Dependence in Lag for Markov Processes on Partially
Ordered State Space

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Abstract
Let \( Y = (Y(t), t \geq 0) \) be a stationary homogeneous Markov process
with partially ordered state space \( E \). In this paper we show that, under
stochastic monotonicity assumptions, a dependence in this process decrease
in time. Our results extend the one existing for linearly ordered case and
allow us to skip an assumption about uniform semigroups. We apply our
results to Jackson networks.

1 Introduction
Let \( Y = (Y(t), t \geq 0) \) be a stationary homogeneous Markov process with partially
ordered state space \( E \). In this paper we show that, under stochastic monotonicity
assumptions, a dependence in this process decrease in time in the sense of super-
modular ordering. This is counterpart to the results in totally ordered case, see

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Fang et al. (1994), Bäuelre and Rolski (1997), Hu and Pan (2000) and Miyoshi and Rolski (2003). Our results allow us also to skip an assumption about uniform semigroups.

The technique for comparing dependence in Markov processes in partially ordered state space was introduced in Daduna and Szekli (2004) and then developed in Daduna et al. (2004). Note that our results mean that starting from equilibrium dependence in time for Markov processes decrease and marginal distributions are the same at any time instant $t$.

2 Supermodular orderings on partially ordered space

Assume that $E$ is a partially ordered Polish space equipped with Borel $\sigma$-algebra $\mathcal{E}$ and a partial ordering $\prec$.

A function $f : E \rightarrow \mathbf{R}$ is increasing if for $x, y \in E, x \prec y$ implies $f(x) \leq f(y)$. We denote by $\mathcal{I}^+(E, \prec)$ the set of all increasing real-valued functions on $E$.

Let $(E_1, \prec_1), (E_2, \prec_2)$ be partially ordered spaces. We say that a function $f : E_1 \times E_2 \rightarrow \mathbf{R}$ has isotone differences if for any $x_1 \prec_1 x_2, y_1 \prec_2 y_2, x_1, x_2 \in E_1, y_1, y_2 \in E_2$, we have

$$f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1).$$

More generally, let $(E_1, \prec_1), \ldots, (E_n, \prec_n)$ be partially ordered sets and $f : \times_{i=1}^n E_i \rightarrow \mathbf{R}$. Suppose that $(x_1, \ldots, x_n) \in \times_{i=1}^n E_i$. If $f$ has isotone differences w.r.t. all pairs $(x_i, x_j), i \neq j$, when another coordinates are fixed, we say that $f$ is supermodular, we write $f \in \mathcal{L}_{\text{sm}}(\times E_i, \prec_*)$, where $\prec_*$ denotes coordinatewise ordering, i.e. for $x = (x_1, \ldots, x_n) \in \times E_i, y = (y_1, \ldots, y_n) \in \times E_i$, we have $x \prec_* y$ if $x_i \prec y_i$ for all $i = 1, \ldots, n$. In particular, if $(E_1, \prec_1) = (E, \prec), i = 1, \ldots, n$, then we denote $\prec_* = \prec$.

**Definition 2.1** Let $Y = (Y_1, \ldots, Y_n), \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_n)$ be random vectors with values in $(E^n, \mathcal{E}^n, \sim^n)$.

(i) $Y$ is said to be smaller than $\tilde{Y}$ in the supermodular order ($Y \sim_{\text{sm}} \tilde{Y}$) if

$$\mathbf{E} [f(Y_1, \ldots, Y_n)] \leq \mathbf{E} [f(\tilde{Y}_1, \ldots, \tilde{Y}_n)]$$

for all $f \in \mathcal{L}_{\text{sm}}(E^n, \sim^n)$.

(ii) Let $Y = (Y(t), t \in T), \tilde{Y} = (\tilde{Y}(t), t \in T), T \subseteq \mathbf{R}$ be stochastic processes. Then we write $Y \prec_\alpha \tilde{Y}$ if for $t_1 < \cdots < t_n \in T$ we have $(Y(t_1), \ldots, Y(t_n)) \sim_{\text{sm}} (\tilde{Y}(t_1), \ldots, \tilde{Y}(t_n))$. 

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3 Discrete time case

Let $Y = (Y_n, n \in \mathbb{N})$ be a stationary homogeneous discrete time Markov process with a state space $E$, a stationary distribution $\pi$ and a one step transition kernel $K^Y : (F \times \mathcal{F}) \rightarrow [0, 1]$. Let $K^*_Y : (F \times \mathcal{F}) \rightarrow [0, 1]$ be a transition kernel for the corresponding time reversed process. Recall that a transition kernel $K$ is monotone if a map $x \rightarrow \int_y K(x, y)f(y)$ is increasing provided that $f : E \rightarrow \mathbb{R}$ is increasing.

Lemma 3.1 Assume that a kernel $K$ is monotone and $f$ is supermodular on $(E^{n+1}, \prec^{n+1})$. Then a function $h : E^n \rightarrow \mathbb{R}$ defined by

$$h(x_0, \ldots, x_{n-1}) = \int_{E^n} K(x_{n-1}, dx_n)f(x_0, \ldots, x_n)$$

is supermodular on $(E^n, \prec^n)$.

Lemma 3.2 Assume that a kernel $K$ is monotone and $f : E^n \rightarrow \mathbb{R}$ is coordinatewise increasing. Then

$$h_n(x_0) = \int_{E^n} K(x_0, dx_1)K(x_1, dx_2)\cdots K(x_{n-1}, dx_n)f(x_1, \ldots, x_n)$$

is increasing on $(E, \prec)$.

Proof. See Daduna et al. (2004).

The following result is a counterpart to the case $E = \mathbb{R}$ (see Müller and Stoyan (2002)).

Lemma 3.3 (Lorentz inequality) Let $Y_0, Y_1$ be random elements in $E$ with the same distributions. Then

$$\mathbb{E}[f(Y_0, Y_1)] \leq \mathbb{E}[f(Y_0, Y_0)]$$

for each $f \in \mathcal{L}_{sm}(E^2, \prec^2)$ for which the expectations exist.

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Proposition 3.4 Assume that a kernel $K^Y$ is monotone and $f \in \mathcal{L}_{sm}(E^2, \prec^2)$. Then

$$\varphi(n) := \mathbb{E}[f(Y_0, Y_n)]$$

is decreasing on $\mathbb{N}$.  

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\[ \text{Proof !!!} \]
Proof. From Lemma 3.3 we have $\varphi(0) \geq \varphi(1)$. Denote by $K_n^X (K_{n+1}^X)$ a $n$-step transition kernel for $Y (Y_*)$. By conditioning on $Y_1$ we obtain the following:

$$
\mathbb{E}[f(Y_0, Y_n)] = \int_y \int_x \pi(dy) K_n^Y(y, dz) \left( \int_x f(x, z) K_n^X(y, dx) \right)
$$

$$
= \int_y \int_x \pi(dy) K_n^Y(y, dz) \left( \int_x f(x, z) \frac{\pi(dx)}{\pi(dy)} K_n^X(x, dy) \right)
$$

$$
= \int_y \int_x \pi(dx) K_n^X(x, dy) \left( \int_z f(x, z) K_{n-1}^Y(y, dz) \right)
$$

$$
= \int_y \int_x \pi(dx) K_n^X(x, dy) \mathbb{E}[f(x, Y_n) | Y_1 = y]
$$

$$
= \int_y \int_x \pi(dx) K_n^X(x, dy) \mathbb{E}[f(x, Y_{n+1}) | Y_2 = y]
$$

$$
= \mathbb{E}[h(Y_1, Y_2)]
$$

$$
= \mathbb{E}[h(Y_0, Y_1)],
$$

where

$$
h(y_1, y_2) = \mathbb{E}[f(y_1, Y_{n+1}) | Y_2 = y_2]
$$

is a regular version of conditional probability. From Lemmas 3.1, 3.2 this function is supermodular, hence by induction

$$
\mathbb{E}[f(Y_0, Y_n)] \geq \mathbb{E}[h(Y_0, Y_2)]
$$

$$
= \int_y \int_x \pi(dx) K_n^X(x, dy) \mathbb{E}[f(x, Y_{n+1}) | Y_2 = y]
$$

$$
= \int_y \int_x \pi(dy) \int_x \frac{\pi(dx)}{\pi(dy)} K_n^X(x, dy) h(x, y)
$$

$$
= \int_y \int_x K_n^X(x, dy) h(x, y)
$$

$$
= \mathbb{E}[f(Y_0, Y_{n+1})].
$$

Remark 3.5 In particularly, the above lemma implies that $(Y_0, Y_{n+1}) \prec_{cc} (Y_0, Y_n)$, i.e.

$$
\mathbb{E}[f(Y_0)g(Y_{n+1})] \leq \mathbb{E}[f(Y_0)g(Y_n)]
$$

for each $f, g \in \mathcal{I}^+(E, \prec)$ and this in turn implies that $\text{Cov}[Y_0, Y_{n+1}] \leq \text{Cov}[Y_0, Y_n]$. In this context, in linearly ordered case, it was proven in Fang et al. (1994).

Remark 3.6 Note that monotonicity assumption in Lemma 3.4 is sufficient but not necessary even in totally ordered case (See Kulik (2004)).
Theorem 3.7 Assume that both $K^Y$ and $K^Y_k$ are monotone and $f \in \mathcal{L}_{sm}(E^{k+1}, \prec^{k+1})$. Then
\[
\varphi(n_1, \ldots, n_k) := \mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k})]
\]
is decreasing, i.e. $Y$ is $\prec_{sm}$-decreasing.

Proof. First we show that
\[
\mathbb{E}[f(Y_0, Y_{n_1+1}, \ldots, Y_{n_k-1}, Y_{n_k+1})] \leq \mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})].
\]
We have by conditioning on $Y_{n_1}$:
\[
\mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})] =
\int_{x_0, \ldots, x_k} \pi(dx_1) K^Y_{\pi, n_1}(x_1, dx_0) K^Y_{n_2-n_1}(x_1, dx_2) \cdots K^Y_{n_k-n_{k-1}}(x_{k-1}, dx_k) f(x_0, x_1, \ldots, x_k)
\]
\[
= \int_{x_0, \pi \neq 0} \pi(dx_1) K^Y_{n_2-n_1}(x_1, dx_2) \cdots K^Y_{n_k-n_{k-1}}(x_{k-1}, dx_k) \int_{x_0} f(x_0, x_1, \ldots, x_k) K^Y_{\pi, n_1}(x_1, dx_0)
\]
\[
= \int_{x_0, x_1} \pi(dx_0) K^Y_{n_1}(x_0, dx_1) \mathbb{E}[f(x_0, x_1, Y_{n_2+1}, \ldots, Y_{n_k+1}) | Y_{n_1} = x_1]
\]
\[
= \int_{x_0, x_1} \pi(dx_0) K^Y_{n_1}(x_0, dx_1) \mathbb{E}[f(x_0, x_1, Y_{n_2+1}, \ldots, Y_{n_k+1}) | Y_{n_1+1} = x_1]
\]
\[
= \mathbb{E}[h(Y_1, Y_{n_1+1})]
\]
\[
= \mathbb{E}[h(Y_0, Y_{n_1})]
\]
with
\[
h(y_1, y_2) = \mathbb{E}[f(y_1, y_2, Y_{n_2+1}, \ldots, Y_{n_k+1}) | Y_{n_1+1} = y_2].
\]
This functions is supermodular by applying iteratively Lemma 3.1, so from Proposition 3.4 we have
\[
\mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})] =
\geq \mathbb{E}[h(Y_0, Y_{n_1+1})]
\]
\[
= \int_{x_0, x_1} \pi(dx_0) K^Y_{n_1+1}(x_0, x_1) \mathbb{E}[f(x_0, x_1, Y_{n_2+1}, \ldots, Y_{n_k+1}) | Y_{n_2} = x_1]
\]
\[
= \int_{x_1} \pi(dx_1) \int_{x_0} \frac{\pi(dx_0)}{\pi(dx_1)} K^Y_{n_1+1}(x_0, dx_1) h(x_0, x_1)
\]
\[
= \int_{x_1} \pi(dx_1) \int_{x_0} K^Y_{\pi, n_1+1}(x_0, dx_1) h(x_0, x_1)
\]
\[
= \mathbb{E}[f(Y_{n_1+1}, \ldots, Y_{n_k-1+1}, Y_{n_k+1})].
\]
Now we show that
\[
\mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k+1})] \leq \mathbb{E}[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})].
\]
Let $E_*$ be an expected value taken w.r.t. a distribution of $Y_*$. We have by conditioning on $Y_{n_k-1}$:

$$
E[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})] =
$$

$$
= \int_{x, i \neq k-1} K_{*,n_k-1-n_k-2}(x_{k-1}, dx_{k-2}) \cdots K_{*,n_k}(x_1, x_0) \times
$$

$$
\int_{x_{k-1}} \pi(x_{k-1}) K_{n_k-n_k-1}(x_{k-1}, dx_{k}) f(x_0, x_1, \ldots, x_k)
$$

$$
= \int_{x, i \neq k-1} K_{*,n_k-1-n_k-2}(x_{k-1}, dx_{k-2}) \cdots K_{*,n_k}(x_1, x_0) \times
$$

$$
\int_{x_{k-1}} \pi(x_k) K_{*,n_k-n_k-1}(x_k, dx_{k-1}) f(x_0, x_1, \ldots, x_k)
$$

$$
= \int_{x_{k-1}, x_k} \pi(x_k) K_{*,n_k-n_k-1}(x_k, dx_{k-1}) E_*[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-2}, dx_{k-1}, x_k) \mid Y_{n_k-1} = x_{k-1}]
$$

$$
= E_*[h(Y_{n_k-1}, Y_{n_k})],
$$

where

$$
h(y_1, y_2) = E_*[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, y_1, y_2) \mid Y_{n_k-1} = y_1].
$$

This function is supermodular by using Lemma 3.1 applied to time-reversed Markov chain $Y_*$. Therefore, from Proposition 3.4 we have

$$
E[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k})] =
$$

$$
\geq E_*[h(Y_{n_k-1}, Y_{n_k+1})]
$$

$$
= E[f(Y_0, Y_{n_1}, \ldots, Y_{n_k-1}, Y_{n_k+1})].
$$

**Remark 3.8** In the linearly ordered case this theorem was proven in Hu and Pan (2000).

### 4 Continuous time case

Let $Y = (Y(t), t \geq 0)$ be a stationary homogeneous continuous time Feller process with a state space $(E, \mathcal{E}, \prec)$ and a transition semigroup $Q^Y := (Q^Y_t : E \times \mathcal{E} \to [0, 1], t \geq 0)$. Denote by $Q^Y_*$ a transition semigroup for a time reversed process. The corresponding Markov semigroup for $Y$ is $T^Y_t = (T^Y_{t, E} : \mathcal{B}_b(E) \to \mathcal{B}_b(E), t \geq 0)$, and similarly $T^*_Y = (T^*_Y_{t, E} : \mathcal{B}_b(E) \to \mathcal{B}_b(E), t \geq 0)$ for $Y_*$, with $\mathcal{B}_b(E)$ being the space of real valued bounded continuous functions on $E$. Recall that $T^Y_*$ is an uniform semigroup if the map $t \to T^*_Y$ is continuous on $\mathbb{R}_+$. Recall also that a Feller process is monotone if for $x, y \in E, x \prec y$ we have $T_t f(x) \leq T_t f(y)$ for all $\prec$-increasing functions in $\mathcal{B}_b(E)$. Equivalently, we say that a family $Q^Y_*$ is monotone.

By applying the standard uniformization technique and using Proposition 3.4 and Theorem 3.4 we obtain the following results.
Proposition 4.1 Assume that $Q^Y$ is a family of monotone kernels and the semigroup $T^Y$ is uniform. Then for all $s \leq t$,

$$(Y_0, Y_t) \prec_{sm} (Y_0, Y_s).$$

Theorem 4.2 Assume that $Q^Y, Q^Y_*$ are families of monotone kernels and the semigroup $T^Y$ is uniform. Then for all $(s_1, \ldots, s_k) \leq (t_1, \ldots, t_k)$,

$$(Y_0, Y_{t_1}, \ldots, Y_{t_k}) \prec_{sm} (Y_0, Y_{s_1}, \ldots, Y_{s_k}),$$

i.e. $Y$ is $\prec_{sm}$-decreasing.

If we want to skip uniform property of the semigroup then we cannot use results for the discrete time case.

Proposition 4.3 Assume that $Y$ is monotone and reversible. Then

$$\mathbb{E}[f(Y_0, Y_t)] \leq \mathbb{E}[f(Y_0, Y_s)]$$

for all $s \leq t$ and all supermodular functions $f$ such that $f(x, \cdot) \in B_b(E)$ for all $x \in E$.

Proof. We proceed by induction, assuming that

$$\mathbb{E}[f(Y_0, Y_u)] \leq \mathbb{E}[f(Y_0, Y_s)]$$

for all $u \leq v \leq s$ and all $f$ which fulfill the above condition. Let $u \leq s$, then by denoting $f_x(y) = f(x, y)$

$$\mathbb{E}[f(Y_0, Y_s)] = \int \int \pi(dy) f(x, z)Q_{s,u}(y, dx)Q_{s-u}(y, dz)$$
$$= \int \int \pi(dy)Q_{s,u}(y, dx)(T_{s-u}f_x)(y)$$
$$= \int \int \pi(dy)Q_{u,s}(y, dx)h(x, y)$$
$$= \mathbb{E}_s[h(Y_0, Y_u)]$$
$$= \mathbb{E}[h(Y_0, Y_u)]$$

with $h(x, y) = (T_{s-u}f_x)(y)$. Now, for each fixed $x, y$ the function $h(x, \cdot)$ is in $B_b(E)$ from the definition of the Feller process. From Lemma 3.1 $h$ is supermodular, hence from induction for $u \leq v$,

$$\mathbb{E}[f(Y_0, Y_s)] = \mathbb{E}[h(Y_0, Y_u)]$$
$$\geq \mathbb{E}[h(Y_0, Y_v)]$$
$$= \mathbb{E}[f(Y_0, Y_{s+v-u})].$$

□

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Using the same argument as in Proposition 4.3 applied to the proof of Theorem 3.7 we have the following statement.

**Theorem 4.4** Assume that $Y$ is monotone and reversible. Then

$$\mathbb{E}[f(Y_0, Y_{t_1}, \ldots, Y_{t_k})] \leq \mathbb{E}[f(Y_0, Y_{s_1}, \ldots, Y_{s_k})]$$

for all $(s_1, \ldots, s_k) \leq (t_1, \ldots, t_k)$ and all supermodular functions $f$ such that $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in \mathcal{B}_b(E)$ for all $i \geq 1$, $x_i \in E$.

**Remark 4.5**

1. If $E$ is countable then both Proposition 4.3 and Theorem 4.4 hold without reversibility assumption. Then, instead of that, we have to assume in Theorem 4.3 that $Y^*$ is monotone. Indeed, in the proof of Proposition 4.4 we can write

$$\mathbb{E}[f(Y_0, Y_s)] = \sum \sum \sum \pi(y) f(x, z) Q_{s-u}(y, x) Q_{s-u}(y, z)$$

$$= \sum \sum \sum \pi(y) Q_{s-u}(y, x) (T_{s-u} f x)(y)$$

$$= \sum \sum \pi(x) Q_{s-u}(x, y) (T_{s-u} f x)(y)$$

$$= \mathbb{E}[h(Y_0, Y_u)].$$

We are aware if this is valid in uncountable state case.

2. In case of concordance ordering we have in Proposition 4.3 and Theorem 4.4 $(Y_0, Y_{t_1}) \ll_{cc} (Y_0, Y_{s_1})$, $(Y_0, Y_{t_1}, \ldots, Y_{t_k}) \ll_{cc} (Y_0, Y_{s_1}, \ldots, Y_{s_k})$, respectively. Indeed, increasing functions can be approximate by the increasing sequences of bounded and continuous functions. However, in the supermodular case we are aware of any of such type result, unless $E = \mathbb{R}$ (see Müller and Denuit).

Assume now that $Y$ has an infinitesimal generator $Q^Y$. Consider now a stationary homogeneous Markov process $\tilde{Y}$ with generator $Q^\tilde{Y}$ and assume that its Markov semigroup $T^\tilde{Y}$ is defined by $T^\tilde{Y}_t = T^Y_{ct}$ for each $t > 0$ and some $c \in (0, 1)$. Hence, transition kernels are related by $Q^\tilde{Y}_t(x, dy) = cQ^Y_{ct}(x, dy)$. Moreover, both $Y$ and $\tilde{Y}$ have the same stationary distribution. However, roughly speaking, $\tilde{Y}$ changes slower than $Y$. In fact, using Hille-Yosida Theorem, we have for the infinitesimal generators

$$Q^Y f = cQ^\tilde{Y} f$$

for each $f$ which belongs to a domain of $Q^\tilde{Y}$, $D(Q^\tilde{Y}) \subseteq \mathcal{B}_b(E)$. Note also that if $Y$ is monotone then the same holds for $\tilde{Y}$.

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3I do not know if the calculation concerning relationship between original process and its time-reversed counterpart as in Proposition 3.4 is still valid in non-uniform case, i.e. $Q^*_{s,t}(x, dy) = \frac{1}{\pi(dx)} Q_s(x, dx)$. Moreover, in the non-uniform case both $Y$ and $Y^*$ have the same $\pi$? So, I assume reversibility.
Corollary 4.6 Assume that $Y$ is monotone and reversible or monotone with countable state space. Suppose that Markov semigroups for $Y$ and $\tilde{Y}$ are related by $T_{tY}^Y = T_{ct}^{\tilde{Y}}$ for some $c \in (0, 1)$ and all $t > 0$. Then

$$E[f(Y_0, Y_t)] \leq E[f(\tilde{Y}_0, \tilde{Y}_t)]$$

for all supermodular functions $f$ such that $f(x, \cdot) \in B_b(E)$ for all $x \in E$. Moreover, if $T^Y$ is uniform then without reversibility assumption

$$(Y_0, Y_t) \prec_{sm} (\tilde{Y}_0, \tilde{Y}_t).$$

Proof. By Proposition 4.3 (see also the remark above) we have for each supermodular function $f$ which fulfills the above properties, by taking $s = ct$ and denoting $f_x(y) = f(x, y)$,

$$\int x \pi(dx) \int y Q^Y_t(x, dy)f(x, y) \leq \int x \pi(dx) \int y Q^Y_s(x, dy)f(x, y) \quad (3)$$

$$= \int x \pi(dx) T^Y_s f_x(x)$$

$$= \int x \pi(dx) T^{\tilde{Y}}_{ct} f_x(x) \quad (4)$$

$$= \int x \pi(dx) T^Y_{ct} f_x(x)$$

$$= \int x \pi(dx) \int y f(x, y) Q^Y_t(x, dy) \quad (5)$$

Hence the result follows.

Corollary 4.7 Assume that $Y$ is monotone and reversible or $Y, Y_\ast$ are monotone with countable state space. Suppose that Markov semigroups for $Y$ and $\tilde{Y}$ are related by $T_{tY}^Y = T_{ct}^{\tilde{Y}}$ for some $c \in (0, 1)$ and all $t > 0$. Then for all $t_1 \leq \cdots \leq t_k,$

$$E[f(Y_0, Y_{t_1}, \ldots, Y_{t_k})] \leq E[f(\tilde{Y}_0, \tilde{Y}_{t_1}, \ldots, \tilde{Y}_{t_k})],$$

all supermodular functions $f$ such that $f(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n) \in B_b(E)$ for all $i \geq 1, x_i \in E$. If $T^Y$ is uniform and $Y, Y_\ast$ are monotone then $Y \prec_{sm} \tilde{Y}$.

5 Applications

5.1 Jackson networks

Consider a Jackson network which consists of $J$ numbered nodes, summarized in $\bar{J} = \{1, \ldots, J\}$. Station $j, j \in \bar{J}$, is a single server queue with infinite waiting room under FCFS regime. Customers in the network are indistinguishable. At
node $j$ there is an external Poisson arrival stream with intensity $\lambda_j$, $\lambda_j \geq 0$. Customers arriving at node $j$ from the outside or from the other nodes request a service which is exponentially distributed with mean 1. Service at node $j$ is provided with intensity $\mu_j(n_j) > 0$, where $n_j$ is the number of customers at node $j$ including the one being served. We shall assume that $\sup\{\mu_j(k), j \in \{1, \ldots, J\}, k \in \mathbb{N}\} \leq K$ for some finite $K$. All service times and arrival processes are assumed to be independent.

A customer departing from node $j$ immediately proceeds to node $i$ with probability $r(j, i) \geq 0$ or departs from the network with probability $r(j, 0)$. The routing is independent of the past of the system given the momentary node customer is.

Let $\bar{J}_0 := \bar{J} \cap \{0\}$. With $\lambda := \int_{j=1}^{J} \lambda_j$, $r(0, j) := \frac{\lambda_j}{\lambda}$ and $r(0, 0) := 0$ we assume that the matrix $R := (r(i, j), i, j \in \bar{J}_0)$ is irreducible.

Let $X_j(t)$ be the number of customers present at node $j$ at time $t \geq 0$. Then $X(t) = (X_1(t), \ldots, X_J(t))$ is the joint queue length vector at time instant $t \geq 0$ and $X := (X(t), t \geq 0)$ is the joint queue length process with the state space $E := \mathbb{N}^J$.

The following theorem is classical.

**Theorem 5.1** Under the above assumptions the queueing process $X$ is a Markov process with transition rates $(q(x, y) : x, y \in E, x \neq y)$ given by

$$q(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i+1, \ldots, n_J) = \lambda_j$$

and for $n_i > 0$

$$q(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i-1, \ldots, n_J) = \mu_i(n_i)r(i, 0),$$

$$q(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i-1, \ldots, n_j+1, \ldots, n_J) = \mu_i(n_i)r(i, j).$$

Furthermore

$$q(x, x) = -\int_{y \in E \setminus \{x\}} q(x, y) \text{ and } q(x, y) = 0 \text{ otherwise.}$$

We assume henceforth $X$ to be ergodic. The unique stationary and limiting distribution $\pi$ of $X$ is then

$$\pi(n_1, \ldots, n_J) = K(J)^{-1} \prod_{j=1}^{J} \prod_{k=1}^{n_j} \left( \frac{\eta_j}{\mu_j(k)} \right), \quad (n_1, \ldots, n_J) \in \mathbb{N}^J$$

(7)

with normalization constant $K(J)$.

Assume now that a network process $X$ with generator $Q^X$ has state-independent service rates. It is well-known that for such Jackson network both the network process and its time-reversal counterpart are stochastically monotone (see Daduna and Szekli (1995) or Daduna (2001)).

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Corollary 5.2 Suppose that a Jackson network has the state-independent service intensities. Then $X$ is $<_{sm}$-decreasing.

Let $\tilde{X}$ be a network process with generator $\tilde{X}$ and the same service intensities as $X$.

Corollary 5.3 Suppose that both Jackson network have the state-independent service intensities. $X <_{sm} \tilde{X}$.

Remark 5.4 The above corollaries can be rewritten for Jackson networks with unreliable servers, see Daduna et al. (2004) for a description and ordering result.

Remark 5.5 The Corollary 5.2 is no longer true if we start a system from $X_0 = 0$, say. If we take $M/M/1$ queue then $X_t$ is stochastically increasing in $t$, so the marginal distribution cannot be the same at any time instant $t$ as required for supermodular ordering (See Stoyan (1983) and Lindvall (2002) for related results).

5.2 Particle systems

Let $Y$ be a Feller process with the state space $E = \{0, 1\}^{\mathbb{Z}^d}$. By adopting notation from Liggett (1985), we denote for $x \in \mathbb{Z}^d$ and $\eta \in E$ transition rates by $c(x, \eta)$. They describe the rate at which the coordinate $\eta(x)$ flips from 0 to 1 or from 1 to 0 when the system is in state $\eta$. Assume that such defined spin system is ergodic, therefore its invariant measure $\pi$ is defined uniquely. Moreover, if $Y$’s monotone then Proposition 4.3 and Corollary 4.6 are applicable.

Example 5.6 • (Liggett, p.131) The stochastic Ising model on $E = \{0, 1\}^{\mathbb{Z}}$ is defined by rates

$$c(x, \eta) = \begin{cases} 
\exp(-2\beta) & \text{if } \eta(x-1) = \eta(x) = \eta(x+1) \\
1 & \text{if } \eta(x-1) \neq \eta(x+1) \\
\exp(2\beta) & \text{if } \eta(x) \neq \eta(x-1) = \eta(x+1)
\end{cases},$$

$\beta > 0$. It is monotone and ergodic with the product measure $\pi\{\eta(x) = 1\} = \frac{1}{2}$.

• (Liggett, p.136) The spin system on $E = \{0, 1\}^{\mathbb{Z}^d}$ with rates

$$c(x, \eta) = \begin{cases} 
\beta + \sum_u p(u - x)\eta(u) & \eta(x) = 0 \\
\delta + \sum_u p(u - x)[1 - \eta(u)] & \eta(x) = 1
\end{cases},$$

$\beta, \delta \geq 0, \beta + \delta > 0$. It is monotone and ergodic with invariant measure $\pi\{\eta(x) = 1\} = \frac{\beta}{\beta + \delta}$. 

$^4$Reversible or uniform?

$^5$Reversible or uniform?
The nearest-neighbour majority vote process on $E = \{0, 1\}^\mathbb{Z}$ with rates

$$c(x, \eta) = \begin{cases} 
1 - \delta & \text{if } \eta(x - 1) = \eta(x + 1) \neq \eta(x) \\
\delta & \text{otherwise}
\end{cases}$$

$$0 < \delta \leq \frac{1}{2}.$$ It is monotone and ergodic. \(^6\)

References


\(^6\)Reversible or uniform?