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A REMARK ON THE EXACT LAWS OF LARGE NUMBERS FOR RATIOS OF INDEPENDENT RANDOM VARIABLES

BY

PRZEMYSŁAW MATUŁA (LUBLIN)

Abstract. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of i.i.d. random variables which are independent of each other and all have the distribution of a positive random variable ξ with density f_ξ. We study weighted strong laws of large numbers for the ratios of the form $\frac{1}{b_n}\sum_{k=1}^n a_k \frac{X_k}{Y_k}$ in the cases when $\mathbb{E}\xi = \infty$ or $\lim_{x\to 0^+} f_\xi(x) = 0$ or f_ξ is unbounded. This research complements some results known so far.

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1. INTRODUCTION

Let X and Y be independent random variables with the distribution of a positive random variable ξ with density f_ξ. Observe that the survival function of the random variable $R = X/Y$ has the following form:

$$\overline{F}_R(r) = 1 - F_R(x) = \mathbb{P}(R \geqslant r) = \iint\limits_{x/y\geqslant r} f_\xi(x) f_\xi(y)\,dx\,dy$$
$$= \int_0^\infty \Big(\int_0^{x/r} f_\xi(y)\,dy\Big) f_\xi(x)\,dx = \frac{1}{r}\int_0^\infty \Big(\int_0^x f_\xi\Big(\frac{s}{r}\Big)\,ds\Big) f_\xi(x)\,dx.$$

If we assume that $\mathbb{E}\xi < \infty$, f_ξ is bounded and $\lim_{x\to 0^+} f_\xi(x) = f_\xi(0)$, then by the Lebesgue dominated convergence theorem, we get

$$r\overline{F}_R(r) \to \int_0^\infty \Big(\int_0^x f_\xi(0)\,ds\Big) f_\xi(x)\,dx = f_\xi(0)\mathbb{E}\xi \quad \text{as } r\to\infty. \tag{1.1}$$

Therefore $\mathbb{E}R = \infty$ and we cannot obtain the strong law of large numbers for a sequence of i.i.d. random variables distributed like R. The only way to obtain nontrivial limits (finite and not 0) is to study weighted convergence. Results of this kind are called *exact laws of large numbers* (strong or weak, depending on the mode of convergence considered – almost sure or in probability).

A direct application of (1.1) and [6, Theorem 1] yields the following exact strong law of large numbers, also appearing in [4].

THEOREM 1.1. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of i.i.d. random variables which are independent of each other and all have the distribution of a positive, integrable random variable ξ with density f_ξ. Assume that f_ξ is bounded and $\lim_{x\to 0^+} f_\xi(x) = f_\xi(0)$. Then, for all $\alpha > -2$,*

$$\lim_{n\to\infty} \frac{1}{\lg^{\alpha+2} n} \sum_{k=1}^{n} \frac{\lg^{\alpha} k}{k} \frac{X_k}{Y_k} = \frac{f_\xi(0)\mathbb{E}\xi}{\alpha+2} \quad \textit{almost surely.}$$

The first result of this type was proved in [2] for uniformly distributed r.v.'s and generalized in [5] to r.v.'s satisfying a technical condition called "sub-exponentiality". In the above theorem we got rid of this condition. The weak exact law under the above conditions was proved in [3]. Also ratios of order statistics attracted interest of many researchers; we refer the reader to [5, 6] for further references. General results concerning exact strong laws may be found in [1].

The goal of this paper is to present some convergence results in the case when the assumptions of Theorem 1.1 are violated. We shall consider the case $\mathbb{E}\xi = \infty$, unbounded f_ξ and $\lim_{x\to 0^+} f_\xi(x) = 0$.

We use the standard notation $\lg(x) = \log(\max(\mathrm{e}, x))$ where $\log$ is the logarithm to the natural base.

2. THE CASE OF INFINITE MEAN

In this section we focus on the case $\mathbb{E}\xi = \infty$ but restricting our attention to Pareto-type random variables ξ, i.e. satisfying

$$\lim_{x\to\infty} x^2 f_\xi(x) = M \quad \text{for some } M > 0. \tag{2.1}$$

We shall examine the tail behavior of R under condition (2.1) because our main result of this section will be based on the following lemma.

LEMMA 2.1. *Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of independent r.v.'s with the distribution of a r.v. R with survival function $\overline{F}_R$ and density function f_R. Suppose that*

$$\lim_{x\to\infty} \left(x\overline{F}_R(x) - x^2 f_R(x)\right) = A \quad \textit{for some } A > 0. \tag{2.2}$$

Then, for all $\alpha > -2$,

$$\lim_{n\to\infty} \frac{1}{\lg^{\alpha+2} n} \sum_{k=1}^{n} \frac{\lg^{\alpha-1} k}{k} R_k = \frac{A}{2(\alpha+2)} \quad \textit{almost surely.} \tag{2.3}$$

Proof. By using de L'Hospital's rule we get

$$\lim_{x\to\infty} \frac{x\mathbb{P}(R > x)}{\lg x} = \lim_{x\to\infty} (x\overline{F}_R(x) - x^2 f_R(x)) = A$$

and the conclusion follows from [1, Example 2]. ∎

The main result of this section is the following.

THEOREM 2.1. *Let* $(X_n)_{n\in\mathbb{N}}$ *and* $(Y_n)_{n\in\mathbb{N}}$ *be two sequences of i.i.d. random variables which are independent of each other and all have the distribution of a positive random variable* ξ *with density* f_ξ. *Assume that* f_ξ *satisfies* (2.1), *is nonincreasing and* $\lim_{x\to 0^+} f_\xi(x) = f_\xi(0)$. *Then, for all* $\alpha > -2$,

$$\lim_{n\to\infty} \frac{1}{\lg^{\alpha+2} n} \sum_{k=1}^{n} \frac{\lg^{\alpha-1} k}{k} \frac{X_k}{Y_k} = \frac{M f_\xi(0)}{2(\alpha+2)} \quad \textit{almost surely.} \tag{2.4}$$

Proof. Observing that

$$f_R(r) = \int_0^\infty x f_\xi(x) f_\xi(rx)\, dx, \qquad \overline{F}_R(r) = \int_0^\infty r F_\xi(x) f_\xi(rx)\, dx,$$

we get

$$r\overline{F}_R(r) - r^2 f_R(r) = \int_0^\infty \frac{F_\xi(x) - x f_\xi(x)}{x^2} (r^2x^2) f_\xi(rx)\, dx. \tag{2.5}$$

From (2.1) and the monotonicity of f_ξ it follows that $x^2 f_\xi(x)$ is uniformly bounded on $[0, +\infty)$. From the monotonicity of f_ξ it also follows that

$$F_\xi(x) - x f_\xi(x) \geqslant 0 \quad \text{for } x \in [0, +\infty).$$

Moreover,

$$\frac{F_\xi(x) - x f_\xi(x)}{x^2} = \frac{d}{dx}\left[-\frac{F_\xi(x)}{x}\right]$$

is integrable on $[0, +\infty)$ and by the right-continuity of f_ξ at 0 we get

$$\int_0^\infty \frac{F_\xi(x) - x f_\xi(x)}{x^2}\, dx = \left[-\frac{F_\xi(x)}{x}\right]_0^\infty = f_\xi(0).$$

In consequence, by the Lebesgue dominated convergence theorem applied to (2.5), we get

$$\lim_{r\to\infty} (r\overline{F}_R(r) - r^2 f_R(r)) = M f_\xi(0)$$

and so the proof is complete according to Lemma 2.1. ∎

Now, let us present two examples in which the above theorem is applied. In the first example we shall consider the folded Cauchy distribution, and in the second one, the Pareto distribution. In both cases neither Theorem 1.1 nor the results of [5] can be applied.

EXAMPLE 2.1. Assume that ξ has the folded Cauchy distribution with density $f_\xi(x) = \frac{2}{\pi(1+x^2)}$ and distribution function $F_\xi(x) = \frac{2}{\pi}\arctan(x)$ for $x \in [0, +\infty)$. Then (2.4) holds with $Mf_\xi(0) = (2/\pi)^2$.

EXAMPLE 2.2. Assume that ξ has the Pareto distribution with density $f_\xi(x) = \frac{a}{(1+ax)^2}$ and distribution function $F_\xi(x) = 1 - \frac{1}{1+ax}$ for $x \in [0, +\infty)$ and some $a > 0$. Then (2.4) holds with $Mf_\xi(0) = a \cdot \frac{1}{a} = 1$.

3. THE CASE OF DENSITY VANISHING AT ZERO

In this section we discuss the case $\lim_{x\to 0^+} f_\xi(x) = 0$. We will present two results. In the first one we will show that if $f_\xi(x)$ approaches 0 with a polynomial rate then the standard strong law of large numbers holds. In the second result we show an example of a rapidly growing density $f_\xi(x)$ in a neighborhood of 0 and in this case weighted convergence should be considered.

THEOREM 3.1. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of i.i.d. random variables which are independent of each other and all have the distribution of a positive random variable ξ with density f_ξ. Assume that $\mathbb{E}\xi < \infty$ and*

$$f_\xi(t)\log^2 t \text{ is bounded for } t \in (0, t_0], \tag{3.1}$$

for some $t_0 > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{X_k}{Y_k} = \mathbb{E}\xi \cdot \mathbb{E}\frac{1}{\xi} \quad \text{almost surely.} \tag{3.2}$$

Proof. It is sufficient to prove that $\mathbb{E}\frac{1}{\xi} < \infty$. Observe that $1/\xi$ has density $f_{1/\xi}(x) = \frac{1}{x^2} f_\xi(x)$. Thus

$$(x^2\log^2 x) f_{1/\xi}(x) = (\log^2 x) f_\xi\left(\frac{1}{x}\right),$$

which is bounded for $x \geqslant 1/t_0$ and the proof is complete. ∎

REMARK 3.1. It is clear that if $\lim_{t\to 0^+} f_\xi(t)/t^\alpha = C > 0$ for some $\alpha > 0$, then $\lim_{t\to 0^+} f_\xi(t)\log^2 t = 0$ and the assumption (3.1) is satisfied. Moreover $t^\alpha \leqslant C_1/\log^2 t$ in a neighborhood of 0 for some $C_1 > 0$.

EXAMPLE 3.1. Assume that ξ has density $f_\xi(x) = (p+1)x^p$ for $x \in [0,1]$ and some $p > 0$. Then

$$\mathbb{E}\xi = \frac{p+1}{p+2} \quad \text{and} \quad \mathbb{E}\frac{1}{\xi} = \frac{p+1}{p}.$$

If $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are as in Theorem 3.1, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{X_k}{Y_k} = \frac{(p+1)^2}{p(p+2)} \quad \text{almost surely.}$$

In the next example we obtain an exact law of large numbers when ξ has density defined by the integral logarithm function.

EXAMPLE 3.2. Consider the density $f_\xi(x) = -\frac{1}{\log x}$ for $x \in (0, x_0]$. The number $x_0 \approx 0.7674$ may be calculated numerically. Then

$$\overline{F}_R(r) = \frac{1}{r} \int_0^{x_0} \left(\int_0^x f_\xi\left(\frac{s}{r}\right) ds \right) f_\xi(x)\, dx$$

and

$$\begin{aligned} r \log r \overline{F}_R(r) &= \int_0^{x_0} \left(\int_0^x f_\xi\left(\frac{s}{r}\right) ds \right) f_\xi(x)\, dx \\ &= \int_0^{x_0} \left(\int_0^x \frac{\log r}{\log r - \log s}\, ds \right) f_\xi(x)\, dx. \end{aligned}$$

Since $\frac{\log r}{\log r - \log s} \to 1$ as $r \to \infty$ uniformly on the interval $s \in (0, x_0]$, we get

$$r \log r \overline{F}_R(r) \to \int_0^{x_0} x f_\xi(x)\, dx = \mathbb{E}\xi.$$

By numerical calculations $\mathbb{E}\xi \approx 0.525\,51$. According to [1, Example 3], for sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ described in Theorem 3.1 and distributed like ξ, we get

$$\lim_{n\to\infty} \frac{1}{\lg^{\alpha+2} n} \sum_{k=1}^{n} \frac{\lg^{\alpha+1} k}{k \lg \lg k} \frac{X_k}{Y_k} = \frac{\mathbb{E}\xi}{\alpha+2} \quad \text{almost surely}$$

for all $\alpha > -2$. Observe that in this case condition (3.1) is not satisfied.

4. THE CASE OF UNBOUNDED DENSITY

In this last section we discuss the case of unbounded density $f_\xi(x)$ in a neighborhood of 0. More precisely, we shall consider densities such that $\lim_{x\to 0^+} f_\xi(x) = +\infty$. To find explicit weights in our main result we shall impose some regularity conditions on $f_\xi(x)$.

THEOREM 4.1. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of i.i.d. random variables which are independent of each other and all have the distribution of a positive random variable ξ with density f_ξ. Assume that $\mathbb{E}\xi < \infty$ and for some $p \in (0, 1)$,*

$$x^p f_\xi(x) \text{ is bounded for } x \in (0, \infty) \tag{4.1}$$

and

$$\lim_{x\to 0^+} x^p f_\xi(x) = A \quad \text{for some } A > 0. \tag{4.2}$$

Then

$$\text{(4.3)}\qquad \lim_{n\to\infty}\frac{1}{\lg^{\alpha+2} n}\sum_{k=1}^{n}\frac{\lg^{\alpha} k}{k}\left(\frac{X_k}{Y_k}\right)^{1-p} = \frac{A\mathbb{E}\xi^{1-p}}{(1-p)(\alpha+2)} \quad \textit{almost surely}$$

for all $\alpha > -2$.

Proof. We shall apply Theorem 1.1 to the r.v.'s $(X_n^{1-p})_{n\in\mathbb{N}}$ and $(Y_n^{1-p})_{n\in\mathbb{N}}$, which are distributed like ξ^{1-p}. Observe that $\mathbb{E}\xi^{1-p} < \infty$ since $\mathbb{E}\xi < \infty$ and $0 < 1-p < 1$. Moreover the density of ξ^{1-p} is

$$f_{\xi^{1-p}}(x) = f_\xi(x^{\frac{1}{1-p}})\frac{1}{1-p}x^{\frac{1}{1-p}-1} = f_\xi(x^{\frac{1}{1-p}})\frac{1}{1-p}(x^{\frac{1}{1-p}})^p.$$

Thus by (4.1), $f_{\xi^{1-p}}(x)$ is bounded and by (4.2) we get

$$\lim_{x\to 0^+} f_{\xi^{1-p}}(x) = \frac{A}{1-p}. \quad \blacksquare$$

The above theorem is applied in our last example.

EXAMPLE 4.1. Consider the density $f_\xi(x) = (1-p)x^{-p}$ for $x \in (0,1)$ and $p \in (0,1)$ corresponding to the distribution function $F_\xi(x) = x^{1-p}$ for $x \in [0,1]$. Then $x^p f_\xi(x) = 1-p$ and $\mathbb{E}\xi^{1-p} = 1/2$. Thus, for sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ as in Theorem 4.1 we have

$$\text{(4.4)}\qquad \lim_{n\to\infty}\frac{1}{\lg^{\alpha+2} n}\sum_{k=1}^{n}\frac{\lg^{\alpha} k}{k}\left(\frac{X_k}{Y_k}\right)^{1-p} = \frac{1}{2(\alpha+2)} \quad \text{almost surely}$$

for all $\alpha > -2$.

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Przemysław Matuła
Institute of Mathematics
Marie Curie-Skłodowska University
Pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: matula@hektor.umcs.lublin.pl

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