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LÉVY PROCESSES, GENERALIZED MOMENTS AND UNIFORM INTEGRABILITY

BY

DAVID BERGER (DRESDEN), FRANZISKA KÜHN (DRESDEN), AND
RENÉ L. SCHILLING (DRESDEN)

Abstract. We give new proofs of certain equivalent conditions for the existence of generalized moments of a Lévy process $(X_t)_{t\geqslant 0}$; in particular, the existence of a generalized g-moment is equivalent to the uniform integrability of $(g(X_t))_{t\in[0,1]}$. As a consequence, certain functions of a Lévy process which are integrable and local martingales are already true martingales. Our methods extend to moments of stochastically continuous additive processes, and we give new, short proofs for the characterization of lattice distributions and the transience of Lévy processes.

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1. INTRODUCTION

A *generalized moment* of a stochastic process $(X_t)_{t\geqslant 0}$ is an expression of the form $\mathbb{E}[g(X_t)]$. Such moments arise naturally when studying Markov semigroups. It is a classical result that for a Lévy process and a submultiplicative function g the g-moment exists if, and only if, $g(x)\mathbb{1}_{\{|x|>1\}}$ is integrable with respect to the jump measure of the process (Section 2). Throughout the analysis and probability literature, further equivalent criteria for the existence of g-moments can be found. In this note we collect these criteria, add a few more equivalences, and give a unified presentation with novel proofs. These proofs frequently exploit martingale techniques and the Markov property, rather than the translation invariance of a Lévy process, which means that many implications – alas, not all – remain valid in more general situations (cf. Remark 2.2). A summary of the existing literature is also given in Remark 2.2. Our arguments are based on Dynkin's formula (which is a consequence of the martingale nature of the process) and Gronwall's lemma, and this technique can also be used (Section 3) to show that certain functions of a Lévy process $(f(X_t))_{t\geqslant 0}$ which are both a local martingale and integrable, i.e. $\mathbb{E}[|f(X_t)|] < \infty$, are already proper martingales. With some minor changes our methods extend to stochastically continuous additive processes (Section 4), and the criteria in Theorem 4.1 seem to be new, extending earlier work by Fujiwara [3] and Klass & Yang [7]. In the last part of the paper (Section 5) we further apply our results to get a very short proof of the characterization of infinitely divisible lattice distributions and a martingale approach to the transience of Lévy processes.

Let us recall a few key concepts and techniques which will be needed later on. Most of our notation is standard or self-explanatory; we use $|x|_{\ell^p}^p := \sum_{k=1}^d |x_k|^p$ with the usual modification if $p = \infty$. We write $\|f\|_{L^1(\mathbb{R}^d,g)} := \int_{\mathbb{R}^d} |f(x)|\, g(x)\, dx$ for the weighted L^1-norm and $L^1(\mathbb{R}^d, g)$ for the corresponding L^1-space (with a nonnegative, measurable weight function $g : \mathbb{R}^d \to [0,\infty)$).

1.1. Lévy processes. A *Lévy process* $X = (X_t)_{t\geqslant 0}$ is a stochastic process with values in $\mathbb{R}^d$, stationary and independent increments and right-continuous sample paths with finite left-hand limits (càdlàg). Our standard references for Lévy processes are Sato [12] (for probabilistic properties) and Jacob [5, 6] (for analytic aspects). It is well known that a stochastic process X is a Lévy process if it has càdlàg paths and its conditional characteristic function is of the form

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi\cdot(X_t - X_s)} \mid \mathscr{F}_s] = \mathrm{e}^{-(t-s)\psi(\xi)}, \qquad 0 \leqslant s \leqslant t,\ \xi \in \mathbb{R}^d,$$

where $\mathscr{F}_s = \sigma(X_r, r \leqslant s)$ is the natural filtration of X. The *characteristic exponent* $\psi : \mathbb{R}^d \to \mathbb{C}$ is uniquely determined by the *Lévy–Khinchin formula*

$$\psi(\xi) = -\mathrm{i}b\cdot\xi + \frac{1}{2}Q\xi\cdot\xi + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(1 - \mathrm{e}^{\mathrm{i}\xi\cdot x} + \mathrm{i}\xi\cdot x \mathbb{1}_{(0,1)}(|x|)\right) \nu(dx). \tag{1.1}$$

The *Lévy triplet* (b, Q, ν) where $b \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ (a positive semidefinite matrix) and ν (a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d\setminus\{0\}} \min\{|x|^2, 1\}\, \nu(dx) < \infty$) uniquely describe ψ.

Using the characteristic exponent we can determine the infinitesimal generator A of the process X either as *pseudo-differential operator*

$$Au(x) = -\psi(D)u(x) = \mathcal{F}^{-1}[-\psi\mathcal{F}u](x), \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{F}u(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\xi\cdot x}u(x)\, dx$ is the Fourier transform and $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing smooth functions, or as an integro-differential operator

$$Au(x) = Lu(x) + Ju(x) + Ku(x) \tag{1.2}$$

where L, J and K are again linear operators: L is the local part, J takes into account the small jumps and K the large jumps, i.e.

$$\begin{aligned}
Lu(x) &= b\cdot\nabla u(x) + \frac{1}{2}\nabla\cdot Q\nabla u(x),\\
Ju(x) &= \int_{0<|y|<1} (u(x+y)-u(x)-y\cdot\nabla u(x))\nu(dy),\\
Ku(x) &= \int_{|y|\geqslant 1} (u(x+y)-u(x))\nu(dy).
\end{aligned}$$

The precise form of the domain $\mathcal{D}(A)$ of A (as a closed operator on the Banach space $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ of continuous functions vanishing at infinity) is not known; but both the test functions $C_c^\infty(\mathbb{R}^d)$ and the Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ are operator cores. On the other hand, the expression (1.2) has a pointwise meaning for every positive $u \in C^2(\mathbb{R}^d)$: using a Taylor expansion, it is easy to see that for every $x \in \mathbb{R}^d$ one has $|(L+J)u(x)| \leqslant C\sup_{|x-y|\leqslant 1}(|\nabla u(y)| + |\nabla^2 u(y)|) < \infty$ for some finite constant $C > 0$ while positivity is needed for $Ku(x) \in (-\infty,\infty]$ [1]. We will continue to use the notation $Au(x)$ despite the fact that $C^2(\mathbb{R}^d) \not\subset \mathcal{D}(A)$; in particular, we allow $Au(x)$ to take values in $(-\infty,\infty]$.

The *transition semigroup* $(P_t)_{t\geqslant 0}$ corresponding to the generator A or the process X is given by $P_t u(x) = \mathbb{E}[u(x+X_t)]$. Its adjoint, $P_t^* u(x) = \mathbb{E}[u(x-X_t)]$ is the transition semigroup of the Lévy process $-X = (-X_t)_{t\geqslant 0}$.

1.2. Additive processes. An *additive process* is a stochastic process $(X_t)_{t\geqslant 0}$ with independent increments, values in $\mathbb{R}^d$ and starting from 0. In particular, every Lévy process is a stochastically continuous additive process. Because of the non-stationarity of increments, an additive process may have fixed jump discontinuities and need not be a semimartingale; a simple – and at the same time typical – example is the process $X_t = b_t$ where $(b_t)_{t\geqslant 0}$ is a deterministic càdlàg function of infinite variation.

For each $t \geqslant 0$, the law of a stochastically continuous additive process $(X_t)_{t\geqslant 0}$ is infinitely divisible, i.e. the law of X_t is uniquely determined by a Lévy–Khinchin formula with a t-dependent characteristic triplet (b_t, Q_t, ν_t). Additivity implies that these triplets satisfy $\langle\xi, Q_s\xi\rangle \leqslant \langle\xi, Q_t\xi\rangle$ and $\nu_s(B) \leqslant \nu_t(B)$ for all $s \leqslant t$ and every $\xi \in \mathbb{R}^d$ and Borel set $B \subseteq \mathbb{R}^d \setminus \{0\}$. Moreover, for each $\xi \in \mathbb{R}^d$ the map

$$t \mapsto \langle\xi, b_t\rangle + \langle\xi, Q_t\xi\rangle + \int_{x\neq 0} \min\{1,|x|^2\}\,\nu_t(dx), \quad t \geqslant 0, \tag{1.3}$$

is continuous. A full discussion is given in Sato [12, Chapter 2.9].

[1] The latter may be replaced by a polynomial growth bound on u, $|u(x)| \leqslant C(1+|x|^p)$ and a moment condition $\int_{|y|>1} |y|^p\,\nu(dy) < \infty$ on ν.

1.3. Dynkin's formula. Let $(X_t)_{t\geqslant 0}$ be a Lévy process (or a strong Markov process) and denote by $\mathscr{F}_t = \sigma(X_s, s \leqslant t)$ its natural filtration and $(A, \mathcal{D}(A))$ the infinitesimal generator. Dynkin's formula states that for every $u \in \mathcal{D}(A)$ and every stopping time σ with $\mathbb{E}[\sigma] < \infty$ we have

$$\mathbb{E}[u(X_\sigma + x)] - u(x) = \mathbb{E}\Big[\int_{[0,\sigma)} Au(X_s)\, ds\Big]. \tag{1.4}$$

There are several ways to prove this result, e.g. using arguments from potential theory (as in [14, Proposition 7.31]), semigroup theory (as in [11, Proposition VII.1.6]) or by Itô's formula. At the heart of the argument is the fact that

$$M_t^{[u]} := u(X_t + x) - u(x) - \int_{[0,t)} Au(X_s)\, ds, \quad u \in \mathcal{D}(A), \tag{1.5}$$

is an $\mathscr{F}_t$-martingale, combined with a stopping argument.

There are various ways to extend the class of functions u for which we have some kind of Dynkin's formula. It is clear that formula (1.4) can be extended to all functions u with $u(X_\sigma) \in L^1(\mathbb{P})$ and $Au(X_{s\wedge\sigma}) \in L^1(ds \otimes \mathbb{P})$. Such moment estimates will be given below.

Here we need a *Dynkin inequality* which we are going to prove for positive $g \in C^2(\mathbb{R}^d)$.

LEMMA 1.1 (Dynkin's inequality). *Let $(X_t)_{t\geqslant 0}$ be a Lévy process with generator $(A, \mathcal{D}(A))$ and extend A using* (1.2) *to $C^2(\mathbb{R}^d)$. For every $g \in C^2(\mathbb{R}^d)$ satisfying $g(x) \geqslant 0$ and every stopping time σ,*

$$\mathbb{E}[g(X_{t\wedge\sigma})] \leqslant g(0) + \mathbb{E}\Big[\int_{[0,t\wedge\sigma)} |Ag(X_s)|\, ds\Big]. \tag{1.6}$$

Proof. Pick for every $R > 0$ some cut-off function $\chi_R \in C^\infty(\mathbb{R}^d)$ with $\mathbb{1}_{\overline{B}_{R+1}(0)} \leqslant \chi_R \leqslant \mathbb{1}_{\overline{B}_{R+2}(0)}$. Since $g\chi_R \in C_c^2(\mathbb{R}^d)$ we know that $g\chi_R \in \mathcal{D}(A)$, and we see that for any stopping time σ the process $(M_{t\wedge\sigma}^{[g\chi_R]})_{t\geqslant 0}$ is a martingale. Therefore we have

$$\mathbb{E}[(g\chi_R)(X_{t\wedge\sigma})] - g(0) = \mathbb{E}\Big[\int_{[0,t\wedge\sigma)} A(g\chi_R)(X_s)\, ds\Big]. \tag{1.7}$$

If we replace σ by the stopping time $\sigma \wedge \tau_R$ where $\tau_R = \inf\{s \geqslant 0 \mid |X_s| \geqslant R\}$, then we can use the fact that $|X_s| \leqslant R$ if $s \in [0, t\wedge\sigma\wedge\tau_R)$. This implies, in particular, that $\partial^\alpha(g\chi_R)(X_s) = \partial^\alpha g(X_s)$, and we see from the integro-differential representation (1.2) of A that for all $0 \leqslant s < t \wedge \sigma \wedge \tau_R$,

$$
\begin{aligned}
(1.8) \qquad A(g\chi_R)(X_s) &= L(g\chi_R)(X_s) + J(g\chi_R)(X_s) + K(g\chi_R)(X_s) \\
&= b \cdot \nabla g(X_s) + \frac{1}{2}\nabla \cdot Q\nabla g(X_s) \\
&\quad + \int_{0<|y|<1} (g(X_s + y) - g(X_s) - y \cdot \nabla g(X_s))\, \nu(dy) \\
&\quad + \int_{|y|\geqslant 1} ((g\chi_R)(X_s + y) - g(X_s))\, \nu(dy).
\end{aligned}
$$

For the second equality observe that $|X_s| \leqslant R$, $|X_s + y| \leqslant R + 1$ for $|y| < 1$ and that L is a local operator. Since g is positive, we have $g\chi_R \leqslant g$, and we conclude that $A(g\chi_R)(X_s) \leqslant Ag(X_s)$. Inserting this into (1.7) gives

$$
\begin{aligned}
\mathbb{E}[(g\chi_R)(X_{t\wedge\sigma\wedge\tau_R})] - g(0) &\leqslant \mathbb{E}\Big[\int_{[0,t\wedge\sigma\wedge\tau_R)} Ag(X_s)\, ds\Big] \\
&\leqslant \mathbb{E}\Big[\int_{[0,t\wedge\sigma)} |Ag(X_s)|\, ds\Big].
\end{aligned}
$$

Since $g \geqslant 0$, we can use Fatou's lemma on the left-hand side and get (1.6). ∎

1.4. Friedrichs mollifiers. Let $j : \mathbb{R}^d \to [0, \infty)$ be a C^∞-function with compact support $\operatorname{supp} j \subset \overline{B}_1(0)$ such that $j(x)$ is rotationally symmetric and the integral $\int j(x)\, dx$ is equal to 1. For every $\epsilon > 0$ we define $j_\epsilon(x) := \epsilon^{-d} j(x/\epsilon)$, i.e. j_ϵ is again smooth, rotationally symmetric and satisfies $\operatorname{supp} j_\epsilon \subset \overline{B}_\epsilon(0)$ and $\int j_\epsilon(x)\, dx = 1$. For any locally bounded function $g : \mathbb{R}^d \to \mathbb{R}$ the following convolution exists and defines a C^∞-function:

$$
g^\epsilon(x) := j_\epsilon * g(x) := \int g(x - y) j_\epsilon(y)\, dy, \quad x \in \mathbb{R}^d.
$$

Moreover, $\operatorname{supp} g^\epsilon \subset \operatorname{supp} g + \operatorname{supp} j_\epsilon \subset \operatorname{supp} g + \overline{B}_\epsilon(0)$. The function g^ϵ is called a *Friedrichs regularization* of g.

1.5. Submultiplicative functions. A function $g : \mathbb{R}^d \to [0, \infty)$ is said to be *submultiplicative* if there exists a constant $c = c(g) \in [1, \infty)$ such that

$$
\forall x, y \in \mathbb{R}^d : \quad g(x + y) \leqslant c g(x) g(y).
$$

In order to avoid pathologies, we consider only measurable submultiplicative functions [2]. Every locally bounded submultiplicative function grows at most exponentially, i.e. there are constants $a, b \in (0, \infty)$ such that $g(x) \leqslant a e^{b|x|}$. Since $1 + g$ inherits submultiplicativity from g, we may assume that $g \geqslant 1$. The following lemma shows that we can even assume that a submultiplicative function is smooth.

[2] An example of a non-measurable submultiplicative function is $g(x) = e^{a(x)}$, $x \in \mathbb{R}$, where a is a non-measurable solution to the functional equation $a(x + y) = a(x) + a(y)$.

LEMMA 1.2. *Let g be a locally bounded submultiplicative function and g^ϵ its Friedrichs regularization. Then $g^\epsilon \in C^\infty$ is submultiplicative and it satisfies*

$$\forall x \in \mathbb{R}^d: \quad c_\epsilon^{-1} g(x) \leqslant g^\epsilon(x) \leqslant c_\epsilon g(x) \tag{1.9}$$

for some constant $c_\epsilon = c_{\epsilon,g}$.

Proof. Submultiplicativity follows immediately from the two-sided estimate (1.9):

$$g^\epsilon(x+y) \leqslant c_\epsilon g(x+y) \leqslant c_\epsilon c g(x) g(y) \leqslant c_\epsilon^3 c g^\epsilon(x) g^\epsilon(y).$$

In order to see (1.9), we use the definition of g^ϵ and fact that g is submultiplicative,

$$g^\epsilon(x) = \int g(x-y) j_\epsilon(y)\, dy \leqslant c g(x) \int g(-y) j_\epsilon(y)\, dy \leqslant c \sup_{|y|\leqslant\epsilon} g(y) g(x)$$

and

$$g(x) = \int g(x) j_\epsilon(y)\, dy \leqslant c \int g(x-y) g(y) j_\epsilon(y)\, dy \leqslant c \sup_{|y|\leqslant\epsilon} g(y) g^\epsilon(x). \quad \blacksquare$$

2. GENERALIZED MOMENTS AND UNIFORM INTEGRABILITY

Let $(X_t)_{t\geqslant 0}$ be a Lévy process with triplet (b, Q, ν). The following moment result for a locally bounded submultiplicative functions g is well-known (cf. Sato [12, Theorem 25.3, p. 159]):

$$\mathbb{E}[g(X_t)] < \infty \text{ for some (hence, all) } t > 0 \iff \int_{|y|\geqslant 1} g(y)\, \nu(dy) < \infty. \tag{2.1}$$

Our aim is to show that this is also equivalent to a certain uniform integrability condition. Although we cast the statement and proof for Lévy processes, an extension to certain Lévy-type processes is possible; see Remark 2.1 below. We denote by $\mathcal{T}$ the family of stopping times for the process X equipped with its natural filtration.

THEOREM 2.1. *Let $(X_t)_{t\geqslant 0}$ be a Lévy process with generator A, transition semigroup $(P_t)_{t\geqslant 0}$, and triplet (b, Q, ν), and let g be a locally bounded submultiplicative function. The following assertions are equivalent:*

(a) $\mathbb{E}[g(X_t)]$ *is finite for some* (*hence, all*) $t > 0$.

(b) $\mathbb{E}[\sup_{s\leqslant t} g(X_s)]$ *is finite for some* (*hence, all*) $t > 0$.

(c) $\{g(X_\sigma)\}_{\sigma\in\mathcal{T},\, \sigma\leqslant t}$ *is uniformly integrable for every $t > 0$, i.e.*

$$\lim_{R\to\infty} \sup_{\sigma\in\mathcal{T},\, \sigma\leqslant t} \int_{g(X_\sigma)>R} g(X_\sigma)\, d\mathbb{P} = 0.$$

(d) $\sup_{\sigma\in\mathcal{T},\,\sigma\leqslant t}\mathbb{E}[g(X_\sigma)]$ *is finite for every* $t>0$.

(e) $\int_{|y|\geqslant 1} g(y)\,\nu(dy)<\infty$.

(f) *The adjoint semigroup* $P_t^* f(x) := \mathbb{E}[f(x-X_t)]$ *is a strongly continuous semigroup on the weighted* L^1*-space* $L^1(\mathbb{R}^d, g)$.

(g) *The adjoint generator* $(A^*\phi)(x) := (A\phi)(-x)$ *satisfies* $A^*\phi\in L^1(\mathbb{R}^d,g)$ *for all* $\phi\in C_c^\infty(\mathbb{R}^d)$.

(h) *There exists a non-negative* $\phi\in C_c^\infty(\mathbb{R}^d)$, $\phi\not\equiv 0$, *with* $A^*\phi\in L^1(\mathbb{R}^d,g)$.

If one of the conditions is satisfied (*hence, all are*)*, then there are constants* $c_i>0$, $i=1,2$, *such that*

$$\mathbb{E}[g(X_t)]\leqslant c_1\mathrm{e}^{c_2 t},\qquad t\geqslant 0, \tag{2.2}$$

and $\|A^*\phi\|_{L^1(\mathbb{R}^d,g)}$ *is bounded by a constant multiple of*

$$\Big(|b|_{\ell^1}+\|Q\|_{\ell^1}+\int_{y\neq 0}(1\wedge|y|^2)\,\nu(dy)+\int_{|y|\geqslant 1}g(y)\,\nu(dy)\Big)\|\phi\|_{C_b^2(\mathbb{R}^d)}.$$

REMARK 2.1. There is another equivalent condition if $x\mapsto g(|x|)$ is locally bounded, submultiplicative and $g(r)$ is increasing:

(i) $\mathbb{E}[g(\sup_{s\leqslant t}|X_s|)]<\infty$ for some (hence, all) $t>0$.

The direction (i)⇒(a) follows from the assumption that g is increasing. The implication (b)⇒(i) does not need monotonicity since we have

$$g\Big(\sup_{s\leqslant t}|X_s|\Big)\leqslant\sup_{s\leqslant t}g(|X_s|)$$

at least if $x\mapsto g(|x|)$ is continuous. This can always be achieved by a Friedrichs regularization.

Let us also point out that we may replace A^* and P_t^* by A, and P_t if either (the law of) X_t is symmetric or if g is even, i.e. $g(x)=g(-x)$.

REMARK 2.2. Many of the equivalent criteria in Theorem 2.1 are scattered throughout the analysis and probability literature, and they are usually proved separately, using ad hoc methods.

The equivalence of (a), (d) (for deterministic times), (e) and (i) can be found in Sato [12, Chapter 25]. The equivalence of (a) and (b) is due to Siebert [15], and variants of (g), (h) appear first in Hulanicki [4]; their proofs are cast in the language of probability on (Lie) groups. The conditions (c), (f), and the condition (d) (with bounded stopping times) are new, and so are the streamlined proofs given in the present paper.

Some of our arguments carry over to Lévy-type processes whose generators have bounded coefficients (see [1, p. 55] for the notation); in particular (e)⇒(a) [using the alternative proof below] ⇒(b)⇒(c)⇒(d) becomes

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d}\int_{|y|\geqslant 1} g(y)\,\nu(x,dy)<\infty \implies & \sup_{x\in\mathbb{R}^d}\mathbb{E}^x[g(X_t-x)]<\infty\\
\implies & \sup_{x\in\mathbb{R}^d}\mathbb{E}^x\Big[\sup_{s\leqslant t} g(X_s-x)\Big]<\infty\\
\implies & \{g(X_\sigma-x)\}_{\sigma\in\mathcal{T},\,\sigma\leqslant t} \text{ is uniformly integrable}\\
\implies & \sup_{\sigma\in\mathcal{T},\,\sigma\leqslant t}\mathbb{E}^x[g(X_\sigma-x)]<\infty,
\end{aligned}$$

while (d)⇒(e) only yields $\inf_{x\in\mathbb{R}^d}\int g(y)\,\nu(x,dy)<\infty$, and an additional condition of the type

$$\sup_{x\in\mathbb{R}^d}\int_{|y|\geqslant 1} g(y)\,\nu(x,dy)\leqslant C\inf_{x\in\mathbb{R}^d}\int_{|y|\geqslant 1} g(y)\,\nu(x,dy)$$

is needed to get equivalences; this is partly worked out in [8].

In order to prove this theorem, we need a few preparations.

LEMMA 2.1. *Let g be a locally bounded submultiplicative function. If $(X_t)_{t\geqslant 0}$ is a Lévy process, then*

$$\mathbb{E}\Big[\sup_{s\leqslant T} g(X_s)\Big]=\kappa_T<\infty \quad \textit{implies} \quad \mathbb{E}\Big[\sup_{s\leqslant 2T} g(X_s)\Big]\leqslant \kappa_T(1+c\kappa_T)<\infty$$

and

$$\mathbb{E}[g(X_t)]<\infty \textit{ for some } t>0 \quad \textit{implies} \quad \mathbb{E}[g(X_t)]<\infty \textit{ for all } t>0.$$

Proof. **1°** We have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\leqslant 2T} g(X_s)\Big] &\leqslant \mathbb{E}\Big[\sup_{s\leqslant T} g(X_s)\Big]+\mathbb{E}\Big[\sup_{T\leqslant s\leqslant 2T} g(X_s)\Big]\\
&=\kappa_T+\mathbb{E}\Big[\sup_{s\leqslant T} g(X_{s+T})\Big].
\end{aligned}$$

Since g is a submultiplicative function, we see $g(X_{s+T})\leqslant cg(X_{s+T}-X_T)g(X_T)$; moreover, X_T and the process $(X_{s+T}-X_T)_{s\geqslant 0}\sim (X_s)_{s\geqslant 0}$ are independent [3],

[3]We may replace this by the (strong) Markov property if $\sup_{y\in\mathbb{R}^d}\sup_{s\leqslant T}\mathbb{E}^y[g(X_s-y)]<\infty$. For Lévy processes the first supremum is always trivial.

and so

$$\mathbb{E}\Big[\sup_{s\leqslant T} g(X_{s+T})\Big] \leqslant c\mathbb{E}\Big[\sup_{s\leqslant T} g(X_{s+T}-X_T)\Big]\mathbb{E}[g(X_T)]$$
$$\leqslant c\mathbb{E}\Big[\sup_{s\leqslant T} g(X_s)\Big]\mathbb{E}[g(X_T)] \leqslant c\kappa_T^2.$$

2° Let $t_0 > 0$ be such that $\mathbb{E}[g(X_{t_0})] < \infty$. Using the Markov property we see that for any $s < t_0$,

$$\mathbb{E}[g(X_{t_0})] = \mathbb{E}[g(X_{t_0} - X_s + X_s)] = \int_{\mathbb{R}^d} \mathbb{E}[g(X_s + y)]\,\mathbb{P}(X_{t_0-s} \in dy).$$

Thus, there is some y such that $\mathbb{E}[g(X_s + y)] < \infty$, and we conclude from the submultiplicative property that $\mathbb{E}[g(X_s)] \leqslant cg(-y)\mathbb{E}[g(X_s + y)] < \infty$ for all $s \leqslant t_0$. As before, we can now show that $\mathbb{E}[g(X_{2t_0})] < \infty$ and, by iteration, we see that $\mathbb{E}[g(X_t)] < \infty$ for all $t > 0$. ■

LEMMA 2.2. *Let g be a locally bounded submultiplicative function and denote by g^ϵ its regularization with a Friedrichs mollifier. If A is the generator of a Lévy process given by* (1.2)*, then $|Ag^\epsilon(x)|$ is bounded by*

$$\Big(C_\epsilon \cdot C(b,Q,\nu;g)\cdot \sup_{|y|\leqslant 1} g(y)\Big)\cdot g(x),$$

where

$$C(b,Q,\nu;g) = \Big(|b|_{\ell^1} + |Q|_{\ell^1} + \int_{y\neq 0} (1\wedge |y|^2)\,\nu(dy) + \int_{|y|\geqslant 1} g(y)\,\nu(dy)\Big).$$

Note that the constant C_ϵ appearing in Lemma 2.2 is, in general, unbounded as $\epsilon \to 0^+$.

Proof of Lemma 2.2. Without loss of generality we can assume that $g \geqslant 1$. Otherwise we use $g+1$ instead of g and observe that $A(g+1)^\epsilon = A(g^\epsilon+1) = Ag^\epsilon$. We set $s_\epsilon := \sup_{|y|\leqslant\epsilon} g(y)$ and as in (1.2) write $A = L + J + K$.

Observe that $|\partial^\alpha g^\epsilon(x)| \leqslant c_{\alpha,\epsilon} s_\epsilon g(x)$ for every multi-index $\alpha \in \mathbb{N}_0^d$. This follows from $\partial^\alpha g^\epsilon(x) = (\partial^\alpha j_\epsilon) * g(x)$ and

$$|\partial^\alpha g^\epsilon(x)| \leqslant \int |\partial^\alpha j_\epsilon(y)| g(x-y)\,dy \leqslant cg(x)s_\epsilon \int |\partial^\alpha j_\epsilon(y)|dy.$$

We can now estimate the three parts of A separately. For the local part we use the above estimate with $|\alpha| = 1$ and $|\alpha| = 2$:

$$|Lg^\epsilon|(x) \leqslant s_\epsilon\bigg(\sum_{i=1}^d |b_i| c_{\epsilon,i} + \frac{1}{2}\sum_{i,k=1}^d |q_{ik}| c_{\epsilon,i,k}\bigg)\cdot g(x)$$
$$\leqslant c_\epsilon s_\epsilon (|b|_{\ell^1} + |Q|_{\ell^1})\cdot g(x).$$

The large-jump part is estimated using $|\alpha| = 0$ and the submultiplicativity of g:

$$\begin{aligned}|Kg^\epsilon(x)| &\leqslant c_\epsilon s_\epsilon \int\limits_{|y|\geqslant 1} (g(x)g(y) + g(x))\,\nu(dy)\\ &= c_\epsilon s_\epsilon \int\limits_{|y|\geqslant 1} (g(y)+1)\,\nu(dy)\cdot g(x).\end{aligned}$$

Using Taylor's formula with integral remainder term we can rewrite the part containing the small jumps and we see that

$$\begin{aligned}|Jg^\epsilon(x)| &= \Big|\sum_{i,k=1}^d \int\limits_{0<|y|<1} \int\limits_0^1 \partial_i\partial_k g^\epsilon(x+ty) y_i y_k (1-t)\,dt\,\nu(dy)\Big|\\ &\leqslant c_\epsilon s_\epsilon \sum_{i,k=1}^d c_{\epsilon,i,k} \int\limits_{0<|y|<1} \int\limits_0^1 g(x+ty)|y_i y_k|(1-t)\,dt\,\nu(dy)\\ &\leqslant c_\epsilon s_\epsilon \sum_{i,k=1}^d c_{\epsilon,i,k} \int\limits_{0<|y|<1} \int\limits_0^1 g(ty)|y_i y_k|(1-t)\,dt\,\nu(dy)\cdot g(x)\\ &\leqslant c'_\epsilon \sup_{|y|\leqslant 1} g(y) \int\limits_{0<|y|<1} |y|^2\,\nu(dy)\cdot g(x).\end{aligned}$$

If we combine these three estimates, the claim follows. ■

Proof of Theorem 2.1. We show (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(g)⇒(h)⇒(a) and then (c)⇒(f)⇒(a) [4]. Throughout the proof we will assume that $g \geqslant 1$ and $g \in C^2(\mathbb{R}^d)$. Otherwise we could replace g by its Friedrichs regularization g^ϵ (see Lemma 1.2), and $g+1$, resp., $g^\epsilon + 1$.

1° (a)⇒(b). If $\mathbb{E}[g(X_t)]$ is finite for some $t>0$, then Lemma 2.1 shows that it is finite for all $t>0$. For $a,b>0$ we define a stopping time

$$\sigma := \sigma_{a,b} := \inf\{s \mid g(X_s) > cg(0)\mathrm{e}^{a+b}\}$$

and observe that we can use the submultiplicativity of g to get

$$\begin{aligned}\mathbb{P}(g(X_t) > \mathrm{e}^a) &\geqslant \mathbb{P}\big(g(X_t) > \mathrm{e}^a,\, g(X_\sigma) \geqslant cg(0)\mathrm{e}^{a+b},\, \sigma \leqslant t\big)\\ &\geqslant \mathbb{P}\big(g(X_\sigma - X_t) < g(0)\mathrm{e}^b,\, g(X_\sigma) \geqslant cg(0)\mathrm{e}^{a+b},\, \sigma\leqslant t\big).\end{aligned}$$

[4] A direct proof of (e)⇒(a) is given in the next section. This is particularly interesting if we want to go beyond Lévy processes.

The strong Markov property yields (for all $t \leqslant T$, with T to be determined in the following step)

$$\text{(2.3)} \qquad \begin{aligned} \mathbb{P}(g(X_t) > \mathrm{e}^a) &\geqslant \int_{\sigma \leqslant t} \mathbb{P}\big(g(-X_{t-\sigma(\omega)}) < g(0)\mathrm{e}^b\big)\, \mathbb{P}(d\omega) \\ &\geqslant \inf_{r \leqslant t} \mathbb{P}\big(g(-X_r) < g(0)\mathrm{e}^b\big) \cdot \mathbb{P}(\sigma \leqslant t) \\ &\geqslant \frac{1}{2}\mathbb{P}\Big(\sup\nolimits_{s \leqslant t} g(X_s) > cg(0)\mathrm{e}^{a+b}\Big). \end{aligned}$$

In the last estimate we use $\{\sup_{s \leqslant t} g(X_s) > cg(0)\mathrm{e}^{a+b}\} \subseteq \{\sigma \leqslant t\}$. The factor $\frac{1}{2}$ comes from the fact that g is locally bounded and $\lim_{t\to 0^+} \mathbb{P}(|X_t| > \epsilon) = 0$ (continuity in probability), which shows that there is some $0 < T \leqslant t_0$ such that

$$\mathbb{P}(g(-X_t) < g(0)\mathrm{e}^b) \geqslant 1/2 \quad \text{for all } t \leqslant T.$$

This proves that for $\mathrm{e}^\gamma := cg(0)\mathrm{e}^b$,

$$\mathbb{P}\Big(\sup_{s \leqslant T} g(X_s) > \mathrm{e}^{a+\gamma}\Big) \leqslant 2\mathbb{P}(g(X_T) > \mathrm{e}^a).$$

We can now use the layer-cake formula to see that

$$\text{(2.4)} \qquad \begin{aligned} \mathbb{E}\Big[\sup_{s \leqslant T} g(X_s)\Big] &= 1 + \int_1^\infty \mathbb{P}\Big(\sup_{s \leqslant T} g(X_s) > y\Big)\, dy \\ &= 1 + \mathrm{e}^\gamma \int_{-\gamma}^\infty \mathbb{P}\Big(\sup_{s \leqslant T} g(X_s) > \mathrm{e}^\gamma \mathrm{e}^a\Big)\mathrm{e}^a\, da \\ &\leqslant 1 + \mathrm{e}^\gamma \int_{-\gamma}^0 \mathrm{e}^a\, da + 2\mathrm{e}^\gamma \int_0^\infty \mathbb{P}(g(X_T) > \mathrm{e}^a)\mathrm{e}^a\, da \\ &\leqslant \mathrm{e}^\gamma + 2\mathrm{e}^\gamma \mathbb{E}[g(X_T)]. \end{aligned}$$

Using Lemma 2.1 we see that $\mathbb{E}[\sup_{s \leqslant nT} g(X_s)] < \infty$ for all $n \in \mathbb{N}$, i.e. (b) holds for all $t > 0$.

2º (b)⇒(c). If $\sup_{s \leqslant t} g(X_s)$ is integrable for some $t > 0$, then it is integrable for all $t > 0$ (cf. Lemma 2.1). For fixed $t > 0$, let $\sigma \in \mathcal{T}$ be a stopping time with $\sigma \leqslant t$. Then we have

$$g(X_\sigma) \leqslant \sup_{r \leqslant t} g(X_r) \in L^1(\mathbb{P}).$$

Consequently, the family $\{g(X_\sigma)\}_{\sigma \in \mathcal{T},\, \sigma \leqslant t}$ is dominated by the integrable random variable $\sup_{r \leqslant t} g(X_r)$; hence, it is uniformly integrable.

3º (c)⇒(d). This is immediate from the fact that uniform integrability implies boundedness in L^1.

4º (d)⇒(e). Recall the definition of τ_R from Lemma 1.1. We rearrange (1.8) and insert it into (1.7) to get

$$\begin{aligned}
\mathbb{E}[(g\chi_R)(X_{t\wedge\tau_R})] - \mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} (L+J)g(X_s)\,ds\Big] \\
&= g(0) + \mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} \int_{|y|\geqslant 1} (g\chi_R(X_s+y) - g(X_s))\,\nu(dy)\,ds\Big] \\
&= g(0) + \mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} \int_{|y|\geqslant 1} g\chi_R(X_s+y)\,\nu(dy)\,ds\Big] \\
&\quad - \nu(|y|\geqslant 1)\cdot\mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} g(X_s)\,ds\Big].
\end{aligned}$$

Now we use Lemma 2.2 for the Lévy generator $L+J$ and the estimates

$$\mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} g(X_s)\,ds\Big] \leqslant \int_{[0,t)} \mathbb{E}[g(X_s)]\,ds \leqslant t\sup_{s\leqslant t}\mathbb{E}[g(X_s)]$$

and

$$\mathbb{E}[(g\chi_R)(X_{t\wedge\tau_R})] \leqslant \mathbb{E}[g(X_{t\wedge\tau_R})] \leqslant \sup_{\sigma\in\mathcal{T},\,\sigma\leqslant t}\mathbb{E}[g(X_\sigma)].$$

The constant appearing in Lemma 2.2 for $L+J$ depends only on $\nu|_{B_1(0)}$, in particular it is independent of $\int_{|y|>1} g(y)\,\nu(dy)$. Because of our assumption (d), there is a constant C_t, not depending on R or $\int_{|y|>1} g(y)\,\nu(dy)$, such that

$$C_t \geqslant \mathbb{E}\Big[\int_{[0,t\wedge\tau_R)} \int_{|y|\geqslant 1} g\chi_R(X_s+y)\,\nu(dy)\,ds\Big].$$

Letting $R\to\infty$, Fatou's lemma and yet another application of submultiplicativity yield

$$\begin{aligned}
C_t &\geqslant \mathbb{E}\Big[\int_{[0,t)} \int_{|y|\geqslant 1} g(X_s+y)\,\nu(dy)\,ds\Big] \\
&\geqslant \frac{1}{c}\mathbb{E}\Big[\int_{[0,t)} \int_{|y|\geqslant 1} \frac{g(y)}{g(-X_s)}\,\nu(dy)\,ds\Big] \\
&\geqslant \frac{1}{c}\mathbb{E}\Big(\int \mathbb{1}_{\{|X_s|\leqslant\delta\}}\,\frac{ds}{g(-X_s)}\Big)\int_{|y|\geqslant 1} g(y)\,\nu(dy)
\end{aligned}$$

for any $\delta \in (0,1)$. Since g is locally bounded, $g \geqslant 1$, and $s \mapsto X_s$ is càdlàg, (e) follows.

5⁰ (e)⇒(g). Let $\phi \in C_c^\infty(\mathbb{R}^d)$. The reflection $\widetilde{\phi}(x) := \phi(-x)$ is again in $C_c^\infty(\mathbb{R}^d)$ and thus $\|A^*\widetilde{\phi}\|_\infty < \infty$ by the representation of A^* as an integro-differential operator (1.2) and Taylor's formula. Choose $R > 0$ such that $\operatorname{supp}\phi$ is contained in the ball $B_R(0)$. Then

$$(A^*\widetilde{\phi})(-x) = \int_{y\neq 0} \phi(x-y)\,\nu(dy), \quad |x| \geqslant 2R.$$

Since g is bounded on compact sets, it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} g(x)|(A^*\widetilde{\phi})(x)|\,dx &= \Big(\int_{|x|\leqslant 2R} + \int_{|x|>2R}\Big) g(x)\,|(A^*\widetilde{\phi})(x)|\,dx \\ &\leqslant |B_{2R}(0)| \sup_{|x|\leqslant 2R} g(x)\|A^*\widetilde{\phi}\|_\infty + \int_{|x|>2R}\int_{y\neq 0} g(x)|\phi(x-y)|\,\nu(dy)\,dx.\end{aligned}$$

It remains to show that the integral on the right-hand side is finite. By Tonelli's theorem and a change of variables ($z = x - y$),

$$\begin{aligned} I &:= \int_{|x|>2R}\int_{y\neq 0} g(x)|\phi(x-y)|\,\nu(dy)\,dx \\ &= \int\int \mathbb{1}_{|z+y|>2R}\mathbb{1}_{|z|\leqslant R}|\phi(z)|g(z+y)\,dz\,\nu(dy),\end{aligned}$$

where we use $\operatorname{supp}\widetilde{\phi} \subseteq B_R(0)$. The estimate $\mathbb{1}_{|z+y|>2R}\mathbb{1}_{|z|\leqslant R} \leqslant \mathbb{1}_{|y|\geqslant R}$ and the submultiplicativity of g now yield

$$I \leqslant c\Big(\int_{|z|\leqslant R} |\phi(z)|\,g(z)\,dz\Big)\Big(\int_{|y|\geqslant R} g(y)\,\nu(dy)\Big) < \infty;$$

the integrals are finite because of (e) and the local boundedness of g, ϕ.

6⁰ (g)⇒(h). Trivial.

7⁰ (h)⇒(a). This part of the proof draws on work by Hulanicki [4]. Take $\phi \in C_c^\infty(\mathbb{R}^d)$ non-negative such that $\phi \not\equiv 0$ and $A^*\phi \in L^1(\mathbb{R}^d, g)$. Let us first assume that g is bounded. Set

$$h(t) := \int_{\mathbb{R}^d} (P_t\phi)(-x)g(x)\,dx, \quad t \geqslant 0,$$

where $(P_t u)(x) = \mathbb{E}[u(x+X_t)]$ is the semigroup. We want to show that $|h'(t)| \leqslant Ch(t)$ for some constant $C > 0$. An application of Tonelli's theorem and a change of variables yield

$$\int_{\mathbb{R}^d} |P_t A\phi(-x)| g(x)\,dx \leqslant \mathbb{E}\Big[\int_{\mathbb{R}^d} |(A\phi)(-y)| g(y+X_t)\,dy\Big] \\ \leqslant c\mathbb{E}[g(X_t)] \int_{\mathbb{R}^d} |(A^*\phi)(y)| g(y)\,dy < \infty;$$

the latter integral is finite because $A^*\phi \in L^1(\mathbb{R}^d, g)$ and g is bounded. Moreover, Dynkin's formula (1.4) entails that $\frac{d}{dt} P_t\phi = P_t A\phi$. Consequently, we can use a version of the differentiation lemma for parameter-dependent integrals (cf. [9, Proposition A.1]) [5] to obtain

$$h'(t) = \int_{\mathbb{R}^d} \frac{d}{dt}(P_t\phi)(-x) g(x)\,dx = \int_{\mathbb{R}^d} (P_t A\phi)(-x) g(x)\,dx$$

and, by the above estimate,

$$|h'(t)| \leqslant c\mathbb{E}[g(X_t)] \int_{\mathbb{R}^d} |(A^*\phi)(y)| g(y)\,dy.$$

The submultiplicativity of g gives

$$\left(\int_{\mathbb{R}^d} \frac{\phi(x)}{g(x)}\,dx\right) g(y) \leqslant c \int_{\mathbb{R}^d} \phi(x) g(y-x)\,dx = c(\phi * g)(y)$$

for all $y \in \mathbb{R}^d$, i.e.

$$g(y) \leqslant \frac{c}{\|\phi\|_{L^1(\mathbb{R}^d, 1/g)}} (\phi * g)(y); \tag{2.5}$$

note that

$$\|\phi\|_{L^1(\mathbb{R}^d, 1/g)} = \int_{\mathbb{R}^d} \frac{\phi(x)}{g(x)}\,dx \in (0, \infty)$$

because $g \geqslant 1$ is locally bounded and $\phi > 0$ on a set of positive Lebesgue measure. Using (2.5) for $y = X_t$, we get

$$|h'(t)| \leqslant C\mathbb{E}[(\phi * g)(X_t)] = C \int_{\mathbb{R}^d} \mathbb{E}[\phi(X_t - x)] g(x)\,dx = Ch(t), \quad t \geqslant 0.$$

---

[5] This is a variant of the classical differentiation lemma for parameter-dependent integrals. If $u = u(t,x)$ is a function such that $t \mapsto u(t,x)$ is differentiable, $\int_a^b \int_{\mathbb{R}^d} |\partial_t u(t,x)|\,dx\,dt < \infty$ and $t \mapsto \int_{\mathbb{R}^d} \partial_t u(t,x)\,dx$ is continuous, then $t \mapsto \int_{\mathbb{R}^d} u(t,x)\,dx$ is differentiable on (a,b) with derivative $\int_{\mathbb{R}^d} \partial_t u(t,x)\,dx$. These conditions are clearly satisfied in our application; see the calculation immediately before the present paragraph.

Hence, by Gronwall's lemma,

$$h(t) \leqslant h(0)\mathrm{e}^{\alpha t}, \quad t \geqslant 0,$$

for some constant $\alpha > 0$. Invoking once more (2.5), we conclude that

$$\mathbb{E}[g(X_t)] \leqslant c'\mathbb{E}[(\phi * g)(X_t)] = c'h(t) \leqslant c'h(0)\mathrm{e}^{\alpha t}. \tag{2.6}$$

So far, we assumed that g is bounded. For unbounded g, we replace g by $\min\{g, n\}$ – which is again submultiplicative – in the above estimates and find that

$$\mathbb{E}[\min\{g(X_t), n\}] \leqslant c''\mathrm{e}^{\alpha t}$$

for some constants $c'' > 0$, $\alpha > 0$, not depending on $n \in \mathbb{N}$. Thus, by Fatou's lemma, $\mathbb{E}[g(X_t)] \leqslant c''\mathrm{e}^{\alpha t}$ for all $t \geqslant 0$.

8° (c)$\Rightarrow$(f). Let $f \in L^1(\mathbb{R}^d, g)$. We see that

$$\begin{aligned}\int_{\mathbb{R}^d} |P_t^* f(x)|\, g(x)dx &\leqslant \mathbb{E}\Big[\int_{\mathbb{R}^d} |f(x - X_t)|g(x - X_t + X_t)\, dx\Big] \\ &\leqslant c\mathbb{E}[g(X_t)]\|f\|_{L^1(\mathbb{R}^d,g)},\end{aligned}$$

which shows that $P_t^* : L^1(\mathbb{R}^d, g) \to L^1(\mathbb{R}^d, g)$ is continuous. Let $\phi \in C_c(\mathbb{R}^d)$ and assume that $\operatorname{supp}\phi$ is contained in some ball $B_R(0)$ with radius $R > 0$. We show that $P_t^*\phi \to \phi$ in $L^1(\mathbb{R}^d, g)$ as $t \to 0^+$. Since ϕ is uniformly continuous, we can pick $\epsilon = \epsilon(\eta) > 0$ in such a way that $|\phi(x + y) - \phi(x)| \leqslant \eta$ for all x and all $|y| < \epsilon$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} |P_t^*\phi(x) - \phi(x)|g(x)\, dx &\leqslant \mathbb{E}\Big[\int_{\mathbb{R}^d} |\phi(x - X_t) - \phi(x)|g(x)\, dx\Big] \\ &= \mathbb{E}\Big[\int_{B_{R+\epsilon}(0)} |\phi(x - X_t) - \phi(x)|\mathbb{1}_{|X_t|<\epsilon}g(x)\, dx\Big] \\ &\quad + \int_{\mathbb{R}^d} \mathbb{E}[|\phi(x - X_t) - \phi(x)|\mathbb{1}_{|X_t|\geqslant\epsilon}]g(x)\, dx \\ &\leqslant |B_{R+\epsilon}(0)| \sup_{|x|<R+\epsilon} g(x)\cdot\eta + \mathbb{P}(|X_t| \geqslant \epsilon)\|\phi\|_{L^1(\mathbb{R}^d,g)} \\ &\quad + c\mathbb{E}[g(X_t)\mathbb{1}_{|X_t|\geqslant\epsilon}]\|\phi\|_{L^1(\mathbb{R}^d,g)}.\end{aligned}$$

Since the family $(g(X_t))_{t\leqslant 1}$ is uniformly integrable, $X_t \to 0$ in probability, and $\eta > 0$ is arbitrary, we obtain $\|P_t^*\phi - \phi\|_{L^1(\mathbb{R}^d,g)} \to 0$ as $t \to 0^+$. Using the fact that $C_c(\mathbb{R}^d)$ is a dense subset of $L^1(\mathbb{R}^d, g)$, we conclude that $(P_t^*)_{t\geqslant 0}$ is a strongly continuous operator semigroup on $L^1(\mathbb{R}^d, g)$.

9$^{\text{o}}$ (f)$\Rightarrow$(a). Let $\phi \in C_c(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, g)$ such that $\phi \geqslant 0$ and $\phi = 1$ on $B_1(0)$. Using $g \geqslant 1$ and the submultiplicativity of g, we see that

$$\|P_t^* \phi\|_{L^1(\mathbb{R}^d, g)} = \mathbb{E}\Big[\int_{\mathbb{R}^d} (g(x + X_t))\phi(x)\,dx\Big] \geqslant \frac{1}{c}\mathbb{E}[g(X_t)] \int_{\mathbb{R}^d} \frac{\phi(x)\,dx}{g(-x)};$$

this implies that $\mathbb{E}[g(X_t)] < \infty$.

Since all conditions are equivalent, the exponential bound (2.2) follows from Gronwall's inequality (see (2.6) in Step 7$^{\text{o}}$). ■

The proof of Theorem 2.1 contains the following moment result for Lévy processes with bounded jumps. Alternative proofs can be found in Sato [12, Theorem 25.3, p. 159] or [6, Lemma 8.2]. If we use in Step 3$^{\text{o}}$ the submultiplicative function $g(x) := \mathrm{e}^{\beta|x|}$, $\beta > 0$, $x \in \mathbb{R}^d$, we get the following corollary.

COROLLARY 2.1. *Let $Y = (Y_t)_{t \geqslant 0}$ be a Lévy process with bounded jumps, i.e. the Lévy measure has bounded support. Then Y has exponential moments, i.e. $\mathbb{E}[\mathrm{e}^{\beta|Y_t|}] < \infty$ for all $\beta > 0$ and $t \geqslant 0$.*

If Y has a non-degenerate jump part, then moments of the type $\mathbb{E}[\mathrm{e}^{\beta|X_t|^{1+\epsilon}}]$ do not exist (cf. [2, Theorem 3.3(c)]).

3. DOOB'S CONDITION (DL) FOR LÉVY PROCESSES

Let Y be a stochastic process and $\mathcal{T}$ be the family of all stopping times with respect to the natural filtration of Y. Recall that Y *satisfies the condition* (DL) if for each fixed $t > 0$ the family $(Y_{t\wedge\sigma})_{\sigma\in\mathcal{T}}$ is uniformly integrable, i.e.

$$\text{(DL)} \qquad \forall t > 0: \quad \lim_{R\to\infty} \sup_{\sigma\in\mathcal{T}} \int |Y_{t\wedge\sigma}| \mathbb{1}_{|Y_{t\wedge\sigma}| \geqslant R}\, d\mathbb{P} = 0.$$

It is well known (cf. [11, Proposition IV.1.7, p. 124]) that a local martingale is a martingale if, and only if, it satisfies (DL).

As a direct consequence of Theorem 2.1, we get the following characterization of the condition (DL) for functions of a Lévy process.

COROLLARY 3.1. *Let $X = (X_t)_{t\geqslant 0}$ be a Lévy process with triplet (b, Q, ν) and g a locally bounded submultiplicative function. The following are equivalent:*

(a) *The process $(g(X_t))_{t\geqslant 0}$ satisfies the condition* (DL).

(b) $\mathbb{E}[g(X_t)]$ *is finite for some $t > 0$.*

(c) $\int_{|y|\geqslant 1} g(y)\,\nu(dy) < \infty$.

COROLLARY 3.2. *Let X be a Lévy process and $f : \mathbb{R}^d \to \mathbb{R}$ such that for some locally bounded submultiplicative function g we have $|f(x)| \leqslant g(x)$. If $\mathbb{E}[g(X_t)]$ is finite and $(f(X_t))_{t\geqslant 0}$ is a local martingale, then $(f(X_t))_{t\geqslant 0}$ is a martingale.*

Proof. Corollary 3.1 shows that $(g(X_t))_{t\geqslant 0}$, hence $(f(X_t))_{t\geqslant 0}$ enjoys the property (DL); therefore the local martingale $(f(X_t))_{t\geqslant 0}$ is already a proper martingale. ■

The setting of Corollary 3.2 is quite natural if one thinks of Itô's formula or the expression (1.5) appearing in the discussion of Dynkin's formula: If A is the (pointwise extension to positive C^2-functions of the) generator of X, and if Af is equal to 0, then $f(X_t)$ is, by Itô's formula, a local martingale. Corollary 3.2 thus gives a condition when this local martingale is a true martingale.

3.1. A direct proof of Corollary 3.1(c)⇒(b)**.** Sometimes it is useful to have a direct proof that existence of moments of the Lévy measure gives generalized moments for the process. The approach below gives a method using standard 'household' techniques from any course on Markov processes, notably Dynkin's formula and Gronwall's inequality; therefore it applies to more general (strong) Markov processes.

Alternative proof for Corollary 3.1(c)⇒(b) *resp. Theorem 2.1*(e)⇒(a). In view of Lemma 1.2 we may replace g by its regularization g^ϵ. Combining Dynkin's inequality (Lemma 1.1) and the estimate from Lemma 2.2 shows

$$\mathbb{E}[g^\epsilon(X_{t\wedge\sigma})] \leqslant g^\epsilon(0) + C_{\epsilon,g}^{b,Q,\nu}\mathbb{E}\Big[\int_{[0,t\wedge\sigma)} g(X_s)\,ds\Big].$$

The constant C depends on ϵ, the triplet (b,Q,ν) and $\int_{|y|\geqslant 1} g(y)\,\nu(dy)$ (see Lemma 2.2). If we replace σ by $\sigma\wedge\tau_R$ with $\tau_R = \inf\{s\geqslant 0 \mid |X_s|\geqslant R\}$ and set $\kappa_R := \sup_{|y|\leqslant R} g(y)$, then we get

$$\begin{aligned}\mathbb{E}[g^\epsilon(X_{t\wedge\sigma\wedge\tau_R})\wedge\kappa_R] &\leqslant g^\epsilon(0) + C_{\epsilon,g}^{b,Q,\nu}\mathbb{E}\Big[\int_{[0,t\wedge\sigma\wedge\tau_R)} g(X_s)\wedge\kappa_R\,ds\Big]\\ &\leqslant g^\epsilon(0) + \tilde{C}_{\epsilon,g}^{b,Q,\nu}\int_{[0,t)}\mathbb{E}[g^\epsilon(X_{s\wedge\sigma\wedge\tau_R})\wedge\kappa_R]\,ds.\end{aligned}$$

We may now appeal to Gronwall's lemma and find

$$\mathbb{E}[g^\epsilon(X_{t\wedge\sigma\wedge\tau_R})\wedge\kappa_R] \leqslant g^\epsilon(0)\mathrm{e}^{tC_{\epsilon,g}^{b,Q,\nu}}.$$

Fatou's lemma proves $\mathbb{E}[g^\epsilon(X_{t\wedge\sigma})] < \infty$, and (b) follows if we take $\sigma\equiv t$. ■

4. MOMENTS OF STOCHASTICALLY CONTINUOUS ADDITIVE PROCESSES

In this section, we apply Theorem 2.1 to characterize the existence of generalized moments of stochastically continuous additive processes $(X_t)_{t\geqslant 0}$.

THEOREM 4.1. *Let $(X_t)_{t\geqslant 0}$ be a stochastically continuous additive process, and denote by (b_t, Q_t, ν_t) its characteristics. Let $g \geqslant 1$ be a locally bounded submultiplicative function. The following assertions are equivalent for any $t > 0$:*

(a) $\int_{|y|\geqslant 1} g(y)\, \nu_t(dy) < \infty$.

(b) $\mathbb{E}[g(X_t)] < \infty$.

(c) $\sup_{s\leqslant t} \mathbb{E}[g(X_s)] < \infty$.

(d) $\mathbb{E}[\sup_{s\leqslant t} g(X_s)] < \infty$.

(e) $(g(X_\sigma))_{\sigma\in\mathcal{T},\sigma\leqslant t}$ *is uniformly integrable.*

If one of these conditions is satisfied (hence, all are), then there are positive constants $C, C_1, C_2(t)$ such that

$$\mathbb{E}\Big[\sup_{s\leqslant t} g(X_s)\Big] \leqslant C\mathbb{E}[g(X_t)] \quad \text{and} \quad \mathbb{E}\Big[\sup_{s\leqslant t} g(X_s)\Big] \leqslant C_1 \mathrm{e}^{C_2(t)}. \tag{4.1}$$

The constants $C, C_1, C_2(t)$ depend on g and (the characteristics of) the additive process.

REMARK 4.1. (a) Klass & Yang [7] prove the inequality

$$\mathbb{E}\Big[\sup_{s\leqslant t} g(X_s)\Big] \leqslant C \sup_{s\leqslant t} \mathbb{E}[g(X_t)]$$

for so-called moderate functions g, satisfying $g(x+y) \leqslant C_g(g(x)+g(y))$ for all $x, y \in \mathbb{R}^d$.

(b) The exponential moment $\mathbb{E}[\exp(X_t)]$ of an additive process can be calculated explicitly; namely, $\mathbb{E}[\exp(X_t)] = \exp(\psi_t)$ for the continuous negative definite function ψ_t with triplet (b_t, Q_t, ν_t). If $(X_t)_{t\geqslant 0}$ is stochastically continuous, this follows from the fact that for each $t > 0$ there is a Lévy process $(Y_s)_{s\geqslant 0}$ with $Y_t = X_t$ in distribution; thus, $\mathbb{E}[\exp(X_t)] = \mathbb{E}[\exp(Y_t)]$ can be calculated using the well known formula for Lévy processes. This observation simplifies part of the proof in Fujiwara [3], who also considers general additive processes which are neither stochastically continuous nor semimartingales.

Proof of Theorem 4.1. We show (b)⇒(a)⇒(c)⇒(d)⇒(e)⇒(b).

1° (b)⇒(a). Since the law of X_t is infinitely divisible, there exists a Lévy process $(Y_s)_{s\geqslant 0}$ such that $Y_t = X_t$ in distribution. Therefore, the assertion follows directly from Theorem 2.1(a)⇒(e).

2° (a)⇒(c). The alternative proof of Corollary 3.1(c)⇒(b) in combination with Lemma 1.2 shows that

$$\mathbb{E}[g(X_s)] \leqslant c_\epsilon g^\epsilon(0) \mathrm{e}^{sC_\epsilon C(b_s, Q_s, \nu_s; g)}$$

for the constant $C(b_s, Q_s, \nu_s; g)$ appearing in Lemma 2.2, where g^ϵ is the Friedrichs regularization of g for some fixed $\epsilon > 0$. Since $(b_s, Q_s, \nu_s)_{s\geqslant 0}$ are the characteristics of an additive process, the constant $C(b_s, Q_s, \nu_s; 1)$ (from Lemma 2.2) depends continuously on $s \geqslant 0$ (see (1.3)); therefore it is bounded on $[0, t]$. The condition (a) ensures that $\int_{|y|\geqslant 1} g(y)\,\nu_s(dy) \leqslant \int_{|y|\geqslant 1} g(y)\,\nu_t(dy) < \infty$. Altogether we see that $\sup_{s\leqslant t} C(b_s, Q_s, \nu_s; g) \leqslant C(t) < \infty$, and

$$\sup_{s\leqslant t} \mathbb{E}[g(X_s)] \leqslant C_1 \mathrm{e}^{tC_\epsilon C(t)}. \tag{4.2}$$

This bound can also be obtained from the estimate in [8, Theorem 4.1].

3° (c)⇒(d). The proof mimics the proof of Theorem 2.1(a)⇒(b). As $(X_s)_{s\geqslant 0}$ is stochastically continuous, it is uniformly stochastically continuous on compact time intervals, and there is some constant $b > 0$ such that

$$\inf_{s\in[0,t]} \mathbb{P}(g(X_s - X_t) < g(0)\mathrm{e}^b) \geqslant \frac{1}{2}.$$

Indeed, for all $s \in [0, t]$ and any $K > 0$ we have

$$\begin{aligned}&\mathbb{P}\big(g(X_s - X_t) \geqslant g(0)\mathrm{e}^b\big)\\ &\qquad\leqslant \mathbb{P}\big(g(X_s - X_t) \geqslant g(0)\mathrm{e}^b,\ |X_t - X_s| \leqslant K\big) + \mathbb{P}(|X_t - X_s| > K).\end{aligned}$$

Since $(X_s)_{s\geqslant 0}$ is uniformly stochastic continuous on the compact interval $[0, t]$, we can pick $K > 0$ so large that the second term on the right is smaller than $\frac{1}{2}$. Since g is locally bounded, we can now choose $b = b(K)$ so large that the first probability becomes zero. Choose a constant $c > 0$ such that $g(x + y) \leqslant cg(x)g(y)$ for all $x, y \in \mathbb{R}^d$. For a fixed $a > 0$ set

$$\sigma := \inf\{s > 0 \mid g(X_s) > cg(0)\mathrm{e}^{a+b}\}.$$

Since $(X_s)_{s\geqslant 0}$ has independent increments, we argue as in the proof of Theorem 2.1 (see (2.3)) to find that

$$\begin{aligned}\mathbb{P}(g(X_t) > \mathrm{e}^a) &\geqslant \mathbb{P}\big(g(X_\sigma - X_t) < g(0)\mathrm{e}^b,\ \sigma \leqslant t\big)\\ &= \int_{\{\sigma\leqslant t\}} \mathbb{P}\big(g(X_s - X_t) < g(0)\mathrm{e}^b\big)\Big|_{s=\sigma}\, d\mathbb{P}\\ &\geqslant \frac{1}{2}\mathbb{P}(\sigma \leqslant t).\end{aligned}$$

As $\{\sup_{s\leqslant t} g(X_s) > cg(0)\mathrm{e}^b\mathrm{e}^a\} \subseteq \{\sigma \leqslant t\}$, this implies

$$\mathbb{P}(g(X_t) > \mathrm{e}^a) \geqslant \frac{1}{2}\mathbb{P}\Big(\sup_{s\leqslant t} g(X_s) > cg(0)\mathrm{e}^b\mathrm{e}^a\Big).$$

If γ is such that $e^\gamma = cg(0)e^b$, then we see as in (2.4) that

$$\mathbb{E}\Big[\sup_{s\leqslant t} g(X_s)\Big] \leqslant e^\gamma + 2e^\gamma \mathbb{E}[g(X_t)] \leqslant 3e^\gamma \mathbb{E}[g(X_t)].$$

4° (d)⇒(e)⇒(b). Trivial.

The moment estimates (4.1) follow from (4.2) and the estimate in Step 3°. ■

REMARK 4.2. If $(X_t)_{t\geqslant 0}$ is an additive process, then $\mathbb{E}[g(X_t)] < \infty$ does not imply, in general, $\mathbb{E}[g(X_r)] > \infty$ for $r > t$. Take e.g. an isotropic α-stable Lévy process $(L_t)_{t\geqslant 0}$ for $\alpha \in (0,2)$ and consider

$$X_t := \begin{cases} 0, & t < 1, \\ L_{t-1}, & t \geqslant 1. \end{cases}$$

Clearly, $(X_t)_{t\geqslant 0}$ is an additive process and $\mathbb{E}[e^{X_t}] < \infty$ for all $t \leqslant 1$ but for $t > 1$ we have $\mathbb{E}[e^{X_t}] = \infty$.

5. MARTINGALE METHODS FOR LÉVY PROCESSES

Let us give a further application of Theorem 2.1 to the recurrence/transience behaviour of Lévy processes. As a warm-up and in order to illustrate the method, we begin with a very short proof for the characterization of infinitely divisible random variables taking values in a lattice (Theorem 5.1). Interestingly, the proof for transience (Corollary 5.1) is essentially the same argument, the difference being that Theorem 5.1 uses the characteristic exponent ψ of the process for $\xi \in \mathbb{R}$, while Corollary 5.1 relies on (the extension of ψ to certain) $\zeta \in i\mathbb{R}$. Both results are, of course, well-known (Sato [12, Section 24], resp., Zhang et al. [16, Theorem 3.4]) but the present proofs are substantially different and much shorter. Recall that a random variable Y is infinitely divisible if, and only if, there is a Lévy process $(X_t)_{t\geqslant 0}$ such that $Y \sim X_1$.

THEOREM 5.1. *Let $(X_t)_{t\geqslant 0}$ be a one-dimensional Lévy process with characteristic exponent ψ and triplet (b, Q, ν) (see (1.1)). The following assertions are equivalent (for fixed $\beta \neq 0$):*

(a) $\psi(\beta) = i\alpha$ *for some* $\alpha \in \mathbb{R}$.

(b) $|\mathbb{E}[e^{i\beta X_t}]| = 1$ *for some, hence for all,* $t > 0$.

(c) $X_{t_0} + \gamma$ *has values in* $2\pi\beta^{-1}\mathbb{Z}$ *for some* $t_0 > 0$ *and some* $\gamma \in \mathbb{R}$.

(d) $X_t + \alpha\beta^{-1}t$ *has values in* $2\pi\beta^{-1}\mathbb{Z}$ *for and* $\alpha \in \mathbb{R}$ *and all* $t > 0$.

(e) $\operatorname{supp}\nu \subset 2\pi\beta^{-1}\mathbb{Z}$, $Q = 0$ *and*

$$b = -\alpha\beta^{-1} + 2\pi\beta^{-1} \sum_{|k|<\beta/(2\pi)} k\nu(\{2\pi\beta^{-1}k\}).$$

REMARK 5.1. Our proof shows that there is also a connection between γ and α in Theorem 5.1(c, d). If we use $t = t_0$ in (d), it turns out that the localization parameter γ is in $\alpha\beta^{-1}t_0 + 2\pi\beta^{-1}\mathbb{Z}$.

Proof of Theorem 5.1. (a)⇒(b). This follows from $\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X_t}] = \mathrm{e}^{-t\psi(\beta)}$ for any $t > 0$.

(b)⇒(c). We fix $t_0 > 0$ such that $|\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X_{t_0}}]| = 1$ and apply Lemma 5.1 to $X = \beta X_{t_0}$. This shows that there exists some $\gamma \in \mathbb{R}$ such that X is supported in $2\pi\beta^{-1}\mathbb{Z} - \gamma$.

(c)⇒(d). Let $t_0 > 0$ be as in (c) and $t > 0$, $t \neq t_0$. We see that $\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X_t}] = (\mathrm{e}^{-t_0\psi(\beta)})^{t/t_0}$. As $|\mathrm{e}^{-t_0\psi(\beta)}| = 1$, we know that $\psi(\beta) \in \mathrm{i}\mathbb{R}$. Lemma 5.1 shows that X_t is supported in $2\pi\beta^{-1}\mathbb{Z} + \mathrm{i}t\psi(\beta)\beta^{-1} = 2\pi\beta^{-1}\mathbb{Z} - t\operatorname{Im}\psi(\beta)\beta^{-1}$ for every $t > 0$.

(d)⇒(e). Let $t > 0$. A direct calculation shows that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X_t}] = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi k - \mathrm{i}\alpha t}\mathbb{P}(X_t = 2\pi\beta^{-1}k - \alpha\beta^{-1}t) = \mathrm{e}^{-\mathrm{i}\alpha t}.$$

On the other hand, infinite divisibility entails

$$\mathrm{e}^{-\mathrm{i}\alpha t} = \mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X_t}] = \mathrm{e}^{-t\psi(\beta)} \quad \text{for all } t > 0.$$

This is only possible if $\psi(\beta) = \mathrm{i}\alpha$. Comparing this with the Lévy–Khinchin formula (1.1), we conclude that

$$b = -\alpha\beta^{-1} + 2\pi\beta^{-1} \sum_{|k|<\beta(2\pi)^{-1}} k\nu(\{2\pi\beta^{-1}k\}),$$

$Q = 0$ and $\operatorname{supp}\nu \subset 2\pi\beta^{-1}\mathbb{Z}$.

(e)⇒(a). This follows from the Lévy–Khinchin formula (1.1). ■

The key step in the proof of Theorem 5.1 is the following well-known result. The standard proof can be found in Lukacs [10, Section 2.1], we prefer to give a(n equally short) martingale argument which appears, again, in the proof of Corollary 5.1 and highlights the connection between Theorem 5.1 and Corollary 5.1.

LEMMA 5.1. *Let X be a real random variable. If there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that $\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta X}] = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$, then the distribution of X is supported on $2\pi\beta^{-1}\mathbb{Z} + \beta^{-1}\theta$.*

Proof. Without loss of generality we may assume that $\beta = 1$, i.e. $|\mathbb{E}[e^{iX}]| = 1$ or $\mathbb{E}[e^{i(X-\theta)}] = 1$ for some $\theta \in [0, 2\pi)$. Let $(X_i)_{i\in\mathbb{N}}$ be iid copies of the random variable X and define, for every $n \in \mathbb{N}$,

$$Y_n := \frac{1}{2^n}\prod_{k=1}^{n}(1+\cos(X_k-\theta)) = \prod_{k=1}^{n}\frac{1+\cos(X_k-\theta)}{2}.$$

Set $\mathscr{F}_n := \sigma(X_1, \dots, X_n)$ and note that $(Y_n)_{n\in\mathbb{N}}$ is adapted to this filtration. We see that

$$\begin{aligned}\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}\left[\prod_{k=1}^{n}\frac{1+\cos(X_k-\theta)}{2}\;\middle|\;\mathcal{F}_{n-1}\right]\\ &= Y_{n-1}\frac{1+\mathbb{E}[\cos(X_n-\theta)]}{2} = Y_{n-1},\end{aligned}$$

so $(Y_n)_{n\in\mathbb{N}}$ is a discrete-time martingale. As $0 \leqslant Y_n \leqslant 1$ and $\mathbb{E}[Y_n] = 1$, we conclude that Y_n converges to 1 in L^1 and a.s. as $n \to \infty$. Thus, $Y_1 = \mathbb{E}[1 \mid \mathcal{F}_1] = 1$ a.s., which implies that $X - \theta$ takes only values in the lattice $2\pi\mathbb{Z}$. ∎

Assume now that $(X_t)_{t\geqslant 0}$ is a Lévy process which admits an exponential moment $\mathbb{E}[e^{\beta X_t}] < \infty$ for some $\beta \neq 0$. By Theorem 2.1, $\int_{|y|\geqslant 1} e^{\beta y}\,\nu(dy) < \infty$, and it is easy to see from the Lévy–Khinchin formula (1.1) that the exponent ψ has a continuous continuation to all complex numbers $\xi + i\eta \in \mathbb{C}$ with $\xi \in \mathbb{R}$ and η between 0 and $-\beta$. In particular,

$$\mathbb{E}[e^{\beta X_t}] = e^{-t\psi(-i\beta)}, \quad t > 0,$$

which shows that the sets $A := \{\beta \in \mathbb{R} \mid \mathbb{E}[e^{\beta X_t}] = 1\}$, where $t > 0$ is fixed, $\{\beta \in \mathbb{R} \mid \psi(-i\beta) = 0\}$ and $\{\beta \in \mathbb{R} \mid \forall t > 0 : \mathbb{E}[e^{\beta X_t}] = 1\}$ coincide. Using the fact that $(X_t)_{t\geqslant 0}$ has stationary and independent increments, it follows with a similar argument to the proof of Lemma 5.1 that $A = \{\beta \in \mathbb{R} \mid (e^{\beta X_t})_{t\geqslant 0}$ is a martingale$\}$.

COROLLARY 5.1. *Let $(X_t)_{t\geqslant 0}$ be a one-dimensional Lévy process. If the set $A \setminus \{0\}$ is not empty, then X_t is transient.*

Proof. Define $Y_t = e^{\beta X_t}$ with $\beta \in A\setminus\{0\}$. The discussion before Corollary 5.1 shows that Y_t is a martingale. Since $t \mapsto e^{\beta Y_t}$ is positive and right-continuous, the martingale convergence theorem shows that $\lim_{t\to\infty} Y_t = Y_\infty$ a.s. for some a.s. finite random variable Y_∞. As βX_t is again a Lévy process, which is either transient or recurrent, we see that $e^{\beta X_t}$ can only converge to a finite limit if $\beta X_t \to -\infty$ as $t \to \infty$; thus, X_t cannot be recurrent. ∎

It is clear that Corollary 5.1 still holds for a d-dimensional Lévy process if we interpret ξX_t and βX_t as scalar products with $\xi, \beta \in \mathbb{R}^d$. By Cauchy's inequality, $|\beta \cdot X_t|/|\beta| \leqslant |X_t|$, and so $\lim_{t\to\infty}|X_t| = \infty$, i.e. $(X_t)_{t\geqslant 0}$ is transient if $\beta \cdot X_t$ is transient.

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David Berger, Franziska Kühn, René L. Schilling
Institut für Mathematische Stochastik
Fakultät Mathematik
TU Dresden
01062 Dresden, Germany
E-mail: david.berger2@tu-dresden.de
franziska.kuehn1@tu-dresden.de
rene.schilling@tu-dresden.de

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