mBm.

PROPOSITION 4.1. *Under the assumption (\mathcal{H}_β) , for any fixed t_0 , the fractional derivative of local time of mBm is LASS with parameter $1 - (1 + \gamma)H(t_0)$, in the sense that, for every $x \in \mathbb{R}$, the family of processes $\{D_\rho(t, x), t \in [0, 1]\}_{\rho \geq 0}$ defined by*

$$D_\rho(t, x) := \frac{D^\gamma L(t_0 + t\rho, \cdot)(\rho^{H(t_0)}x + B(t_0)) - D^\gamma L(t_0, \cdot)(\rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)(1+\gamma)}}$$

converges in law to the fractional derivative $\{D^\gamma l_t(x), t \in [0, 1]\}$ of local time of the fBm $B^{H(t_0)}$.

P r o o f. To prove the convergence in law, we proceed in two steps. First, we prove the tightness of the family $\{D_\rho(t, x), t \in [0, 1]\}_{\rho \geq 0}$ in the space of continuous functions. According to Theorem 3.1, we obtain

$$\begin{aligned} & \mathbb{E}|D_\rho(t, x) - D_\rho(s, x)|^m \\ &= \frac{\mathbb{E}|D^\gamma L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - D^\gamma L(t_0 + s\rho, \rho^{H(t_0)}x + B(t_0))|^m}{\rho^{(1-(1+\gamma)H(t_0))m}} \\ &\leq C \frac{|t\rho - s\rho|^{(1-(1+\gamma)H(t_0))m}}{\rho^{(1-(1+\gamma)H(t_0))m}} \leq C|t - s|^{m(1-H(t_0)(1+\gamma))}. \end{aligned}$$

Then to prove the tightness, it suffices to take $m > 1/(1 - H(t_0)(1 + \gamma))$.

It now remains to prove the convergence of the finite-dimensional distributions of D_ρ , as ρ tends to zero, to those of the fractional derivative of local time of the fBm $B^{H(t_0)}$ with Hurst parameter $H(t_0)$.

We note $C_\gamma = \gamma/(2\pi\Gamma(1 - \gamma))$ and $Y^\rho(t_0) = \rho^{H(t_0)}x + B(t_0)$. In view of the definition of $D^\gamma L$, we get

$$\begin{aligned} & D^\gamma L(t_0 + t\rho, \cdot)(Y^\rho(t_0)) - D^\gamma L(t_0, \cdot)(Y^\rho(t_0)) \\ &= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_{t_0}^{t_0+t\rho} e^{iuB(s)} [e^{-iu(Y^\rho(t_0)+a)} - e^{-iu(Y^\rho(t_0)-a)}] \frac{1}{a^{1+\gamma}} da duds \\ &= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{iuB(\rho r+t_0)} [e^{-iu(Y^\rho(t_0)+a)} - e^{-iu(Y^\rho(t_0)-a)}] \frac{1}{a^{1+\gamma}} \rho da duds \\ &= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t \exp\left[i\frac{v}{\rho^{H(t_0)}}B(\rho r+t_0)\right] \left\{ \exp\left[-i\frac{v}{\rho^{H(t_0)}}(Y^\rho(t_0)+a)\right] \right. \\ &\quad \left. - \exp\left[-i\frac{v}{\rho^{H(t_0)}}(Y^\rho(t_0)-a)\right] \right\} \frac{\rho^{1-H(t_0)}}{a^{1+\gamma}} da dv dr \\ &= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t \exp\left[iv\frac{B(\rho r+t_0)-B(t_0)}{\rho^{H(t_0)}}\right] \\ &\quad \times [e^{-iv(x+b)} - e^{-iv(x-b)}] \frac{1}{b^{1+\gamma}} \rho^{1-(1+\gamma)H(t_0)} db dv dr. \end{aligned}$$

We have used the change of variables: $s = \rho r + t_0$, $u = v/\rho^{H(t_0)}$ and $b = a/\rho^{H(t_0)}$. Consequently, we have

$$\begin{aligned} D_\rho(t, x) &= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t \exp\left[iv\frac{B(\rho r+t_0)-B(t_0)}{\rho^{H(t_0)}}\right] \\ &\quad \times [e^{-iv(x-b)} - e^{-iv(x+b)}] \frac{1}{b^{1+\gamma}} dr dv db, \end{aligned}$$

which is the fractional derivative of local time of the Gaussian process B^ρ .

Finally, by using Lemma 5.2 in [8], concerning the continuity of the fractional derivative, we complete the proof of this result. ■

4.2. Local limit theorems. Limit theorems for fractional derivative of local time were studied by Yamada [21] for Brownian motion, Fitzsimmons and Gettoor [10] for symmetric stable process of index $1 < \alpha \leq 2$ and Shieh [20] for the fBm with Hurst parameter $H \in (0, 1)$ where this last author proved that if f is in the range of fractional derivative transform, i.e., $f = D_{\pm}^{\gamma}g$ with $g \in C^{\beta}$ and $\int g(x)dx \neq 0$, then the family of processes,

$$\left\{ \frac{1}{A^{1-(1+\gamma)H}} \int_0^{At} f(X_s)ds, t \geq 0 \right\},$$

converges in law, in the space of continuous functions, as $A \rightarrow +\infty$, to the process

$$\left\{ \int g(x)dx D_{\pm}^{\gamma}L_t(0) \right\}.$$

Ait Ouahra and Eddahbi [2] generalized the result of Fitzsimmons and Gettoor [10] in Hölder space and Ait Ouahra and Ouali [3] extended the result of Shieh [20] to Besov space.

The main tool in the proof of all these results was the self-similarity of the process. In this section, by using the LASS property, we investigate local limit theorems for the occupation time of the mBm.

THEOREM 4.1. *Suppose $f = D_{\pm}^{\gamma}g$, where $g \in C^{\beta}$ with compact support for some β such that $0 < \gamma < \beta < \xi < \min(1, (1 - \lambda)/(2\lambda))$. Under the assumption (\mathcal{H}_{β}) , the following convergence in law holds:*

$$\lim_{A \rightarrow +\infty} \lim_{\rho \rightarrow 0^+} \frac{1}{A^{1-(1+\gamma)H(t_0)}} \int_0^{At} f \left(\frac{B(\rho s + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds = \int_{\mathbb{R}} g(x)dx D_{\mp}^{\gamma}l_t(0).$$

THEOREM 4.2. *Suppose $f = D_{\pm}^{\gamma}g$, where $g \in C^{\beta}$ with compact support for some β such that $0 < \gamma < \beta < \xi < \min(1, (1 - \lambda)/(2\lambda))$. Under the assumption (\mathcal{H}_{β}) , we have*

$$\frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} f \left(\frac{B(s) - B(t_0) - \rho^{H(t_0)}y}{\theta(\rho)} \right) ds \xrightarrow[\rho \rightarrow 0^+]{\mathcal{L}} \int_{\mathbb{R}} g(x)dx D_{\mp}^{\gamma}l_t(y)$$

with $\int |g(x)||x|^{\xi-\gamma}dx < \infty$, $\psi(\rho)/\theta(\rho) = \rho^{1-(1+\gamma)H(t_0)}$ and $\theta(\rho)/\rho^{H(t_0)} = o(1)$.

Proof of Theorem 4.1. Combining the LASS property of mBm and Theorem VI.4.2 in Gihman and Skorohod [12], we obtain

$$\int_0^{At} f \left(\frac{B(s\rho + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds \xrightarrow[\rho \rightarrow 0^+]{\mathcal{L}} \int_0^{At} f(B^{H(t_0)}(s))ds,$$

and by using the result of Shieh [20], the family of processes

$$\left\{ \frac{1}{A^{1-(1+\gamma)H(t_0)}} \int_0^{At} f(B^{H(t_0)}(s)) ds \right\}_{t \geq 0}$$

converges in law to the process

$$\left\{ \int_{\mathbb{R}} g(x) dx D_{\mp}^{\gamma} l_t(0) \right\}_{t \geq 0}.$$

Then the theorem is proved. ■

Proof of Theorem 4.2. We prove only the case $f = D_{+}^{\gamma} g$ since the proof of the other case is similar. We put $A_{\rho} = \theta(\rho)x + \rho^{H(t_0)}y + B(t_0)$ and $B_{\rho} = \rho^{H(t_0)}y + B(t_0)$. By the occupation time formula and the switching identity, we have

$$\begin{aligned} & \frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} D_{+}^{\gamma} g \left(\frac{B(s) - B(t_0) - \rho^{H(t_0)}y}{\theta(\rho)} \right) ds \\ &= \int_{\mathbb{R}} f(x) \frac{L(\rho t + t_0, A_{\rho}) - L(t_0, A_{\rho})}{\rho^{1-(1+\gamma)H(t_0)}} dx \\ &= \int_{\mathbb{R}} g(x) \frac{D_{-}^{\gamma} L(\rho t + t_0, \cdot)(A_{\rho}) - D_{-}^{\gamma} L(t_0, \cdot)(A_{\rho})}{\rho^{1-(1+\gamma)H(t_0)}} dx. \end{aligned}$$

Moreover,

$$\begin{aligned} & D_{-}^{\gamma} L(\rho t + t_0, \cdot)(A_{\rho}) - D_{-}^{\gamma} L(t_0, \cdot)(A_{\rho}) \\ &= D_{-}^{\gamma} L(\rho t + t_0, \cdot)(A_{\rho}) - D_{-}^{\gamma} L(\rho t + t_0, \cdot)(B_{\rho}) - D_{-}^{\gamma} L(t_0, \cdot)(A_{\rho}) \\ &\quad - D_{-}^{\gamma} L(t_0, \cdot)(B_{\rho}) + D_{-}^{\gamma} L(\rho t + t_0, \cdot)(B_{\rho}) + D_{-}^{\gamma} L(t_0, \cdot)(B_{\rho}) \\ &= D_{-}^{\gamma} L(\rho t + t_0, \cdot)(B_{\rho}) - D_{-}^{\gamma} L(t_0, \cdot)(B_{\rho}) + D_{-}^{\gamma} L(J_0, \cdot)(A_{\rho}) - D_{-}^{\gamma} L(J_0, \cdot)(B_{\rho}) \end{aligned}$$

with $J_0 = [t_0, t_0 + \rho t]$. Therefore,

$$\begin{aligned} & \frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} D_{+}^{\gamma} g \left(\frac{B(s) - B(t_0) - \rho^{H(t_0)}y}{\theta(\rho)} \right) ds \\ &= \frac{D_{-}^{\gamma} L(\rho t + t_0, \cdot)(B_{\rho}) - D_{-}^{\gamma} L(t_0, \cdot)(B_{\rho})}{\rho^{1-(1+\gamma)H(t_0)}} \int_{\mathbb{R}} g(x) dx \\ &\quad - \int_{\mathbb{R}} \frac{D_{-}^{\gamma} L(J_0, \cdot)(B_{\rho}) - D_{-}^{\gamma} L(J_0, \cdot)(A_{\rho})}{\rho^{1-(1+\gamma)H(t_0)}} g(x) dx \\ &=: (*) - (**). \end{aligned}$$

By Proposition 4.1 we have $(*) \xrightarrow{\mathcal{L}} D^\gamma l_t(y) \int_{\mathbb{R}} g(x) dx$ as $\rho \rightarrow 0$. It only remains to prove that $(**)$ converges to zero. By the Hölder inequality we have

$$(4.2) \quad \mathbb{E} \left| \int_{\mathbb{R}} \frac{D^\gamma_- L(J_0, \cdot)(B_\rho) - D^\gamma_- L(J_0, \cdot)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} g(x) dx \right| \\ \leq \int_{\mathbb{R}} |g(x)| \left\| \frac{D^\gamma_- L(J_0, \cdot)(B_\rho) - D^\gamma_- L(J_0, \cdot)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} \right\|_2 dx.$$

Applying Lemma 2.1 for the process $B(t) - B(t_0)$ instead of mBm, we have

$$\left\| \frac{D^\gamma_- L(J_0, \cdot)(B_\rho) - D^\gamma_- L(J_0, \cdot)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} \right\|_2 \\ \leq C(t\rho)^{1-H(t_0)(1+\xi)} |x|^{\xi-\gamma} \frac{\theta(\rho)^{\xi-\gamma}}{\rho^{1-(1+\gamma)H(t_0)}} \\ \leq C t^{1-H(t_0)(1+\xi)} |x|^{\xi-\gamma} \left(\frac{\theta(\rho)}{\rho^{H(t_0)}} \right)^{\xi-\gamma}$$

for sufficiently small ρ and $0 < \gamma < \xi < \min(1, (1-\lambda)/(2\lambda))$. Hence, (4.2) is dominated by

$$C t^{1-H(t_0)(1+\xi)} \int_{\mathbb{R}} |g(x)| |x|^{\xi-\gamma} dx \left(\frac{\theta(\rho)}{\rho^{H(t_0)}} \right)^{\xi-\gamma}.$$

This last integral is finite by the assumption of the theorem and then $(**)$ tends to zero as ρ tends to zero. ■

5. APPENDIX

Proof of Lemma 4.1. To prove the convergence in law, we proceed in two steps. First we prove the tightness of the family $\{Y_\rho(t, x), t \in [0, 1]\}_{\rho>0}$ in the space of continuous functions. By using (2.2) and (2.3), for ρ small enough, we obtain

$$\mathbb{E} |Y_\rho(t, x) - Y_\rho(s, x)|^m \\ = \frac{\mathbb{E} [L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0 + \rho s, \rho^{H(t_0)}x + B(t_0))]}{\rho^{(1-H(t_0))m}} \\ \leq C_m |t - s|^{(1-H(t_0))m}.$$

We can take $m > 1/(1 - H(t_0))$, to get the tightness.

Now, we prove the convergence of the finite-dimensional distributions of Y_ρ , as ρ tends to zero, to those of the local time ℓ of the fBm $B^{H(t_0)}$ with Hurst parameter $H(t_0)$. We need to show that, for any $d \geq 1$, $a_1, \dots, a_d \in \mathbb{R}$ and $t_1, \dots, t_d \in [0, 1]$, the following convergence holds:

$$\sum_{j=1}^d a_j Y_\rho(t_j, x) \xrightarrow{\mathcal{L}} \sum_{j=1}^d a_j \ell(t_j, x) \quad \text{as } \rho \rightarrow 0.$$

We will show the convergence of the corresponding characteristic function. More precisely, we prove that

$$\left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right| \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

Let us introduce the following notation:

$$B^\rho(t) = \frac{B(t_0 + \rho t) - B(t_0)}{\rho^{H(t_0)}},$$

$$\phi_{\varepsilon, x}(X)(t) = \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[x, x+\varepsilon]}(X(s)) ds,$$

$$I_1^{\varepsilon, \rho} = \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^\rho)(t_j) \right] \right|,$$

$$I_2^{\varepsilon, \rho} = \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^{H(t_0)})(t_j) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^\rho)(t_j) \right] \right|$$

and

$$I_3^{\varepsilon, \rho} = \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^{H(t_0)})(t_j) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right|.$$

Therefore,

(5.1)

$$\left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right| \leq I_1^{\varepsilon, \rho} + I_2^{\varepsilon, \rho} + I_3^{\varepsilon, \rho}.$$

On the other hand, Y_ρ is the local time of B^ρ and by using the mean value theorem and the occupation density formula we obtain

$$\begin{aligned} (5.2) \quad I_1^{\varepsilon, \rho} &\leq C \max_{1 \leq j \leq d} \mathbb{E} |Y_\rho(t_j, x) - \phi_{\varepsilon, x}(B^\rho)(t_j)| \\ &= C \max_{1 \leq j \leq d} \mathbb{E} \left| \frac{1}{\varepsilon} \int_x^{x+\varepsilon} Y_\rho(t_j, y) dy - Y_\rho(t_j, x) \right| \\ &\leq C \max_{1 \leq j \leq d} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} \|Y_\rho(t_j, y) - Y_\rho(t_j, x)\|_{L^m(\Omega)} dy \end{aligned}$$

and

$$\begin{aligned} & \|Y_\rho(t_j, y) - Y_\rho(t_j, x)\|_{L^m(\Omega)} \\ &= \left\| \frac{L(t_0 + \rho t_j, \rho^{H(t_0)}y + B(t_0)) - L(t_0, \rho^{H(t_0)}y + B(t_0))}{\rho^{1-H(t_0)}} \right. \\ & \quad \left. - \frac{L(t_0 + \rho t_j, \rho^{H(t_0)}x + B(t_0)) - L(t_0, \rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)}} \right\|_{L^m(\Omega)}. \end{aligned}$$

On the other hand, putting $I_j = [t_0, t_0 + \rho t_j]$, we get

$$\begin{aligned} & \left\| \frac{L(I_j, \rho^{H(t_0)}y + B(t_0)) - L(I_j, \rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)}} \right\|_{L^m(\Omega)} \\ & \leq \frac{C_m(\rho t_j)^{1-H(t_0)(1+\xi)} \rho^{H(t_0)\xi} |x - y|^\xi}{\rho^{1-H(t_0)}} \leq \frac{C_m t_j^{1-H(t_0)(1+\xi)} \rho^{1-H(t_0)} |x - y|^\xi}{\rho^{1-H(t_0)}} \\ & \leq C_m |x - y|^\xi. \end{aligned}$$

Therefore, (5.2) is dominated by

$$C_m \max_{1 \leq j \leq d} \frac{1}{\epsilon} \int_x^{x+\epsilon} |x - y|^\xi dy = C_m \epsilon^\xi.$$

Then (5.2) converges to zero as ϵ tends to zero uniformly in ρ .

We deal now with $I_2^{\epsilon, \rho}$. Since the family of processes $B^\rho(t), t \in [0, 1]_{\rho > 0}$, converges in distribution to the fBm $B^{H(t_0)}(t), t \in [0, 1]$, with Hurst parameter $H(t_0)$, the second term converges to zero as ρ tends to zero, by Lemma 5.1 in Boufoussi et al. [7]. The last term in (5.1) is treated in a similar way to the first one and the proof of the finite-dimensional convergence is complete.

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REFERENCES

- [1] M. Ait Ouahra, *Weak convergence to fractional Brownian motion in some anisotropic Besov space*, Ann. Math. Blaise Pascal 11 (1) (2004), pp. 1–17.
- [2] M. Ait Ouahra and M. Eddahbi, *Théorèmes limites pour certaines fonctionnelles associées aux processus stables sur l'espace de Hölder*, Publ. Mat. 45 (2) (2001), pp. 371–386.
- [3] M. Ait Ouahra and M. Ouali, *Occupation time problems for fractional Brownian motion and some other self-similar processes*, Random Oper. Stoch. Equ. 17 (1) (2009), pp. 69–89.
- [4] A. Benassi, S. Jaffard, and D. Roux, *Elliptic Gaussian random processes*, Rev. Mat. Iberoam. 13 (1) (1997), pp. 19–90.

- [5] S. M. Berman, *Local nondeterminism and local times of Gaussian processes*, Indiana Univ. Math. J. 23 (1973/74), pp. 69–94.
- [6] B. Boufoussi, M. Dozzi, and R. Guerbaz, *On the local time of multifractional Brownian motion*, Stochastics 78 (1) (2006), pp. 33–49.
- [7] B. Boufoussi, M. Dozzi, and R. Guerbaz, *Sample path properties of the local time of multifractional Brownian motion*, Bernoulli 13 (3) (2007), pp. 849–867.
- [8] M. Eddahbi and J. Vives, *Chaotic expansion and smoothness of some functionals of the fractional Brownian motion*, J. Math. Kyoto Univ. 43 (2) (2003), pp. 349–368.
- [9] H. Ezawa, J. R. Klauder, and L. A. Shepp, *Vestigial effects of singular potentials in diffusion theory and quantum mechanics*, J. Math. Phys. 16 (1975), pp. 783–799.
- [10] P. J. Fitzsimmons and R. K. Gettoor, *Limit theorems and variation properties for fractional derivatives of the local time of a stable process*, Ann. Inst. H. Poincaré Probab. Statist. 28 (2) (1992), pp. 311–333.
- [11] M. Fukushima, *A decomposition of additive functionals of finite energy*, Nagoya Math. J. 74 (1979), pp. 137–168.
- [12] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes. I*, Springer, New York–Heidelberg 1974.
- [13] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. I*, Math. Z. 27 (1) (1928), pp. 565–606.
- [14] M. Jolis and N. Viles, *Continuity in law with respect to the Hurst parameter of the local time of the fractional Brownian motion*, J. Theoret. Probab. 20 (2) (2007), pp. 133–152.
- [15] J. Lévy-Véhel, *Introduction to the multifractal analysis of images*, in: *Fractal Image Encoding and Analysis*, Y. Fisher (Ed.), Springer, New York 1995.
- [16] J. Lévy-Véhel and R. F. Peltier, *Multifractional Brownian motion: Definition and preliminary results*, [Research Report], RR-2645, INRIA, 1995.
- [17] M. Meerschaert, D. Wu, and Y. Xiao, *Local times of multifractional Brownian sheets*, Bernoulli 14 (3) (2008), pp. 865–898.
- [18] B. Pesquet-Popescu and J. Lévy-Véhel, *Stochastic fractal models for image processing*, IEEE Signal Processing Magazine 19 (5) (2002), pp. 48–62.
- [19] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon 1993.
- [20] N. Shieh, *Limit theorems for local times of fractional Brownian motions and some other self-similar processes*, J. Math. Kyoto Univ. 36 (4) (1996), pp. 641–652.
- [21] T. Yamada, *On some limit theorems for occupation times of one-dimensional Brownian motion and its continuous additive functionals locally of zero energy*, J. Math. Kyoto Univ. 26 (2) (1986), pp. 309–322.

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