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ONE-DIMENSIONAL REFLECTED BSDES WITH TWO BARRIERS UNDER LOGARITHMIC GROWTH AND APPLICATIONS

BY

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Abstract. We deal with the problem of existence and uniqueness of a solution for one-dimensional reflected backward stochastic differential equations with two strictly separated barriers when the generator has logarithmic growth $|y|\,|\ln|y|| + |z|\sqrt{|\ln|z||}$ in the state variables y and z. The terminal value ξ and the obstacle processes $(L_t)_{0\leqslant t\leqslant T}$ and $(U_t)_{0\leqslant t\leqslant T}$ are L^p-integrable for a suitable $p > 2$. The main idea is to use the concept of local solution to construct a global one. As applications, we broaden the class of functions for which mixed zero-sum stochastic differential games admit an optimal strategy and the related double-obstacle partial differential equation problem has a unique viscosity solution.

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1. INTRODUCTION

In this paper we are concerned with the problem of existence and uniqueness of a solution for one-dimensional reflected backward stochastic differential equations (BSDEs for short) driven by Brownian motion $(B_t)_{t\leqslant T}$ with two continuous reflecting barriers $L := (L_t)_{t\leqslant T}$ and $U := (U_t)_{t\leqslant T}$ and whose coefficient and terminal values are f and ξ respectively. This means that we want to show the existence of a unique quadruple

$$(Y, Z, K^+, K^-)$$

of $\mathcal{F}_t$-adapted processes such that

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$$
(1.1)\qquad \begin{cases} Y_t = \xi + \int\limits_t^T f(s, Y_s, Z_s)\, ds \\ \qquad\qquad + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int\limits_t^T Z_s\, dB_s, \quad t \in [0,T]; \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [0,T]; \\ \int\limits_0^T (Y_s - L_s)\ dK_s^+ = \int\limits_0^T (U_s - Y_s)\ dK_s^- = 0. \end{cases}
$$

In the framework of a Brownian filtration, the notion of BSDEs was first introduced by Pardoux and Peng [18]. Then, in [11], El-Karoui et al. introduced BSDEs with a lower obstacle $L := (L_t)_{t \leqslant T}$ where the solution Y is assumed to be above L; subsequently, Cvitanić and Karatzas [4] generalized these results to BSDEs with two barriers (upper and lower). Due to their appearance in many finance problems such as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance, as well as their many applications in several other problems: optimal switching, stochastic games, non-linear PDEs etc. (see [7, 11, 12, 17] and the references therein), many authors have attempted to improve the result of [4] and establish the existence and uniqueness of a solution by focusing on weakening the Lipschitz property of the coefficient or the square integrability of the data (see [8] for the latter).

The main objective of this paper is to show the existence and uniqueness of a solution for BSDEs with two reflecting barriers with a generator allowing logarithmic growth in the state variables y and z:

$$
\begin{aligned} &|f(t,\omega,y,z)| \\ &\quad \leqslant |\eta_t| + c_0|y|\,|\ln|y|| + c_1|z|\sqrt{|\ln|z||}, \qquad \forall (t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d, \end{aligned}
$$

with the terminal data ξ and the barriers being merely p-integrable (with $p > 2$). For example, let C be a constant and $f(y) = -Cy \ln|y|$, and consider the BSDE

$$
(1.2)\qquad \begin{cases} Y_t = \xi + \int\limits_t^T f(Y_s)\, ds \\ \qquad\qquad + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int\limits_t^T Z_s\, dB_s, \quad t \in [0,T]; \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [0,T]; \\ \int\limits_0^T (Y_s - L_s)\ dK_s^+ = \int\limits_0^T (U_s - Y_s)\ dK_s^- = 0. \end{cases}
$$

The generator in (1.2) is not locally monotone or of sublinear growth in the y-variable; moreover, it grows as a large power of y. The logarithmic nonlinearity $y \ln|y|$ which appears in (1.2) is interesting in itself and to our knowledge it has not been covered yet; the same for $f(z) = |z|\sqrt{|\ln|z||}$. As we can see, our assumption covers both $f(y) = -Cy\ln|y|$ and $f(z) = |z|\sqrt{|\ln|z||}$.

Moreover, we also impose another assumption on f (see (**H.4**) below) which is local in y, z and also in ω; this enables us to cover certain BSDEs with stochastic monotone generators.

There are two main reasons why we study this kind of problem. The first one is zero-sum games, of Dynkin type or of mixed type, where we broaden the class of data for which those games have a value. It is well known that double-barrier reflected BSDEs are connected with mixed zero-sum games, which we describe briefly.

Assume that we have a stochastic system whose dynamic $(x_t)_{t\leqslant T}$ satisfies

$$x_t = x_0 + \int_0^t \varphi(s, x_s, u_s, v_s)\, ds + \int_0^t \sigma(s, x_s)\, dB_s, \qquad t \in [0, T],\ x_0 \in \mathbb{R}^d,$$

φ is the drift of the system and the stochastic processes $(u_t)_{t\leqslant T}$ and $(v_t)_{t\leqslant T}$ are adapted and stand for, respectively, the intervention functions of two agents A_1 and A_2 on that system (the system could be for example a stock market and A_1 and A_2 are two traders). Moreover, the two agents can exit the system whenever they want, meaning that they can stop controlling at stopping times τ and σ. However, their actions are not free and their advantages are antagonistic, i.e., there is a payoff $J(u, \tau; v, \sigma)$ between them such that

$$\begin{aligned} &J(u, \tau; v, \sigma) \\ &\quad = \mathbb{E}^{(u,v)}\Big[\int_0^{\tau\wedge\sigma} h(s, x, u_s, v_s)\, ds + L_\sigma \mathbb{1}_{\{\sigma\leqslant\tau<T\}} + U_\tau \mathbb{1}_{\{\tau<\sigma\}} + \xi \mathbb{1}_{\{\tau\wedge\sigma=T\}}\Big], \end{aligned}$$

where h is the instantaneous reward of A_2, L (resp. U) is the reward if A_2 decides to stop at σ (resp. τ) before the terminal time T, and ξ is the reward if he decides to stay until T.

The first (resp. second) player chooses a pair (u, τ) (resp. (v, σ)) of continuous control and stopping time, and looks for minimizing (resp. maximizing) this payoff, meaning we aim to find a pair of strategies (u^*, τ^*) and (v^*, σ^*) for A_1 and A_2 respectively such that $J(u^*, \tau^*; v, \sigma) \leqslant J(u^*, \tau^*; v^*, \sigma^*) \leqslant J(u, \tau; v^*, \sigma^*)$. The main idea is to characterize the value function as a solution of a specific reflected BSDE with two barriers. This problem has already been studied, for example, in [15] when σ^{-1}, φ and h are bounded and in [13] when $\sigma^{-1}\varphi$ is bounded and h is of linear growth with respect to the x-variable. We consider the case when both h and φ are of linear growth in x.

The second reason for considering this problem is to weaken the hypotheses under which the two-obstacle parabolic partial differential variational inequality has a unique solution in the viscosity sense. We consider for example the Markovian of the BSDE (1.2), which is defined by the SDE-BSDE system

$$
(1.3)\quad \begin{cases} X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})\, du + \int_t^s \sigma(u, X_u^{t,x})\, dB_u; \\ Y_s^{t,x} = g(X_T^{t,x}) - C \int_s^T Y_u^{t,x} \ln |Y_u^{t,x}|\, du \\ \qquad + \int_s^T dK_u^{+,t,x} - \int_s^T dK_u^{-,t,x} - \int_s^T Z_u^{t,x} dB_u; \\ h(s, X_s^{t,x}) \leqslant Y_s^{t,x} \leqslant h'(s, X_s^{t,x}), \quad \forall s \in [t,T]; \\ \int_t^T (Y_s^{t,x} - h(s, X_s^{t,x}))\, dK_s^{+,t,x} = \int_t^T (h'(s, X_s^{t,x}) - Y_s^{t,x})\, dK_s^{-,t,x} = 0. \end{cases}
$$

The system of double-obstacle variational inequality associated with (1.3) is given by

(1.4)

$$
\begin{cases} \min\big[u(t,x) - h(t,x), \max\big\{-\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) \\ \qquad + Cu(t,x) \ln |u(t,x)|, u(t,x) - h'(t,x)\big\}\big] = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d; \\ u(T,x) = g(x), \quad \forall x \in \mathbb{R}^d, \end{cases}
$$

where

$$
\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} ((\sigma\sigma^*)(t,x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} (b(t,x))_i \frac{\partial}{\partial x_i}.
$$

The logarithmic nonlinearity $u \ln |u|$ is interesting in its own right, since it is neither locally Lipschitz nor uniformly continuous.

This paper is organized as follows. In Section 2, we present the notations and the assumptions used throughout the paper. Moreover, we give some preliminary results that will be useful in this paper. In Section 3, we show the existence of a local solution for the two-barrier reflected BSDE. Later we show the existence and uniqueness of a solution for (1.1). In Section 4, we apply the results obtained to prove that the value function of a mixed zero-sum stochastic differential game problem can be characterized as the solution of a specific BSDE with two barriers. In Section 5, we show that, provided the problem is formulated within a Markovian framework, the solution of the reflected BSDE provides a probabilistic representation for the unique viscosity solution of the related obstacle parabolic partial differential variational inequality.

2. NOTATIONS, ASSUMPTIONS AND PRELIMINARY RESULTS

2.1. Notations. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0 \leqslant t \leqslant T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leqslant t\})_{0 \leqslant t \leqslant T}$. Let $\mathbf{F} = (\mathbf{F}_t)_{0 \leqslant t \leqslant T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0 \leqslant t \leqslant T}$ with the P-null sets of $\mathcal{F}$.

Next for any $p > 0$,

- let $\mathcal{S}^p$ be the space of $\mathbb{R}$-valued $\mathbf{F}_t$-adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that
$$\|Y\|^p_{\mathcal{S}^p} = \mathbb{E}\Big[\sup_{t\leqslant T}|Y_t|^p\Big] < +\infty;$$

- let $\mathcal{M}$ denote the set of $\mathcal{P}$-measurable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$ such that
$$\int_0^T |Z_s|^2\,ds < +\infty \qquad P\text{-a.s.},$$
and let $\mathcal{M}^p$ the subset of $\mathcal{M}$ such that
$$\|Z\|^p_{\mathcal{M}^p} = \mathbb{E}\Big[\Big(\int_0^T |Z_s|^2\,ds\Big)^{p/2}\Big] < +\infty;$$

- let $\mathcal{A}$ be the set of adapted continuous non-decreasing processes $(K_t)_{t\in[0,T]}$ such that $K_0 = 0$ and $K_T < +\infty$ P-a.s. and $\mathcal{A}^p$ is the subset of $\mathcal{A}$ such that $\mathbb{E}[K_T^p] < +\infty$.

2.2. Assumptions. Suppose we are given four data:

- ξ is an $\mathbb{R}$-valued and $\mathbf{F}_T$-measurable random variable.
- $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is a random function which is measurable for any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$ and the process $(f(t,\omega,y,z))_{0\leqslant t\leqslant T}$ is progressively measurable.
- $L := (L_t)_{0\leqslant t\leqslant T}$ and $U := (U_t)_{0\leqslant t\leqslant T}$ are continuous progressively measurable $\mathbb{R}$-valued processes.

We make the following assumptions on the data ξ, f, L and U:

(**H.1**) There exists a positive constant λ large enough such that
$$\mathbb{E}[|\xi|^{e^{\lambda T}+1}] < +\infty.$$

(**H.2**) $L_t < U_t$ for all $t\in[0,T]$ and $L_T \leqslant \xi \leqslant U_T$. In addition for $p \in\,]1,2[$ we have
$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big((L_t^+)^{e^{\lambda T}+1}\big)^{\frac{p}{p-1}}\Big] < +\infty, \quad \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big((U_t^-)^{e^{\lambda T}+1}\big)^{\frac{p}{p-1}}\Big] < +\infty,$$
where $L^+ = L\vee 0$ and $U^- = (-U)\vee 0$.

(**H.3**) (i) f is continuous in (y,z) for almost all (t,ω).

(ii) There exist positive constants c_0, λ (large enough) and c_1 and a process $(\eta_t)_{t\leqslant T}$ such that

$$|f(t,\omega,y,z)| \leqslant |\eta_t| + c_0|y|\,|\ln|y|| + c_1|z|\sqrt{|\ln(|z|)|}, \quad \forall(t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d,$$

and

$$\mathbb{E}\Big[\int_0^T |\eta_s|^{e^{\lambda T}+1}\,ds\Big] < +\infty.$$

(**H.4**) There exist $v \in \mathbb{L}^{q'}(\Omega\times[0,T];\mathbb{R}_+)$ (for some $q' > 0$), a real valued sequence $(A_N)_{N>1}$ and constants $M \in \mathbb{R}_+$ and $r > 0$ such that

(i) $1 < A_N \leqslant N^r, \forall N > 1$;

(ii) $\lim_{N\to\infty} A_N = +\infty$;

(iii) for all $N \in \mathbb{N}$ and y, y', z, z' such that $|y|, |y'|, |z|, |z'| \leqslant N$, we have

$$(y-y')\big(f(t,\omega,y,z) - f(t,\omega,y',z')\big)\mathbb{1}_{\{v_t(\omega)\leqslant N\}} \leqslant M\left(|y-y'|^2\ln A_N + |y-y'|\,|z-z'|\sqrt{\ln A_N} + \frac{\ln A_N}{A_N}\right).$$

2.3. Preliminary results. Now let us define the notions of local and global solutions of the reflected BSDE associated with the quadruple (ξ, f, L, U) which we consider in this paper. We start with a global solution.

DEFINITION 2.1. We say that $\{(Y_t, Z_t, K_t^+, K_t^-); 0 \leqslant t \leqslant T\}$ is a *solution* of the reflected BSDE associated with two continuous barriers L and U, a terminal condition ξ and a generator f if

$$\begin{cases} Y \in \mathcal{S}^{e^{\lambda T}+1}, \quad Z \in \mathcal{M}, \quad K^{\pm} \in \mathcal{A}; \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds \\ \qquad + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s\,dB_s, \quad t \in [0,T]; \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [0,T]; \\ \int_0^T (Y_s - L_s)\,dK_s^+ = \int_0^T (U_s - Y_s)\,dK_s^- = 0. \end{cases} \tag{2.1}$$

Since in many applications, especially in stochastic games or mathematical finance, we do not need strong integrability conditions on Z and $K^{\pm}$, we do not require them in Definition 2.1.

Now we define a local solution. In the following, $p \in\ ]1,2[$.

DEFINITION 2.2. Let τ and γ be stopping times such that $\tau \leqslant \gamma$ P-a.s. We say that $(Y_t, Z_t, K_t^+, K_t^-)_{0\leqslant t\leqslant T}$ is a *local solution* on $[\tau,\gamma]$ of the reflected BSDE associated with two continuous barriers L and U, a terminal condition ξ and a generator f if

(2.2)
$$\begin{cases} Y \in \mathcal{S}^{e^{\lambda T}+1}, \quad Z \in \mathcal{M}^2, \quad K^{\pm} \in \mathcal{A}^p; \\ Y_t = Y_\gamma + \int_t^\gamma f(s, Y_s, Z_s)\,ds \\ \qquad + (K_\gamma^+ - K_t^+) - (K_\gamma^- - K_t^-) - \int_t^\gamma Z_s\,dB_s, \quad \forall t \in [\tau,\gamma]; \\ Y_T = \xi; \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [\tau,\gamma], \quad \int_\tau^\gamma (Y_s - L_s)\,dK_s^+ = \int_\tau^\gamma (U_s - Y_s)\,dK_s^- = 0. \end{cases}$$

We first begin with an estimation of f which can be easily proved.

LEMMA 2.1. *If* (**H.3**) *holds, then for any* $\alpha \in\,]1,2[$,

$$\mathbb{E}\Big[\int_0^T |f(s,Y_s,Z_s)|^{2/\alpha}\,ds\Big] \leqslant C\mathbb{E}\Big[\int_0^T |\eta_s|^2\,ds + \sup_{s\leqslant T}|Y_s|^{4/\alpha} + \int_0^T |Z_s|^2\,ds\Big], \tag{2.3}$$

where C *is a positive constant that depends on* c_0 *and* T.

Proof. From assumption (**H.3**) we can see that there exists $\varepsilon > 0$ such that

$$|f(s,Y_s,Z_s)| \leqslant |\eta_s| + c_0|Y_s|^2 + \frac{c_1}{\sqrt{2\varepsilon}}|Z_s|^{1+\varepsilon}.$$

Then there exists a constant $C > 0$ (that changes from line to line) such that

$$|f(s,Y_s,Z_s)|^{2/\alpha} \leqslant C(|\eta_s|^{2/\alpha} + |Y_s|^{4/\alpha} + |Z_s|^{2(1+\varepsilon)/\alpha}).$$

Next, we choose $0 < \varepsilon < 1$ and we put $\alpha = 1+\varepsilon$. Then

$$|f(s,Y_s,Z_s)|^{2/\alpha} \leqslant C(|\eta_s|^{2/\alpha} + |Y_s|^{4/\alpha} + |Z_s|^2).$$

Hence, (2.3) follows. ■

We now introduce the comparison result established in [9, Theorem 4.1], which also holds in our setting.

PROPOSITION 2.1. *Let* (ξ, f, L) *and* (ξ', f', L') *satisfy all the assumptions* (**H.1**)–(**H.4**). *Suppose in addition that:*

(i) $\xi \leqslant \xi'$ *P-a.s.*

(ii) $f(t,y,z) \leqslant f'(t,y,z)$ $dP \times dt$ *a.e.*, $\forall (t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$.

(iii) $L_t \leqslant L'_t$ *for all* $t \in [0,T]$ *P-a.s.*

Let (Y, Z, K^+) be the solution of the reflected BSDE with one lower barrier associated with (ξ, f, L), i.e.

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + K_T^+ - K_t^+ - \int_t^T Z_s\, dB_s, \quad t \in [0, T]; \\ L_t \leqslant Y_t, \quad \forall t \in [0, T]; \\ \int_0^T (Y_s - L_s)\, dK_s^+ = 0, \end{cases}$$

and (Y', Z', K'^+) the solution of the reflected BSDE with one lower barrier associated with (ξ', f', L'). Then

$$Y_t \leqslant Y_t', \quad 0 \leqslant t \leqslant T, \quad P\text{-a.s.}$$

REMARK 2.1. The comparison result also holds for reflected BSDEs with one upper barrier U, that is, if (ξ, f, U) and (ξ', f', U') satisfy **(H.1)**–**(H.4)**, and moreover $\xi \leqslant \xi'$, $f(t, y, z) \leqslant f'(t, y, z)$ and $U \leqslant U'$, then P-a.s., $Y_t \leqslant Y_t'$ for $0 \leqslant t \leqslant T$, where (Y, Z, K^-) is the solution of the one upper barrier reflected BSDEs associated with (ξ, f, U), i.e.

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - K_T^- + K_t^- - \int_t^T Z_s\, dB_s, \quad t \in [0, T]; \\ U_t \geqslant Y_t, \quad \forall t \in [0, T]; \\ \int_0^T (U_s - Y_s)\, dK_s^- = 0, \end{cases}$$

and (Y', Z', K'^-) is the solution of the one upper barrier reflected BSDEs associated with (ξ', f', U').

If $L = -\infty$, then $K^+ = 0$ and the comparison theorem holds in the case of BSDE without a barrier.

3. EXISTENCE AND UNIQUENESS OF A SOLUTION

In this section we are going to show the existence and uniqueness of a solution for (2.1), but first we show that it has a local solution in the sense of Definition 2.2; then we show that this solution is in fact global when the barriers are completely separated. The main difficulty in this section is to show that the solution of the one-barrier reflected BSDE studied in [9] can be obtained using the penalization method and the comparison theorem, since the authors of [9] used the localization technique to get that result. Actually, we have the following theorem.

THEOREM 3.1. *For each $n \geqslant 0$, let $(y_t^n, z_t^n)_{t\leqslant T}$ be the unique solution (existing due to* [2, *Theorem* 2.1]) *of the BSDE*

$$y_t^n = \xi + \int_t^T \big(f(s, y_s^n, z_s^n) + n(L_s - y_s^n)^+\big)\, ds - \int_t^T z_s^n\, dB_s, \quad t \in [0,T]. \tag{3.1}$$

Then the processes $(y_s^n, z_s^n, \int_0^s n(L_r - y_r^n)^+ dr)_{s\leqslant T}$ converge to $(y_s, z_s, k_s)_{s\leqslant T}$ solving the following reflected BSDE with one barrier:

$$\begin{cases} \text{(a) } \mathbb{E}\Big[\sup\limits_{0\leqslant s\leqslant T} |y_s|^{e^{\lambda T}+1} + \int_0^T |z_s|^2\, ds + k_T^p\Big] < +\infty; \\ \text{(b) } y_t = \xi + \int_t^T f(s, y_s, z_s)\, ds + k_T - k_t - \int_t^T z_s\, dB_s, \quad t \in [0,T]; \\ \text{(c) } L_t \leqslant y_t, \quad \forall t \in [0,T]; \\ \text{(d) } \int_0^T (y_s - L_s)\, dk_s = 0. \end{cases} \tag{3.2}$$

Proof. Part (a) of (3.2) is a direct consequence of [9, Proposition 3.1]; that is, there exists a positive constant $C(\lambda, p, c_0, c_1, T)$ such that for all $p \in\,]1,2[$,

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]} |y_t|^{e^{\lambda t}+1} + \int_0^T |z_s|^2\, ds + k_T^p\Big] \\ &\leqslant C(\lambda, p, c_0, c_1, T)\mathbb{E}\Big[1 + |\xi|^{e^{\lambda T}+1} + \int_0^T |\eta_s|^{e^{\lambda s}+1}\, ds + \sup_{0\leqslant t\leqslant T}\big((L_t^+)^{e^{\lambda t}}\big)^{\frac{p}{p-1}}\Big]. \end{aligned} \tag{3.3}$$

Next, we define

$$k_t^n = n\int_0^t (L_s - y_s^n)^+\, ds, \quad t \in [0,T].$$

Hence, from (3.3) we infer that for $p \in\,]1,2[$,

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} |y_s^n|^{e^{\lambda T}+1} + \int_0^T |z_s^n|^2\, ds + (k_T^n)^p\Big] < +\infty, \quad \forall n \geqslant 0. \tag{3.4}$$

Note that if we define $f_n(t,y,z) = f(t,x,y) + n(L_t - y)^+$, then $f_n(t,y,z) \leqslant f_{n+1}(t,y,z)$. Using the comparison theorem of [9], it follows that $y_t^n \leqslant y_t^{n+1}$, $0 \leqslant t \leqslant T$, a.s. Thus, y_t^n has a limit $\bar{y}_t$. Therefore, by dominated convergence we have

$$\mathbb{E}\Big[\int_0^T (\bar{y} - y_t^n)^{e^{\lambda T}+1}\, dt\Big] \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

The rest of the proof will be divided into two steps.

STEP 1. We will show that for $p \in \left]\frac{e^{\lambda T+1}}{e^{\lambda T+1}-1}, 2\right[$,

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}\big(\big(L_s - y_s^n\big)^+\big)^{\frac{p}{p-1}}\Big] \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

For any $n \geqslant 0$ and $t \leqslant T$, we have

$$y_t^n = \xi + \int_t^T f(s, y_s^n, z_s^n)\, ds + k_t^n - \int_t^T z_s^n\, dB_s. \tag{3.7}$$

Putting $g_s^n = f(s, y_s^n, z_s^n)$ and writing (3.7) in the forward form we get

$$k_t^n = y_0^n - y_t^n - \int_0^t g_s^n\, ds + \int_0^t z_s^n\, dB_s.$$

Since from Lemma 2.1 and (3.4), for any $n \geqslant 0$,

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}|y_t^n|^{e^{\lambda T}+1} + \int_0^T |g_s^n|^{2/\alpha}\, ds + \int_0^T |z_s^n|^2\, ds\Big] < +\infty,$$

there exist processes $(g_t)_{0\leqslant t\leqslant T}$ and $(z_t)_{0\leqslant t\leqslant T}$ which are the weak limits of (subsequences of) $(g_t^n)_{0\leqslant t\leqslant T}$ and $(z_t^n)_{0\leqslant t\leqslant T}$ respectively. Hence, for any stopping time $\bar{\tau} \leqslant T$ the following weak convergences hold:

$$\int_0^{\bar{\tau}} z_s^n\, dB_s \to \int_0^{\bar{\tau}} z_s\, dB_s \quad \text{and} \quad \int_0^{\bar{\tau}} g_s^n\, ds \to \int_0^{\bar{\tau}} g_s\, ds.$$

It follows that

$$k_{\bar{\tau}}^n \to k_{\bar{\tau}} = \bar{y}_0 - \bar{y}_{\bar{\tau}} - \int_0^{\bar{\tau}} g_s\, ds + \int_0^{\bar{\tau}} z_s\, dB_s.$$

Now for any stopping times $\bar{\sigma} \leqslant \bar{\tau} \leqslant T$ we have $k_{\bar{\sigma}}^n \leqslant k_{\bar{\tau}}^n$, so $k_{\bar{\sigma}} \leqslant k_{\bar{\tau}}$. Hence, $(k_t)_{0\leqslant t\leqslant T}$ is an increasing process. Additionally, $\mathbb{E}[(k_T)^p] \leqslant \liminf_{\to\infty} \mathbb{E}[(k_T^n)^p] < +\infty$. Hence, thanks to the monotonic limit of Peng [19, Lemma 2.2], the processes $(\bar{y}_t)_{0\leqslant t\leqslant T}$ and $(k_t)_{0\leqslant t\leqslant T}$ are RCLL.

Next, due to the fact that $\mathbb{E}[(k_T^n)^p] < +\infty$ for any $n \geqslant 0$, we deduce, taking the limit $n \to \infty$, that

$$\mathbb{E}\Big[\int_0^T (L_s - \bar{y}_s)^+\, ds\Big] = 0.$$

Therefore, P-a.s. $\bar{y}_t \geqslant L_t$ for any $t < T$. But $\xi \geqslant L_T$, so $\bar{y} \geqslant L$. Hence, $(L_t - y_t^n)^+ \downarrow 0$ for $0 \leqslant t \leqslant T$ a.s., and from Dini's theorem the convergence is uniform in t. Since $(L_t - y_t^n)^+ \leqslant |L_t| + |\bar{y}_t^0|$, the result follows.

STEP 2. We will show that (y^n, z^n, k^n) converges to (y, z, k) solving (3.2).

Let $0 \leqslant T' \leqslant T$ and put $\Delta_t := |y_t^n - y_t^m|^2 + A_N^{-1}$ and $\Phi(s) = |y_s^n| + |y_s^m| + |z_s^n| + |z_s^m| + v_s$. Then by Itô's formula, for $C > 0$ and $1 < \beta < \min(3-\alpha, 2)$,

$$\begin{aligned}
(3.8)\qquad & e^{Ct}\Delta_t^{\beta/2} + C\int_t^{T'} e^{Cs}\Delta_s^{\beta/2}\,ds = e^{CT'}\Delta_{T'}^{\beta/2} \\
&+ \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)\big(f(s,y_s^n,z_s^n) - f(s,y_s^m,z_s^m)\big)\mathbb{1}_{\Phi(s)>N}\,ds \\
&+ \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)\big(f(s,y_s^n,z_s^n) - f(s,y_s^m,z_s^m)\big)\mathbb{1}_{\Phi(s)\leqslant N}\,ds \\
&- \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n - z_s^m|^2\,ds - \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)(z_s^n - z_s^m)\,dB_s \\
&- \beta\frac{\beta-2}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-2}\big((y_s^n - y_s^m)(z_s^n - z_s^m)\big)^2\,ds \\
&+ \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)(dk_s^n - dk_s^m).
\end{aligned}$$

First let us deal with

$$B := \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)(dk_s^n - dk_s^m).$$

We have

$$\begin{aligned}
B = {} & \beta\int_t^{T'} e^{Cs}(|y_t^n - y_t^m|^2 + A_N^{-1})^{\beta/2-1}(y_s^n - y_s^m)\,dk_s^n \\
& + \beta\int_t^{T'} e^{Cs}(|y_t^m - y_t^n|^2 + A_N^{-1})^{\beta/2-1}(y_s^m - y_s^n)\,dk_s^m.
\end{aligned}$$

Since $dk_s^n = \mathbb{1}_{\{y_s^n \leqslant L_s\}}\,dk_s^n$ and $dk_s^m = \mathbb{1}_{\{y_s^m \leqslant L_s\}}\,dk_s^m$ and the function $x \mapsto \beta e^{Cs}(|x-y|^2 + A_N^{-1})^{\beta/2-1}(x-y)$ is non-decreasing, it follows that

$$\begin{aligned}
B \leqslant {} & \beta\int_t^{T'} e^{Cs}(|L_s - y_s^m|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^m)\,dk_s^n \\
& + \beta\int_t^{T'} e^{Cs}(|L_s - y_s^n|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^n)\,dk_s^m \\
\leqslant {} & \beta e^{CT'}\sup_{0\leqslant s\leqslant T}\Big((|L_s - y_s^m|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^m)^+\Big)\int_t^{T'} dk_s^n \\
& + \beta e^{CT'}\sup_{0\leqslant s\leqslant T}\Big((|L_s - y_s^n|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^n)^+\Big)\int_t^{T'} dk_s^m.
\end{aligned}$$

Therefore,

$$B \leqslant 2\beta e^{CT'} \sup_{0\leqslant s\leqslant T} \big((|L_s - y_s^m|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^m)^+\big) k_T^n$$
$$+ 2\beta e^{CT'} \sup_{0\leqslant s\leqslant T} \big((|L_s - y_s^n|^2 + A_N^{-1})^{\beta/2-1}(L_s - y_s^n)^+\big) k_T^m,$$

which is due to the fact that the processes $(k_t^n)_{t\leqslant T}$ and $(k_t^m)_{t\leqslant T}$ are both increasing. Now, since $\beta/2 - 1 < 0$ and since for all $t \in [0, T]$,

$$A_N^{-1} \leqslant |L_t - y_t^m|^2 + A_N^{-1} \quad \text{and} \quad A_N^{-1} \leqslant |L_t - y_t^n|^2 + A_N^{-1},$$

we get

$$(|L_t - y_t^n|^2 + A_N^{-1})^{\beta/2-1} \leqslant A_N^{1-\beta/2},$$
$$(|L_t - y_t^m|^2 + A_N^{-1})^{\beta/2-1} \leqslant A_N^{1-\beta/2}.$$

It follows that

$$B \leqslant 2A_N^{1-\beta/2}\beta e^{CT'} \sup_{0\leqslant s\leqslant T} (L_s - y_s^m)^+ k_T^n + 2A_N^{1-\beta/2}\beta e^{CT'} \sup_{0\leqslant s\leqslant T} (L_s - y_s^n)^+ k_T^m. \tag{3.9}$$

Next we put

$$J_1 := \beta \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)\big(f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)\big)\mathbb{1}_{\Phi(s)>N}\, ds,$$
$$J_2 := \beta \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)\big(f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)\big)\mathbb{1}_{\Phi(s)\leqslant N}\, ds.$$

Let $\kappa = 3 - \alpha - \beta$. Since $\frac{\beta-1}{2} + \frac{\kappa}{2} + \frac{\alpha}{2} = 1$, we use Hölder's inequality to obtain

$$J_1 \leqslant \beta e^{CT'} \frac{1}{N^\kappa} \Big[\int_t^{T'} \Delta_s\, ds\Big]^{(\beta-1)/2} \times \Big[\int_t^{T'} \Phi(s)^2\, ds\Big]^{\kappa/2} \times \Big[\int_t^{T'} |f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)|^{2/\alpha}\, ds\Big]^{\alpha/2}. \tag{3.10}$$

For J_2 we use assumption (**H.4**) to obtain

$$J_2 \leqslant \beta M \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1} \times \Big[|y_s^n - y_s^m|^2 \ln A_N + \frac{\ln A_N}{A_N} + |y_s^n - y_s^m|\,|z_s^n - z_s^m|\sqrt{\ln A_N}\Big]\mathbb{1}_{\{\Phi(s)\leqslant N\}}\, ds \tag{3.11}$$
$$\leqslant \beta M \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}\big[\Delta_s \ln A_N + |y_s^n - y_s^m|\,|z_s^n - z_s^m|\sqrt{\ln A_N}\big]\mathbb{1}_{\{\Phi(s)\leqslant N\}}\, ds.$$

We combine (3.8) with (3.9)–(3.11) to get

$$
\begin{aligned}
(3.12)\qquad & e^{Ct}\Delta_t^{\beta/2} + C\int_t^{T'} e^{Cs}\Delta_s^{\beta/2}\,ds \leqslant e^{CT'}\Delta_{T'}^{\beta/2} \\
& + \beta e^{CT'}\frac{1}{N^\kappa}\Big[\int_t^{T'}\Delta_s\,ds\Big]^{(\beta-1)/2} \times \Big[\int_t^{T'}\Phi(s)^2\,ds\Big]^{\kappa/2} \\
& \times \Big[\int_t^{T'}|f(s,y_s^n,z_s^n)-f(s,y_s^m,z_s^m)|^{2/\alpha}\,ds\Big]^{\alpha/2} \\
& + \beta M\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(\Delta_s\ln A_N + |y_s^n-y_s^m|\,|z_s^n-z_s^m|\sqrt{\ln A_N})\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n-z_s^m|^2\,ds - \beta\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(y_s^n-y_s^m)(z_s^n-z_s^m)\,dB_s \\
& - \beta\frac{\beta-2}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-2}\big((y_s^n-y_s^m)(z_s^n-z_s^m)\big)^2\,ds \\
& + 2A_N^{1-\beta/2}\beta e^{CT'}\sup_{0\leqslant s\leqslant T}(L_s-y_s^m)^+k_T^n \\
& + 2A_N^{1-\beta/2}\beta e^{CT'}\sup_{0\leqslant s\leqslant T}(L_s-y_s^n)^+k_T^m.
\end{aligned}
$$

But

$$
\begin{aligned}
&\beta M\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(\Delta_s\ln A_N + |y_s^n-y_s^m|\,|z_s^n-z_s^m|\sqrt{\ln A_N})\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds \\
&-\frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n-z_s^m|^2\,ds-\beta\frac{\beta-2}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-2}((y_s^n-y_s^m)(z_s^n-z_s^m))^2\,ds \\
&\leqslant \beta\Big(\int_t^{T'} Me^{Cs}\Delta_s^{\beta/2}\ln A_N\,ds + \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}\big(M|y_s^n-y_s^m|\,|z_s^n-z_s^m|\sqrt{\ln A_N} \\
&\qquad -\frac{1}{2}|z_s^n-z_s^m|^2 + \frac{2-\beta}{2}|y_s^n-y_s^m|^{-2}|y_s^n-y_s^m|^2|z_s^n-z_s^m|^2\big)\,ds\Big).
\end{aligned}
$$

We now apply [1, Lemma 4.6] to obtain

(3.13)

$$
\begin{aligned}
&\beta M\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}(\Delta_s\ln A_N + |y_s^n-y_s^m|\,|z_s^n-z_s^m|\sqrt{\ln A_N})\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds \\
&-\frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n-z_s^m|^2\,ds-\beta\frac{\beta-2}{2}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-2}\big((y_s^n-y_s^m)(z_s^n-z_s^m)\big)^2\,ds \\
&\leqslant \beta\Big(\int_t^{T'} Me^{Cs}\Delta_s^{\beta/2}\ln A_N\,ds + \int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}\Big(\tfrac{M^2\ln A_N}{\beta-1}|y_s^n-y_s^m|^2 - \tfrac{\beta-1}{4}|z_s^n-z_s^m|^2\Big)\,ds\Big)
\end{aligned}
$$

$$\leqslant \beta M^2 \ln A_N \int_t^{T'} e^{Cs}\Delta_s^{\beta/2}\left(1+\frac{1}{\beta-1}\right) ds - \frac{\beta(\beta-1)}{4}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n - z_s^m|^2\, ds$$
$$\leqslant \frac{2M^2\beta}{\beta-1}\ln A_N \int_t^{T'} e^{Cs}\Delta_s^{\beta/2}\, ds - \frac{\beta(\beta-1)}{4}\int_t^{T'} e^{Cs}\Delta_s^{\beta/2-1}|z_s^n - z_s^m|^2\, ds$$

since $\beta \leqslant 2$.

Choosing $C = C_N = \frac{2M^2\beta}{\beta-1}\ln A_N$, we then have

$$\begin{aligned}(3.14)\qquad & e^{C_N t}\Delta_t^{\beta/2} + \frac{\beta(\beta-1)}{4}\int_t^{T'} e^{C_N s}\Delta_s^{\beta/2-1}|z_s^n - z_s^m|^2\, ds \\ &\leqslant e^{C_N T'}\Delta_{T'}^{\beta/2} - \beta\int_t^{T'} e^{C_N s}\Delta_s^{\beta/2-1}(y_s^n - y_s^m)(z_s^n - z_s^m)\, dB_s \\ &\quad + \beta e^{C_N T'}\frac{1}{N^\kappa}\Big[\int_t^{T'}\Delta_s\, ds\Big]^{(\beta-1)/2} \times \Big[\int_t^{T'}\Phi(s)^2\, ds\Big]^{\kappa/2} \\ &\quad \times \Big[\int_t^{T'} |f(s,y_s^n,z_s^n) - f(s,y_s^m,z_s^m)|^{2/\alpha}\, ds\Big]^{\alpha/2} \\ &\quad + 2A_N^{1-\beta/2}\beta e^{C_N T'}\sup_{0\leqslant s\leqslant T}(L_s - y_s^m)^+ k_T^n \\ &\quad + 2A_N^{1-\beta/2}\beta e^{C_N T'}\sup_{0\leqslant s\leqslant T}(L_s - y_s^n)^+ k_T^m.\end{aligned}$$

Therefore, we take the expectation on both sides of (3.14) and we use Burkholder's inequality to find that there exists a universal constant $\ell > 0$ (which may change hereafter) such that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{(T'-\delta')^+\leqslant t\leqslant T'}|y_t^n - y_t^m|^\beta\Big] + \mathbb{E}\Big[\int_{(T'-\delta')^+}^{T'} \frac{|z_s^n - z_s^m|^2}{(|y_s^n - y_s^m|^2 + \nu_R)^{1-\beta/2}}\, ds\Big] \\ &\leqslant \ell e^{C_N\delta'}\mathbb{E}[|y_{T'}^n - y_{T'}^m|^\beta + A_N^{-\beta/2}] \\ &\quad + \ell\frac{e^{C_N\delta'}}{N^\kappa}\mathbb{E}\Big\{\Big[\int_0^T \Delta_s\, ds\Big]^{(\beta-1)/2} \times \Big[\int_0^T \Phi(s)^2\, ds\Big]^{\kappa/2} \\ &\quad \times \Big[\int_0^T |f(s,y_s^n,z_s^n) - f(s,y_s^m,z_s^m)|^{2/\alpha}\, ds\Big]^{\alpha/2}\Big\} \\ &\quad + \ell A_N^{1-\beta/2}e^{C_N\delta'}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}(L_s - y_s^m)^+ k_T^n\Big] \\ &\quad + \ell A_N^{1-\beta/2}e^{C_N\delta'}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}(L_s - y_s^n)^+ k_T^m\Big],\end{aligned}$$

where $\nu_R = \sup\{A_N^{-1} : N \geqslant R\}$. Thus, by Hölder's inequality, (3.4), Lemma 2.1, (**H.4**)(i) and the fact that $C_N = \frac{2M^2\beta}{\beta-1}\ln A_N$ we get, for $p \in \left]\frac{e^{\lambda T+1}}{e^{\lambda T+1}-1}, 2\right[$,

$$\mathbb{E}\Big[\sup_{(T'-\delta')^+\leqslant t\leqslant T'}|y_t^n - y_t^m|^\beta\Big] + \mathbb{E}\Big[\int_{(T'-\delta')^+}^{T'}\frac{|z_s^n - z_s^m|^2}{(|y_s^n - y_s^m|^2+\nu_R)^{1-\beta/2}}\,ds\Big]$$
$$\leqslant \ell\bigg(e^{C_N\delta'}\mathbb{E}[|y_{T'}^n - y_{T'}^m|^\beta] + \frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\beta/2}} + \frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\kappa/r}}\bigg)$$
$$+\,\ell A_N^{1-\beta/2}e^{C_N\delta'}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}((L_s - y_s^m)^+)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}$$
$$+\,\ell A_N^{1-\beta/2}e^{C_N\delta'}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}((L_s - y_s^n)^+)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}.$$

Hence for $\delta' < (\beta-1)\min\left(\frac{1}{4M^2}, \frac{\kappa}{2rM^2\beta}\right)$ we derive

$$\lim_{N\to+\infty}\frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\beta/2}} = 0 \quad \text{and} \quad \lim_{N\to+\infty}\frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\kappa/r}} = 0.$$

It then follows from (3.6) that, for any $\varepsilon > 0$,

$$\limsup_{n,m\to\infty}\mathbb{E}\Big[\sup_{(T'-\delta')^+\leqslant t\leqslant T'}|y_t^n - y_t^m|^\beta\Big] \leqslant \varepsilon + \ell e^{C_N\delta'}\limsup_{n,m\to\infty}\mathbb{E}[|y_{T'}^n - y_{T'}^m|^\beta]. \tag{3.15}$$

Taking successively $T' = T$, $T' = (T-\delta')^+$, $T' = (T-2\delta')^+, \ldots$ in (3.15) we get

$$\lim_{n,m\to\infty}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}|y_t^n - y_t^m|^\beta\Big] = 0. \tag{3.16}$$

Therefore, there exists a $\mathcal{P}$-measurable process y such that

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}|y_t^n - y_t|\Big] \to 0 \quad \text{as } n\to\infty.$$

Next, we prove that

$$\lim_{n,m\to\infty}\mathbb{E}\Big[\int_0^T|z_s^n - z_s^m\Big|^2\,ds] = 0. \tag{3.17}$$

It follows from Itô's formula that

$$(3.18)\quad |y_0^n - y_0^m|^2 + \int_0^T |z_s^n - z_s^m|^2\, ds = 2\int_0^T (y_s^n - y_s^m)\big(f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)\big)\, ds + 2\int_0^T (y_s^n - y_s^m)(dk_s^n - dk_s^m) - 2\int_0^T (y_s^n - y_s^m)(z_s^n - z_s^m)\, dB_s.$$

First we argue that the third term of the right side in (3.18) is a martingale. We can deduce from Burkholder–Davis–Gundy's inequality and (3.4) that there exists a constant $c > 0$ such that

$$(3.19)\quad \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\Big|\int_0^t (y_s^n - y_s^m)(z_s^n - z_s^m)\, dB_s\Big|\Big] \leqslant c\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} |y_s^n - y_s^m|^2\Big] + c\mathbb{E}\Big[\int_0^T |z_s^n - z_s^m|^2\, ds\Big] < +\infty.$$

Now we deal with the term $\int_0^T (y_s^n - y_s^m)(dk_s^n - dk_s^m)$. Actually, since $dk_s^n = \mathbb{1}_{\{y_s^n \leqslant L_s\}}\, dk_s^n$ and $dk_s^m = \mathbb{1}_{\{y_s^m \leqslant L_s\}}\, dk_s^m$ we obtain

$$(3.20)\quad \begin{aligned}\int_0^T (y_s^n - y_s^m)(dk_s^n - dk_s^m) &= \int_0^T (y_s^n - y_s^m)\, dk_s^n + \int_0^T (y_s^m - y_s^n)\, dk_s^m \\ &\leqslant \int_0^T (L_s - y_s^m)\, dk_s^n + \int_0^T (L_s - y_s^n)\, dk_s^m \\ &\leqslant \int_0^T (L_s - y_s^m)^+\, dk_s^n + \int_0^T (L_s - y_s^n)^+\, dk_s^m.\end{aligned}$$

Combining (3.18)–(3.20) we find that there exists a constant c such that

$$(3.21)\quad \mathbb{E}\Big[\int_0^T |z_s^n - z_s^m|^2\, ds\Big] \leqslant c\mathbb{E}\Big[\int_0^T |y_s^n - y_s^m|\, |f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)|\, ds\Big] + c\mathbb{E}\Big[\int_0^T (L_s - y_s^m)^+\, dk_s^n + \int_0^T (L_s - y_s^n)^+\, dk_s^m\Big].$$

Next by Hölder's inequality we have

$$\begin{aligned}(3.22)\quad &\mathbb{E}\Big[\int_0^T |y_s^n - y_s^m|\, |f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)|\, ds\Big] \\ &\leqslant \mathbb{E}\Big[\Big(\int_0^T |y_s^n - y_s^m|^{\frac{2}{2-\alpha}}\, ds\Big)^{\frac{2-\alpha}{2}} \Big(\int_0^T |f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)|^{\frac{2}{\alpha}}\, ds\Big)^{\frac{\alpha}{2}}\Big] \\ &\leqslant \Big(\mathbb{E}\Big[\int_0^T |y_s^n - y_s^m|^{\frac{2}{2-\alpha}}\, ds\Big]\Big)^{\frac{2-\alpha}{2}} \Big(\mathbb{E}\Big[\int_0^T |f(s, y_s^n, z_s^n) - f(s, y_s^m, z_s^m)|^{\frac{2}{\alpha}}\, ds\Big]\Big)^{\frac{\alpha}{2}}.\end{aligned}$$

We plug the last inequality into (3.21) to get

$$\text{(3.23)}\quad \mathbb{E}\Big[\int_0^T |z_s^n - z_s^m|^2\,ds\Big]$$
$$\leqslant c\Big(\mathbb{E}\Big[\int_0^T |y_s^n - y_s^m|^{\frac{2}{2-\alpha}}\,ds\Big]\Big)^{\frac{2-\alpha}{2}} \times \Big(\mathbb{E}\Big[\int_0^T |f(s,y_s^n,z_s^n) - f(s,y_s^m,z_s^m)|^{\frac{2}{\alpha}}\,ds\Big]\Big)^{\frac{\alpha}{2}}$$
$$+\, c\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} |(L_s - y_s^m)^+|^{\frac{p}{p-1}}\Big]\mathbb{E}[(k^n)^p]$$
$$+\, c\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} |(L_s - y_s^n)^+|^{\frac{p}{p-1}}\Big]\mathbb{E}[(k^m)^p].$$

Then, from Lemma 2.1 and (3.4)–(3.6) (for λ large enough and $1<\alpha<2-\frac{2}{e^{\lambda T+1}}$),

$$\text{(3.24)}\qquad \mathbb{E}\Big[\int_0^T |z_s^n - z_s^m|^2\,ds\Big] \to 0 \quad \text{as } (n,m)\to\infty.$$

Consequently, from (3.1),

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |k_t^n - k_t^m|\Big] \to 0 \quad \text{as } (n,m)\to\infty.$$

Therefore, there exists a pair (z,k) of progressively measurable processes such that

$$\mathbb{E}\Big[\int_0^T |z_s^n - z_s|^2\,ds + \sup_{0\leqslant t\leqslant T} |k_t^n - k_t|\Big] \to 0 \quad \text{as } n\to\infty.$$

It remains to show that

$$\int_0^T (y_s - L_s)\,dk_s = 0.$$

Clearly, $(k_t)_{0\leqslant t\leqslant T}$ is increasing. Moreover, (y^n,k^n) tends to (y,k) uniformly in t in probability. Then

$$\int_0^T (y_s^n - L_s)\,dk_s^n \to \int_0^T (y_s - L_s)\,dk_s$$

in probability as $n\to\infty$. Therefore, since $\int_0^T (y_s^n - L_s)\,dk_s^n \leqslant 0,\ n\in\mathbb{N}$, we have $\int_0^T (y_s - L_s)\,dk_s \leqslant 0$. On the other hand, $\int_0^T (y_s - L_s)\,dk_s \geqslant 0$. Thus,

$$\int_0^T (y_s - L_s)\,dk_s = 0 \quad \text{a.s.}$$

Hence, (y,z,k) solves the reflected BSDE associated with (ξ,f,L). ■

We now focus on the uniqueness of a solution for BSDE (2.1):

PROPOSITION 3.1. *Assume that* (**H.1**)–(**H.4**) *are satisfied. Then the reflected BSDE associated with* (ξ, f, L, U) *has at most one solution.*

Proof. Suppose that (Y, Z, K^+, K^-) and (Y', Z', K'^+, K'^-) are two solutions of (2.1), and for $N \in \mathbb{N}^\star$ set $\Delta_t := |Y_t - Y'_t|^2 + A_N^{-1}$.

Following the same argument as in Step 2 of the proof of Theorem 3.1, one can prove that for every $R \in \mathbb{N}$ and every $\varepsilon > 0$ there exists N_0 such that for every $N > N_0$,

$$\mathbb{E}\Big[\sup_{(T'-\delta')^+\leqslant t\leqslant T'} |Y_t - Y'_t|^\beta\Big] + \mathbb{E}\Big[\int_{(T'-\delta')^+}^{T'} \frac{|Z_s - Z'_s|^2}{(|Y'_s - Y'_s|^2 + \nu_R)^{1-\beta/2}}\, ds\Big] \leqslant \ell e^{C_N\delta'}\mathbb{E}[|Y_{T'} - Y'_{T'}|^\beta] + \varepsilon, \tag{3.25}$$

where $\nu_R = \sup\{A_N^{-1} : N \geqslant R\}$ and ℓ is a universal constant. Taking successively $T' = T, T' = (T-\delta')^+, T' = (T-2\delta')^+, \ldots$ in (3.25), we obtain

$$Y = Y', \quad Z = Z', \quad K^+ - K^- = K'^+ - K'^-.$$

Finally, let us show that $K^+ = K'^+$ and $K^- = K'^-$. For any $t \leqslant T$,

$$\int_0^t (Y_s - L_s)\, dK_s = \int_0^t (Y_s - L_s)\, dK'_s,$$

where $K = K^+ - K^-$ and $K' = K'^+ - K'^-$. But

$$\int_0^t (Y_s - L_s)\, dK_s = -\int_0^t (U_s - L_s)\, dK_s^-, \quad \int_0^t (Y_s - L_s)\, dK'_s = -\int_0^t (U_s - L_s)\, dK'^-_s.$$

Then

$$\int_0^t (U_s - L_s)\, dK_s^- = \int_0^t (U_s - L_s)\, dK'^-_s, \quad \forall t \leqslant T.$$

Since $K_0^- = K'^-_0 = 0$ and $L_t < U_t$ for all $t \leqslant T$ it follows that $K^- = K'^-$, and we also find that $K^+ = K'^+$, which completes the proof. ■

Having overcome the main difficulty of this section (Theorem 3.1), we can now address the existence of a local solution for (2.1):

THEOREM 3.2. *There exists a unique continuous process* $Y = (Y_t)_{t\in[0,T]}$ *such that:*

(i) $\mathbb{E}\big[\sup_{s\leqslant T} |Y_s|^{e^{\lambda T}+1}\big] < +\infty$ *and* $L \leqslant Y \leqslant U$ *and* $Y_T = \xi$.

(ii) *For any stopping time* τ *there exists another stopping time* $\lambda_\tau \geqslant \tau$, P-*a.s., and a triplet* $(Z^\tau, K^{\tau,+}, K^{\tau,-}) \in \mathcal{M}^2 \times \mathcal{A}^p \times \mathcal{A}^p$ $(K_\tau^{\tau,\pm} = 0)$ *such that* P-*a.s.*

$$(3.26)\quad \begin{cases} Y_t = Y_{\lambda_\tau} + \int\limits_t^{\lambda_\tau} f(s, Y_s, Z_s^\tau)\, ds + (K_{\lambda_\tau}^{\tau,+} - K_t^{\tau,+}) - (K_{\lambda_\tau}^{\tau,-} - K_t^{\tau,-}) \\ \qquad - \int\limits_t^{\lambda_\tau} Z_s^\tau\, dB_s, \quad t \in [\tau, \lambda_\tau]; \\ \int\limits_\tau^{\lambda_\tau} (Y_s - L_s)\, dK_s^{\tau,+} = \int\limits_\tau^{\lambda_\tau} (U_s - Y_s)\, dK_s^{\tau,-} = 0. \end{cases}$$

(iii) *If ν_τ and π_τ are stopping times such that*

$$\nu_\tau = \inf\{s \geqslant \tau : Y_s = U_s\} \wedge T \quad \text{and} \quad \pi_\tau = \inf\{s \geqslant \tau : Y_s = L_s\} \wedge T,$$

then P-a.s., $\nu_\tau \vee \pi_\tau \leqslant \lambda_\tau$.

Proof. Having proved Theorem 3.1, the remaining steps to prove Theorem 3.2 are actually the same as in [8]. Thus, to avoid repetition, we only give a sketch of the proof and for more details we refer the reader to [8, pp. 914–924].

First, we analyze the following increasing penalization scheme: for any $n \geqslant 0$,

$$(3.27)\quad \begin{cases} Y^n \in \mathcal{S}^{e^{\lambda T}+1}, \quad Z^n \in \mathcal{M}^2, \quad K^{n,-} \in \mathcal{A}^p; \\ Y_t^n = \xi + \int\limits_t^T (f(s, Y_s^n, Z_s^n) + n(L_s - Y_s^n)^+)\, ds \\ \qquad - (K_T^{n,-} - K_t^{n,-}) - \int\limits_t^T Z_s^n\, dB_s, \quad t \in [0,T]; \\ Y_t^n \leqslant U_t, \quad \forall t \in [0,T], \quad \int\limits_0^T (U_s - Y_s^n)\, dK_s^{n,-} = 0. \end{cases}$$

First, $(Y^n, Z^n, K^{n,-})$ exists due to Theorem 3.1 and the fact that (Y, Z, K) is a solution of the reflected BSDE with a lower obstacle associated with (ξ, f, L) iff $(-Y, -Z, K)$ is a solution of the reflected BSDE with an upper obstacle associated with $(-\xi, -f(t, -Y, -Z), -L)$.

Next, since the sequence $f_n(t, y, z) = f(t, y, z) + n(L_t - y)^+$ is increasing, from Remark 2.1 we know that $Y^n \leqslant Y^{n+1} \leqslant U$ for any $n \geqslant 0$. Then $(Y_t^n)_{n\geqslant 0}$ converges to a lower semicontinuous optional process $Y = (Y_t)_{0\leqslant t\leqslant T}$ that satisfies $Y_t \leqslant U_t$ for all $t \leqslant T$ P-a.s., and $\mathbb{E}\big[\sup_{t\leqslant T} |Y_t|^{e^{\lambda T}+1}\big] < +\infty$.

Next, we put

$$\theta_\tau^n = \inf\{s \geqslant \tau : Y_s^n = U_s\} \wedge T, \quad \theta_\tau = \lim_{n\to\infty} \theta_\tau^n, \quad g_s^n = f(s, Y_s^n, Z_s^n),$$

and we show that Y is RCLL on $[\tau, \theta_\tau]$. Indeed, since $K^{n,-}$ does not increase before θ_τ, (Y_t^n, Z_t^n) satisfy (3.27) with $K_T^{n,-} = K_t^{n,-} = 0$ on $[\tau, \theta_\tau]$. Then, as a consequence of Lemma 2.1 and (3.4), there exist subsequences of $((g_s^n \mathbb{1}_{[\tau,\theta_\tau]}(s))_{s\leqslant T})_{n\geqslant 0}$ and $((Z_s^n \mathbb{1}_{[\tau,\theta_\tau]}(s))_{s\leqslant T})_{n\geqslant 0}$, which we still index by n, and processes $(g_s \mathbb{1}_{[\tau,\theta_\tau]}(s))_{s\leqslant T}$ and $(Z_s \mathbb{1}_{[\tau,\theta_\tau]}(s))_{s\leqslant T}$ such that for any stopping time $\bar{\gamma}$ satisfying $\tau \leqslant \bar{\gamma} \leqslant \theta_\tau$ the following weak convergences hold:

$$\int_\tau^{\bar{\gamma}} Z_s^n \, dB_s \rightharpoonup \int_\tau^{\bar{\gamma}} Z_s \, dB_s \quad \text{and} \quad \int_\tau^{\bar{\gamma}} g_s^n ds \rightharpoonup \int_\tau^{\bar{\gamma}} g_s ds, \quad \text{as } n \to \infty.$$

It follows that

$$K_{\bar{\gamma}}^{n,+} \to K_{\bar{\gamma}}^{+} \quad \text{and} \quad Y_t = Y_\tau - \int_\tau^t g_s ds - K_t^+ + \int_\tau^t Z_s \, dB_s,$$

so that $\mathbb{E}[(K_{\theta_\tau}^+)^p] \leqslant \liminf_{n\to+\infty} \mathbb{E}[(K_{\theta_\tau}^{n,+})^p] < +\infty$. Since $Y^n \leqslant Y^{n+1}$, we can deduce from a result of S. Peng [19, Lemma 2.2] that Y is RCLL on $[\tau, \theta_\tau]$. Next, we can show as in [8] that we have the following proposition which can be considered as a step of the proof.

PROPOSITION 3.2. *Assume that* (**H.1**)–(**H.4**) *are satisfied. Then the following holds true:*

(i) *P-a.s., $Y_{\theta_\tau}\mathbb{1}_{\{\theta_\tau<T\}} = U_{\theta_\tau}\mathbb{1}_{\{\theta_\tau<T\}}$ and P-a.s., $L_t \leqslant Y_t$ for all $t \leqslant T$.*

(ii) *There exist adapted processes $(\bar{K}_t^{\tau,+})_{0\leqslant t\leqslant T}$ and $(\bar{Z}_t^\tau)_{0\leqslant t\leqslant T}$ such that $(Y_t, \bar{Z}_t^\tau, \bar{K}_t^{\tau,+}, 0)_{0\leqslant t\leqslant T}$ is a local solution of the reflected BSDE* (2.1) *on $[\tau, \theta_\tau]$, which means that it satisfies*

$$\text{(3.28)} \qquad \begin{cases} \bar{Z}^\tau \in \mathcal{M}^2, \quad \bar{K}^{\tau,+} \in \mathcal{A}^p; \\ Y_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} f(s, Y_s, \bar{Z}_s^\tau) \, ds + (\bar{K}_{\theta_\tau}^{\tau,+} - \bar{K}_t^{\tau,+}) \\ \qquad - \int_t^{\theta_\tau} \bar{Z}_s \, dB_s, \quad \forall t \in [\tau, \theta_\tau]; \\ Y_T = \xi; \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [\tau, \theta_\tau], \quad \int_\tau^{\theta_\tau} (Y_s - L_s) \, d\bar{K}_s^{\tau,+} = 0. \end{cases}$$

(iii) *If $\upsilon_\tau = \inf\{s \geqslant \tau : Y_s = U_s\} \wedge T$, then $\upsilon_\tau \leqslant \theta_\tau$.*

Now by analyzing the decreasing penalization scheme, that is, for any $m \geqslant 0$,

$$\text{(3.29)} \qquad \begin{cases} \mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} |\tilde{Y}_s^m|^{e^{\lambda T}+1} + \int_0^T |\tilde{Z}_s^m|^2 \, ds + (K_T^{m,+})^p\Big] < +\infty; \\ \tilde{Y}_t^m = \xi + \int_t^T (f(s, \tilde{Y}_s^m, \tilde{Z}_s^m) - m(\tilde{Y}_s^m - U_s)^+) \, ds \\ \qquad + (K_T^{m,+} - K_t^{m,+}) - \int_t^T \tilde{Z}_s^m \, dB_s, \quad t \in [0, T]; \\ \tilde{Y}_t^m \geqslant L_t, \quad \forall t \in [0, T], \quad \int_0^T (\tilde{Y}_s^m - L_s) \, dK_s^{m,+} = 0 \end{cases}$$

($(\tilde{Y}^m, \tilde{Z}^m, K^{m,+})$ exists due to Theorem 3.1) we can also prove

PROPOSITION 3.3. *The following hold:*

(i) *P-a.s., $\tilde{Y}_{\delta_\tau} \mathbb{1}_{\{\delta_\tau < T\}} = L_{\delta_\tau} \mathbb{1}_{\{\delta_\tau < T\}}$ and P-a.s., $\tilde{Y}_t \leqslant U_t$ for all $t \leqslant T$.*

(ii) *There exists a pair of adapted processes $(\tilde{Z}_t^\tau, \tilde{K}_t^{\tau,-})_{t \leqslant T}$ such that the quadruple $(\tilde{Y}_t, \tilde{Z}_t^\tau, 0, \tilde{K}_t^{\tau,-})_{t \leqslant T}$ satisfies*

$$
\begin{cases}
\tilde{Z}^\tau \in \mathcal{M}^2, \quad \tilde{K}^{\tau,-} \in \mathcal{A}^p; \\
\tilde{Y}_t = \tilde{Y}_{\delta_\tau} + \int_t^{\delta_\tau} f(s, \tilde{Y}_s, \tilde{Z}_s^\tau)\, ds - (\tilde{K}_{\delta_\tau}^{\tau,-} - \tilde{K}_t^{\tau,-}) \\
\qquad - \int_t^{\delta_\tau} \tilde{Z}_s^\tau \, dB_s, \quad \forall t \in [\tau, \delta_\tau]; \\
\tilde{Y}_T = \xi; \\
L_t \leqslant \tilde{Y}_t \leqslant U_t, \quad \forall t \in [\tau, \delta_\tau], \quad \int_\tau^{\delta_\tau} (U_s - \tilde{Y}_s)\, d\tilde{K}_s^{\tau,-} = 0.
\end{cases}
\tag{3.30}
$$

(iii) *Put $\mu_\tau = \inf\{s \geqslant \tau : \tilde{Y}_s = L_s\} \wedge T$. Then $\mu_\tau \leqslant \delta_\tau$.*

Here $\tilde{Y} = \lim_{m\to\infty} \tilde{Y}^m$ and $\delta_\tau = \lim_{m\to\infty} \delta_\tau^m$ with $\delta_\tau^m = \inf\{s \geqslant \tau : \tilde{Y}_s^m = L_s\} \wedge T$ for all $m \geqslant 0$.

Next using the comparison result and the technique in [8, p. 923], we can prove that P-a.s., $Y_t = \tilde{Y}_t$ for any $t \leqslant T$. Finally, we proceed once again as in [8, p. 924] to finish the proof.

Next we can proceed as in [14, Theorem 3.7] to show that the local solution is actually a global one:

THEOREM 3.3. *Under* (**H.1**)–(**H.4**)*, the reflected BSDE* (2.1) *associated with (ξ, f, L, U) has a unique solution that is the quadruple (Y, Z, K^+, K^-).*

4. MIXED ZERO-SUM STOCHASTIC DIFFERENTIAL GAME PROBLEM

Now we deal with an application of the double-barrier reflected BSDEs to solving stochastic mixed games problems. First, let us briefly describe the setting of the problem. Write $\Omega = \mathcal{C}([0,T], \mathbb{R}^d)$ for the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$.

Put $\|\omega\|_t = \sup_{s \leqslant t} |\omega_s|$ and consider a mapping $\sigma : [0,T] \times \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying the following assumptions

(**A1**) (i) σ is $\mathcal{P}$-measurable and invertible.

(ii) There exists a constant $C > 0$ such that for all $(t, \omega, \omega') \in [0,T] \times \Omega \times \Omega$,

$$
|\sigma(t,\omega) - \sigma(t,\omega')| \leqslant C \|\omega - \omega'\|_t,
$$
$$
|\sigma(t,\omega)| \leqslant C(1 + \|\omega\|_t), \quad |\sigma^{-1}(t,\omega)| \leqslant C.
$$

Let $x_0 \in \mathbb{R}^d$ and $x = (x_t)_{t \leqslant T}$ be the solution of the following standard functional differential equation:

$$x_t = x_0 + \int_0^t \sigma(s, x)\, dB_s, \quad t \leqslant T. \tag{4.1}$$

The assumptions on σ imply that (4.1) has a unique solution x (see [20, p. 375]). Moreover,

$$\mathbb{E}[(\|x\|_T)^n] < +\infty, \quad \forall n \in [1, +\infty[\quad \text{([16, p. 306])}. \tag{4.2}$$

Let now $\bar{U}$ (resp. V) be a compact metric space and $\mathcal{U}$ (resp. $\mathcal{V}$) the space of all $\mathcal{P}$-measurable processes with values in $\bar{U}$ (resp. V), and let $\varphi : [0, T] \times \mathbb{R}^d \times \bar{U} \times V \to \mathbb{R}^d$ and $h : [0, T] \times \mathbb{R}^d \times \bar{U} \times V \to \mathbb{R}^d$ be such that:

(**A2**) (i) For each $(u, v) \in \bar{U} \times V$, the function $(t, x) \mapsto \varphi(t, x, u, v)$ is predictable.

(ii) For each $(t, x) \in [0, T] \times \mathbb{R}^d$, $\varphi(t, x, \cdot, \cdot)$ and $h(t, x, \cdot, \cdot)$ are continuous on $\bar{U} \times V$.

(iii) There exists a real constant $K > 0$ such that

$$\begin{aligned} &|h(t, x, u, v)| + |\varphi(t, x, u, v)| \\ &\qquad \leqslant K(1 + \|x\|_t), \quad \forall (t, x, u, v) \in [0, T] \times \mathbb{R}^d \times \bar{U} \times V. \end{aligned} \tag{4.3}$$

Under this assumption, for any $(u, v) \in \mathcal{U} \times \mathcal{V}$, we define a probability on $(\Omega, \mathcal{F})$ by

$$\begin{aligned} &\frac{dP^{(u,v)}}{dP} \\ &\quad = \exp\Big\{ \int_0^T \sigma^{-1}(s, x)\varphi(s, x, u_s, v_s)\, dB_s - \frac{1}{2}\int_0^T |\sigma^{-1}(s, x)\varphi(s, x, u_s, v_s)|^2\, ds \Big\}. \end{aligned}$$

We now consider the payoff

$$\begin{aligned} &J(u, \tau; v, \sigma) \\ &\quad = \mathbb{E}^{(u,v)}\Big[\int_0^{\tau\wedge\sigma} h(s, x, u_s, v_s)\, ds + L_\sigma \mathbb{1}_{\{\sigma \leqslant \tau < T\}} + U_\tau \mathbb{1}_{\{\tau < \sigma\}} + \xi \mathbb{1}_{\{\tau\wedge\sigma = T\}} \Big], \end{aligned} \tag{4.4}$$

where L, U and ξ are those of the previous sections. The problem we are interested in is to find a saddle point for the payoff functional $J(u, \tau; v, \sigma)$, that is, we are looking for two intervention strategies (u^*, τ^*) and (v^*, σ^*) that satisfy

$$J(u^*, \tau^*; v, \sigma) \leqslant J(u^*, \tau^*; v^*, \sigma^*) \leqslant J(u, \tau; v^*, \sigma^*). \tag{4.5}$$

Now we define the Hamiltonian associated with this mixed stochastic game problem by

$$H(t,x,z,u,v) := z\sigma^{-1}(t,x)\varphi(t,x,u,v) + h(t,x,u,v), \qquad (t,x,z,u,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \bar{U} \times V.$$

Under Isaacs's condition, and by Beneš' theorem [3], there exists a couple of $\mathcal{P}\otimes\mathcal{B}$-measurable functions $u^* \equiv u^*(t,x,z)$ and $v^* \equiv v^*(t,x,z)$ with values in $\bar{U}$ and V respectively such that for all $(t,x,u,v) \in [0,T] \times \mathbb{R}^d \times \bar{U} \times V$,

$$\begin{aligned} H^*(t,x,z) &= H(t,x,z,u^*(t,x,z),v^*(t,x,z)) \\ &= \inf_{u\in\bar{U}} \sup_{v\in V} H(t,x,z,u,v) = \sup_{v\in V} \inf_{u\in\bar{U}} H(t,x,z,u,v). \end{aligned}$$

THEOREM 4.1. *Under assumptions* (**A1**) *and* (**A2**)*, there exists a quadruple of adapted processes* $(Y^*, Z^*, K^{*,+}, K^{*,-})$ *that is the unique solution of the finite horizon reflected BSDE associated with* (ξ, H^*, L, U)*. Define stopping times by*

$$\sigma^* = \inf\{t \geqslant 0 : Y_t^* = L_t\} \wedge T \quad \textit{and} \quad \tau^* = \inf\{t \geqslant 0 : Y_t^* = U_t\} \wedge T.$$

Then $Y_0^* = J(u^*, \tau^*; v^*, \sigma^*)$ *and* $(u^*, \tau^*; v^*, \sigma^*)$ *is a saddle point for the mixed stochastic game problem.*

Proof. Since H^* satisfy **(H.3)** and **(H.4)** (see [9]), the quadruple $(Y^*, Z^*, K^{*,+}, K^{*,-})$ exists and is unique. The rest of the proof is classical and left to the reader.

5. CONNECTION WITH DOUBLE OBSTACLE VARIATIONAL INEQUALITIES

Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be two globally Lipschitz functions and consider the following SDE:

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t, \qquad t \leqslant T.$$

We denote by $(X_s^{t,x})_{s\geqslant t}$ the unique solution of this SDE starting from x at time $s = t$.

Now suppose we are given four functions

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \qquad g : \mathbb{R}^d \to \mathbb{R}, \qquad h, h' : [0,T] \times \mathbb{R}^d \to \mathbb{R}$$

such that the following holds:

(**H′.1**) f satisfies assumptions (**H.3**) and (**H.4**), and there exists $p > 1$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\mathbb{E}\Big[\int_0^T |f(s, X_s^{t,x}, 0, 0)|^p\,ds\Big] < +\infty.$$

(H′.2) For all $(t,x) \in [0,T] \times \mathbb{R}^d$, $h(t,x) < h'(t,x)$ and $h(T,x) \leqslant g(x) \leqslant h'(T,x)$, and there exists a constant $C > 0$ such that

$$|h'(t,x)| + |h(t,x)| + |g(x)| \leqslant C(1 + \|x\|_t).$$

5.1. Connection with one-obstacle variational inequalities. Let us denote by $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ the solution of the following reflected BSDE:

$$\text{(5.1)} \quad \begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\, dr + K_T^{t,x} - K_s^{t,x} \\ \qquad - \int_s^T Z_r^{t,x}\, dB_r; \\ Y_s^{t,x} \geqslant h(s, X_s^{t,x}), \quad \forall s \in [t,T], \quad \int_t^T (Y_r^{t,x} - h(r, X_r^{t,x}))\, dK_r^{t,x} = 0. \end{cases}$$

Moreover, on $[0,t]$ we set $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = K_s^{t,x} = 0$.

We will show that $Y_t^{t,x}$ is deterministic for every (t,x) and we define

$$\text{(5.2)} \qquad u(t,x) = Y_t^{t,x}.$$

First we will prove that u is continuous and is a viscosity solution of the following obstacle problem: for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

(5.3)

$$\min\left[u(t,x) - h(t,x), -\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),\sigma(t,x)\nabla u(t,x))\right] = 0,$$

with $u(T,x) = g(x)$ for $x \in \mathbb{R}^d$ and

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^d ((\sigma\sigma^*)(t,x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d (b(t,x))_i \frac{\partial}{\partial x_i}.$$

Then we will prove that it is the unique continuous viscosity solution that belongs to some class of functions.

5.1.1. *Continuity*

THEOREM 5.1. *For every $(t,x) \in [0,T] \times \mathbb{R}^d$, the function $u(t,x) = Y_t^{t,x}$ is continuous and of polynomial growth.*

Proof. Let $(t_n, x_n) \to (t,x)$. Since $|Y_t^{t,x} - Y_{t_n}^{t_n,x_n}|$ is deterministic, we have

$$\begin{aligned} |Y_t^{t,x} - Y_{t_n}^{t_n,x_n}| &= \mathbb{E}(|Y_t^{t,x} - Y_{t_n}^{t_n,x_n}|) \\ &\leqslant \mathbb{E}(|Y_t^{t,x} - Y_{t_n}^{t,x}|) + \mathbb{E}(|Y_{t_n}^{t,x} - Y_{t_n}^{t_n,x_n}|). \end{aligned}$$

Then from (3.3) and Lemma 2.1 we get $\lim_{n\to\infty} \mathbb{E}[|Y_t^{t,x} - Y_{t_n}^{t,x}|] = 0$.

Now we shall show that $\lim_{n\to\infty}\mathbb{E}[|Y_{t_n}^{t,x}-Y_{t_n}^{t_n,x_n}|]=0$, for which we use the fact that

$$\mathbb{E}[|Y_{t_n}^{t,x}-Y_{t_n}^{t_n,x_n}|]\leqslant \mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}-Y_s^{t_n,x_n}|\Big].$$

We proceed as in Step 2 of the proof of Theorem 3.1. For $\beta\in\,]1,\min(3-\alpha,2)[$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(T'-\delta')^+\leqslant s\leqslant T'}|Y_s^{t,x}-Y_s^{t_n,x_n}|^\beta\Big]\\
&\qquad+\mathbb{E}\Big[\int_{(T'-\delta')^+}^{T'}\frac{|Z_s^{t,x}-Z_s^{t_n,x_n}|^2}{(|Y_s^{t,x}-Y_s^{t_n,x_n}|^2+\nu_R)^{1-\beta/2}}\Big]\,ds\\
&\qquad\leqslant \ell e^{C_N\delta'}\mathbb{E}[|g(X_{T'}^{t,x})-g(X_{T'}^{t_n,x_n})|^\beta]+\ell\bigg(\frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\beta/2}}+\frac{A_N^{\frac{2M^2\delta'\beta}{\beta-1}}}{A_N^{\kappa/r}}\bigg)\\
&\qquad+\ell e^{C_N\delta'}\beta[2N^2+\nu_1]^{\frac{\beta-1}{2}}\\
&\qquad\qquad\times\mathbb{E}\Big[\int_t^{T'}|f(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x})-f(u,X_u^{t_n,x_n},Y_u^{t,x},Z_u^{t,x})|\,du\Big]\\
&\qquad+\ell e^{C_N\delta'}\bigg(\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}|\Big(|h(s,X_s^{t,x})-h(s,X_s^{t_n,x_n})|^2+A_N^{-1}\Big)^{\beta/2-1}\\
&\qquad\qquad\qquad\times(h(s,X_s^{t,x})-h(s,X_s^{t_n,x_n}))|^{\frac{p}{p-1}}\Big]\bigg)^{\frac{p-1}{p}}.
\end{aligned}$$

Since f, g and h are continuous in x, for $\delta'<(\beta-1)\min\big(\frac{1}{4M^2},\frac{\kappa}{2rM^2\beta}\big)$ we pass to the limit as $n\to\infty$ and then as $N\to\infty$, and by taking successively $T'=T$, $T'=(T-\delta')^+$, $T'=(T-2\delta')^+,\dots$, for every $\beta\in\,]1,\min(3-\alpha,2)[$ we get

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}-Y_s^{t_n,x_n}|^\beta\Big]=0.$$

Finally, since $\beta>1$, the result follows by using Hölder's inequality. The polynomial growth of u follows from (3.3). ■

5.1.2. *Existence of the solution*

THEOREM 5.2. *Assume that* **(H′.1)** *and* **(H′.2)** *are satisfied. Then the function* $u:(t,x)\mapsto u(t,x)=Y_t^{t,x}$ *is a viscosity solution of the obstacle problem* (5.3).

Proof. Consider the following reflected BSDE:

$$Y_s^{t,x,n}=g(X_T^{t,x})+\int_s^T f_n(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\,dr-\int_s^T Z_r^{t,x,n}\,dB_r,\tag{5.4}$$

where

$$f_n(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})=f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})+n(Y_r^{t,x,n}-h(r,X_r^{t,x}))^-.$$

Then, from [2], $u_n(t,x) = Y_t^{t,x,n}$ is a viscosity solution of

$$\frac{\partial u_n}{\partial t}(t,x) + \mathcal{L}u_n(t,x) + f_n(t,x,u_n(t,x),\sigma(t,x)\nabla u_n(t,x)) = 0. \tag{5.5}$$

From the comparison theorem we know that u_n is increasing, and we can argue as in [11] to show that u_n converges to u solving (5.3). ∎

5.2. Connection with double-obstacle variational inequalities. Let $(Y_s^{t,x}, Z_s^{t,x}, K_s^{+,t,x}, K_s^{-,t,x})_{t\leqslant s\leqslant T}$ be a solution of the following reflected BSDE:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})\,du \\ \qquad\qquad + \int_s^T dK_u^{+,t,x} - \int_s^T dK_u^{-,t,x} - \int_s^T Z_u^{t,x}\,dB_u; \\ h(s,X_s^{t,x}) \leqslant Y_s^{t,x} \leqslant h'(s,X_s^{t,x}), \quad \forall s\in[t,T]; \\ \int_t^T (Y_u^{t,x} - h(u,X_u^{t,x}))\,dK_u^{+,t,x} = \int_t^T (h'(u,X_u^{t,x}) - Y_u^{t,x})\,dK_u^{-,t,x} = 0. \end{cases} \tag{5.6}$$

The objective of this section is to show that $u(t,x) = Y_t^{t,x}$ is continuous and it is a viscosity solution of the obstacle problem

$$\begin{cases} \min\Big[u(t,x) - h(t,x), \max\Big\{-\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) \\ \qquad -f(t,x,u(t,x),\sigma(t,x)\nabla u(t,x)), u(t,x) - h'(t,x)\Big\}\Big] = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,T)\times\mathbb{R}^d; \\ u(T,x) = g(x), \quad \forall x\in\mathbb{R}^d. \end{cases} \tag{5.7}$$

5.3. The continuity of the viscosity solution

PROPOSITION 5.1. *For every $(t,x)\in[0,T]\times\mathbb{R}^d$, the function $u(t,x) = Y_t^{t,x}$ is continuous.*

Proof. For any $n \geqslant 0$ let $(\underline{Y}_s^{t,x,n})_{s\leqslant T}$ (resp. $(\overline{Y}_s^{t,x,n})_{s\leqslant T}$) be the first component of the unique solution of the BDSE with one reflecting lower (resp. upper) barrier associated with $(g(X_T^{t,x}), f(s,X_s^{t,x},y,z) - n(h'(s,X_s^{t,x}) - y)^-, h(s,X_s^{t,x}))$ (resp. $(g(X_T^{t,x}), f(s,X_s^{t,x},y,z) + n(h(s,X_s^{t,x}) - y)^+, h'(s,X_s^{t,x}))$). As shown in the previous subsection, for any $n\geqslant 0$ there exist deterministic functions $\underline{u}^n(t,x) = \underline{Y}_t^{t,x,n}$ and $\bar{u}^n(t,x) = \overline{Y}_t^{t,x,n}$ that are viscosity solutions of

$$\min\Big[u(t,x) - h(t,x), -\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),\sigma(t,x)\nabla u(t,x)) \\ + n(h'(t,x) - u(t,x))^-\Big] = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^d; \quad u(T,x) = g(x), \tag{5.8}$$

and

(5.9)

$$\max\Big[u(t,x)-h'(t,x),-\frac{\partial u}{\partial t}(t,x)-\mathcal{L}u(t,x)-f(t,x,u(t,x),\sigma(t,x)\nabla u(t,x)) \\ -n(h(t,x)-u(t,x))^{+}\Big]=0, \quad (t,x)\in[0,T)\times\mathbb{R}^d; \quad u(T,x)=g(x),$$

respectively. Now thanks to the results of the previous sections, the sequence $(\underline{Y}^{t,x,n})_{n\geqslant 0}$ converges increasingly to $Y^{t,x}$ and the sequence $(\overline{Y}^{t,x,n})_{n\geqslant 0}$ converges decreasingly to the same $Y^{t,x}$, meaning that $\underline{u}^n(t,x) \searrow u(t,x)$ and $\bar{u}^n(t,x) \nearrow u(t,x)$. Since $\underline{u}^n$ and $\bar{u}^n$ are both continuous, u is both lower and upper semicontinuous, and hence continuous.

5.4. Existence of the solution

THEOREM 5.3. *Assume that* $(\mathbf{H'.1})$ *and* $(\mathbf{H'.2})$ *are satisfied. Then the function* $u:(t,x)\mapsto u(t,x)=Y_t^{t,x}$ *is a viscosity solution of the obstacle problem* (5.7).

Proof. First note that since $\underline{u}^n$, $\bar{u}^n$ and u are continuous, Dini's lemma shows that they converge uniformly to u on compact subsets of $[0,T]\times\mathbb{R}^d$.

Let us now show that u is a viscosity subsolution of (5.7). Let $\phi \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R}^d)$, and (t_n,x_n) be a sequence of local maximum points of $\underline{u}^n-\phi$ that converges to (t,x). For n large enough we have $\underline{u}^n(t_n,x_n)>h(t_n,x_n)$, and since $\underline{u}^n$ is a viscosity solution of (5.8) we have

$$-\frac{\partial\phi}{\partial t}(t_n,x_n)-\mathcal{L}\phi(t_n,x_n)-f(t_n,x_n,\underline{u}^n(t_n,x_n),\sigma(t_n,x_n)\nabla\phi(t_n,x_n)) \\ +n(h'(t_n,x_n)-\underline{u}^n(t_n,x_n))^{-}\leqslant 0.$$

Then

$$-\frac{\partial\phi}{\partial t}(t_n,x_n)-\mathcal{L}\phi(t_n,x_n)-f(t_n,x_n,\underline{u}^n(t_n,x_n),\sigma(t_n,x_n)\nabla\phi(t_n,x_n))\leqslant 0.$$

Now due to the continuity of the functions and the uniform convergence of $\underline{u}^n$ we obtain

$$-\frac{\partial\phi}{\partial t}(t,x)-\mathcal{L}\phi(t,x)-f(t,x,u(t,x),\sigma(t,x)\nabla\phi(t,x))\leqslant 0.$$

Since $u(T,x)=g(x)$ and $h(t,x)\leqslant u(t,x)\leqslant h'(t,x)$, u is a viscosity subsolution of (5.7). In the same way, with reverse inequalities, we show that u is also a viscosity supersolution of (5.7).

5.5. Uniqueness of the viscosity solution. Before addressing the question of uniqueness of the viscosity solution of (5.7), we recall the following proposition.

PROPOSITION 5.2. *w is a viscosity solution of*

$$\begin{cases} \min\big[w(t,x)-h(t,x),-\frac{\partial w}{\partial t}(t,x)-\mathcal{L}w(t,x) \\ \quad -f(t,x,w(t,x),\sigma(t,x)\nabla w(t,x))\big]=0, \quad (t,x)\in[0,T[\times\mathbb{R}^d; \\ w(T,x)=g(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{5.10}$$

iff $\overline{w}(t,x):=e^t w(t,x)$, for $t\in[0,T]$ and $x\in\mathbb{R}^d$, is a viscosity solution of

(5.11)
$$\begin{cases} \min\big[\overline{w}(t,x)-e^t h(t,x),-\frac{\partial \overline{w}}{\partial t}(t,x)+\overline{w}(t,x)-\mathcal{L}\overline{w}(t,x) \\ \quad -e^t f(t,x,e^{-t}\overline{w}(t,x),\sigma(t,x)\nabla(e^{-t}\overline{w}(t,x)))\big]=0, \quad (t,x)\in[0,T[\times\mathbb{R}^d; \\ \overline{w}(T,x)=e^T g(x), \quad x\in\mathbb{R}^d. \end{cases}$$

We now have the following theorem.

THEOREM 5.4. *Under* (**H′.1**) *and* (**H′.2**)*, equation* (5.7) *has at most one solution.*

Proof. It is enough to show that if v and u are a viscosity supersolution and a viscosity subsolution of (5.7) respectively, then

$$u(t,x)\leqslant v(t,x), \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d.$$

First, note that $v\geqslant h$ and $u\leqslant h'$, and set $\bar{v}:=v\wedge h'$ and $\bar{u}:=u\vee h$. Then $\bar{u}$ (resp. $\bar{v}$) is a viscosity subsolution (resp. supersolution) of (5.7). It follows that $\bar{u}$ (resp. $\bar{v}$) is a viscosity subsolution (resp. supersolution) of (5.3).

Now we show that $\bar{v}$ and $\bar{u}$ satisfy $\bar{u}\leqslant\bar{v}$. Indeed, suppose for some $R>0$ there exists $(\bar{t},\overline{x})\in[0,T]\times B_R$ ($B_R:=\{x\in\mathbb{R}^d:|x|<R\}$) such that

$$\max_{t,x}(u'(t,x)-v'(t,x))=u'(\bar{t},\overline{x})-v'(\bar{t},\overline{x})=\eta>0, \tag{5.12}$$

where $v'(t,x):=e^t\bar{v}(t,x)$ and $u'(t,x):=e^t\bar{u}(t,x)$ for $t\in[0,T]$ and $x\in\mathbb{R}^d$.

Take θ, λ and $\beta\in(0,1]$ small enough. Then, for a small $\epsilon>0$, define

(5.13)
$$\Phi_\epsilon(t,x,y)=(1-\lambda)u'(t,x)-v'(t,y)-\frac{1}{2\epsilon}|x-y|^4-\theta(|x-\overline{x}|^4+|y-\overline{x}|^4)-\beta(t-\bar{t})^2.$$

Since u' and v' are bounded for R large enough, there exists a $(t_\epsilon,x_\epsilon,y_\epsilon)\in[0,T]\times B_R\times B_R$ such that

$$\Phi_\epsilon(t_\epsilon,x_\epsilon,y_\epsilon)=\max_{(t,x,y)}\Phi_\epsilon(t,x,y).$$

On the other hand, from $2\Phi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \geqslant \Phi_\epsilon(t_\epsilon, x_\epsilon, x_\epsilon) + \Phi_\epsilon(t_\epsilon, y_\epsilon, y_\epsilon)$, we have

$$\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^4 \leqslant (1-\lambda)(u'(t_\epsilon, x_\epsilon) - u'(t_\epsilon, y_\epsilon)) + (v'(t_\epsilon, x_\epsilon) - v'(t_\epsilon, y_\epsilon)),$$

and consequently $\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^4$ is bounded, and $|x_\epsilon - y_\epsilon| \to 0$ as $\epsilon \to 0$. Since u' and v' are uniformly continuous on $[0,T] \times \overline{B}_R$, we have $\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^4 \to 0$ as $\epsilon \to 0$. Since

$$(1-\lambda)u'(\bar{t}, \overline{x}) - v'(\bar{t}, \overline{x}) \leqslant \Phi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \leqslant (1-\lambda)u'(t_\epsilon, x_\epsilon) - v'(t_\epsilon, y_\epsilon),$$

it follows by letting $\lambda \to 0$ and using the continuity of u' and v' that, up to a subsequence,

$$(t_\epsilon, x_\epsilon, y_\epsilon) \to (\bar{t}, \overline{x}, \overline{x}). \tag{5.14}$$

Next let us show that $t_\epsilon < T$. Indeed, if $t_\epsilon = T$ then

$$\Phi_\epsilon(\bar{t}, \overline{x}, \overline{x}) \leqslant \Phi_\epsilon(T, x_\epsilon, y_\epsilon)$$

and

$$(1-\lambda)u'(\bar{t}, \overline{x}) - v'(\bar{t}, \overline{x}) \leqslant (1-\lambda)e^T g(x_\epsilon) - e^T g(y_\epsilon) - \beta(T - t_\epsilon)^2,$$

since $u'(T, x_\epsilon) = e^T g(x_\epsilon)$, $v'(T, y_\epsilon) = e^T g(y_\epsilon)$ and g is uniformly continuous on $\overline{B}_R$. Then as $\lambda \to 0$ we have

$$\eta \leqslant -\beta(T - \bar{t})^2 < 0,$$

which is a contradiction and so $t_\epsilon \in [0, T)$.

Now we claim that

$$u'(t_\epsilon, x_\epsilon) - e^{t_\epsilon} h(t_\epsilon, x_\epsilon) > 0. \tag{5.15}$$

If not, there exists a subsequence such that $u'(t_\epsilon, x_\epsilon) - e^{t_\epsilon} h(t_\epsilon, x_\epsilon) \leqslant 0$. Then as $\lambda \to 0$ we have $u'(\bar{t}, \overline{x}) - e^{\bar{t}} h(\bar{t}, \overline{x}) \leqslant 0$, but from the assumption $u'(\bar{t}, \overline{x}) - v'(\bar{t}, \overline{x}) > 0$ we deduce that $0 \geqslant u'(\bar{t}, \overline{x}) - e^{\bar{t}} h(\bar{t}, \overline{x}) > v'(\bar{t}, \overline{x}) - h(\bar{t}, \overline{x})$. Therefore we have $v'(\bar{t}, \overline{x}) - e^{\bar{t}} h(\bar{t}, \overline{x}) < 0$, which leads to a contradiction with (5.11). Next set

$$\psi_\epsilon(t, x, y) = \frac{1}{2\epsilon}|x - y|^4 + \theta(|x - \overline{x}|^4 + |y - \overline{x}|^4) + \beta(t - \bar{t})^2.$$

Then we have

$$\begin{cases} D_t \psi_\epsilon(t, x, y) = 2\beta(t - \bar{t}), \\ D_x \psi_\epsilon(t, x, y) = \frac{2}{\epsilon}(x - y)|x - y|^2 + 4\theta(x - \overline{x})|x - \overline{x}|^2, \\ D_y \psi_\epsilon(t, x, y) = -\frac{2}{\epsilon}(x - y)|x - y|^2 + 4\theta(y - \overline{x})|y - \overline{x}|^2, \\ B(t, x, y) = D^2_{x,y}\psi_\epsilon(t, x, y) = \frac{1}{\epsilon}\begin{pmatrix} a_1(x,y) & -a_1(x,y) \\ -a_1(x,y) & a_1(x,y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix}, \end{cases} \tag{5.16}$$

with $a_1(x,y) = 2|x-y|^2 I + 4(x-y)(x-y)^*$ and $a_2(x) = 4\theta|x-\overline{x}|^2 I + 8xx^*$. Taking into account (5.15) and then applying the result by Crandall et al. [5, Theorem 8.3]) to the function

$$(1-\lambda)u'(t,x) - v'(t,y) - \psi_\epsilon(t,x,y)$$

at the point $(t_\epsilon, x_\epsilon, y_\epsilon)$, for any $\epsilon_1 > 0$ we can find $c, c_1 \in \mathbb{R}$ and $X, Y \in S(d)$ such that

(5.17)
$$\begin{cases} \big(c, \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^2 + 4\theta(x_\epsilon - \overline{x})|x_\epsilon - \overline{x}|^2, X\big) \in J^{2,+}((1-\lambda)u'(t_\epsilon, x_\epsilon)), \\ \big(-c_1, \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^2 - 4\theta(y_\epsilon - \overline{x})|y_\epsilon - \overline{x}|^2, Y\big) \in J^{2,-}(v'(t_\epsilon, y_\epsilon)), \\ c + c_1 = D_t\psi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) = 2\beta(t_\epsilon - \overline{t}), \\ -\big(\frac{1}{\epsilon_1} + \|B(t_\epsilon, x_\epsilon, y_\epsilon)\|\big)I \leqslant \big(\begin{smallmatrix} X & 0 \\ 0 & -Y \end{smallmatrix}\big) \leqslant B(t_\epsilon, x_\epsilon, y_\epsilon) + \epsilon_1 B(t_\epsilon, x_\epsilon, y_\epsilon)^2. \end{cases}$$

Taking now into account (5.15) and the definition of viscosity solution, we get

$$\begin{aligned} &-c - \tfrac{1}{2}\operatorname{Tr}[\sigma^*(t_\epsilon, x_\epsilon)X\sigma(t_\epsilon, x_\epsilon)] \\ &\quad - \big\langle \tfrac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^2 + 4\theta(x_\epsilon - \overline{x})|x_\epsilon - \overline{x}|^2, b(t_\epsilon, x_\epsilon)\big\rangle + (1-\lambda)u'(t_\epsilon, x_\epsilon) \\ &\qquad - (1-\lambda)e^{t_\epsilon} f(t_\epsilon, x_\epsilon, e^{-t}u'(t_\epsilon, x_\epsilon), \sigma(t_\epsilon, x_\epsilon)\nabla(e^{-t_\epsilon}u'(t_\epsilon, x_\epsilon))) \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} &c_1 - \tfrac{1}{2}\operatorname{Tr}[\sigma^*(t_\epsilon, y_\epsilon)Y\sigma(t_\epsilon, y_\epsilon)] \\ &\quad - \big\langle \tfrac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^2 - 4\theta(y_\epsilon - \overline{x})|y_\epsilon - \overline{x}|^2, b(t_\epsilon, y_\epsilon)\big\rangle + v'(t_\epsilon, y_\epsilon) \\ &\qquad - e^{t_\epsilon} f(t_\epsilon, y_\epsilon, e^{-t}v'(t_\epsilon, y_\epsilon), \sigma(t_\epsilon, y_\epsilon)\nabla(e^{-t_\epsilon}v'(t_\epsilon, y_\epsilon))) \geqslant 0, \end{aligned}$$

which implies that

$$\begin{aligned} (5.18)\qquad &(1-\lambda)u'(t_\epsilon, x_\epsilon) - v'(t_\epsilon, y_\epsilon) - c - c_1 \\ &\leqslant \tfrac{1}{2}\operatorname{Tr}[\sigma^*(t_\epsilon, x_\epsilon)X\sigma(t_\epsilon, x_\epsilon) - \sigma^*(t_\epsilon, y_\epsilon)Y\sigma(t_\epsilon, y_\epsilon)] \\ &\quad + \langle \tfrac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^2, b(t_\epsilon, x_\epsilon) - b(t_\epsilon, y_\epsilon)\rangle \\ &\quad + \langle 4\theta(x_\epsilon - \overline{x})|x_\epsilon - \overline{x}|^2, b(t_\epsilon, x_\epsilon)\rangle + \langle 4\theta(y_\epsilon - \overline{x})|y_\epsilon - \overline{x}|^2, b(t_\epsilon, y_\epsilon)\rangle \\ &\quad + (1-\lambda)e^{t_\epsilon} f(t_\epsilon, x_\epsilon, e^{-t}u'(t_\epsilon, x_\epsilon), \sigma(t_\epsilon, x_\epsilon)\nabla(e^{-t_\epsilon}u'(t_\epsilon, x_\epsilon)) \\ &\quad - e^{t_\epsilon} f(t_\epsilon, y_\epsilon, e^{-t}v'(t_\epsilon, y_\epsilon), \sigma(t_\epsilon, y_\epsilon)\nabla(e^{-t_\epsilon}v'(t_\epsilon, y_\epsilon)). \end{aligned}$$

But from (5.16) there exist constants C and C_1 such that

$$\|a_1(x_\epsilon, y_\epsilon)\| \leqslant C|x_\epsilon - y_\epsilon|^2 \quad \text{and} \quad \|a_2(x_\epsilon)\| \vee \|a_2(y_\epsilon)\| \leqslant C_1\theta.$$

As

$$B = B(t_\epsilon, x_\epsilon, y_\epsilon) = \frac{1}{\epsilon}\begin{pmatrix} a_1(x_\epsilon, y_\epsilon) & -a_1(x_\epsilon, y_\epsilon) \\ -a_1(x_\epsilon, y_\epsilon) & a_1(x_\epsilon, y_\epsilon) \end{pmatrix} + \begin{pmatrix} a_2(x_\epsilon) & 0 \\ 0 & a_2(y_\epsilon) \end{pmatrix}$$

we have

$$B \leqslant \frac{C}{\epsilon}|x_\epsilon - y_\epsilon|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1\theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

It follows that

$$B + \epsilon_1 B^2 \leqslant C\left(\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 + \frac{\epsilon_1}{\epsilon^2}|x_\epsilon - y_\epsilon|^4\right)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1\theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

where C and C_1 may change from line to line. Choosing now $\epsilon_1 = \epsilon$ yields the relation

$$B + \epsilon_1 B^2 \leqslant \frac{C}{\epsilon}(|x_\epsilon - y_\epsilon|^2 + |x_\epsilon - y_\epsilon|^4)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1\theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{5.19}$$

Now, from the Lipschitz continuity of σ, (5.17) and (5.19) we get

$$\begin{aligned} &\frac{1}{2}\operatorname{Tr}[\sigma^*(t_\epsilon, x_\epsilon) X \sigma(t_\epsilon, x_\epsilon) - \sigma^*(t_\epsilon, y_\epsilon) Y \sigma(t_\epsilon, y_\epsilon)] \\ &\qquad\qquad\qquad \leqslant \frac{C}{\epsilon}(|x_\epsilon - y_\epsilon|^4 + |x_\epsilon - y_\epsilon|^6) + C_1\theta. \end{aligned}$$

Next by plugging into (5.18) we obtain

$$\begin{aligned} (1-\lambda)&u'(t_\epsilon, x_\epsilon) - v'(t_\epsilon, y_\epsilon) - 2\beta(t_\epsilon - \overline{t}) \\ &\leqslant (1-\lambda)e^{t_\epsilon} f(t_\epsilon, x_\epsilon, e^{-t}u'(t_\epsilon, x_\epsilon), \sigma(t_\epsilon, x_\epsilon)\nabla(e^{-t_\epsilon}u'(t_\epsilon, x_\epsilon)) \\ &\quad - e^{t_\epsilon} f(t_\epsilon, y_\epsilon, e^{-t}v'(t_\epsilon, y_\epsilon), \sigma(t_\epsilon, y_\epsilon)\nabla(e^{-t_\epsilon}v'(t_\epsilon, y_\epsilon)) \\ &\quad + \frac{C}{\epsilon}(|x_\epsilon - y_\epsilon|^4 + |x_\epsilon - y_\epsilon|^6) + C_1\theta. \end{aligned}$$

By letting $\epsilon \to 0$, $\lambda \to 0$, $\theta \to 0$ and taking into account the continuity of f, we obtain $\eta < 0$, which is a contradiction.

Now we have $u \leqslant u \vee h \leqslant v \wedge h' \leqslant v$, which means that if u and $\breve{u}$ are two solutions of (5.7) then $u \leqslant \breve{u}$ and $\breve{u} \leqslant u$. Hence, obviously, $u = \breve{u}$.

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