Markov processes conditioned to never exit a subspace of the state space.

Zbigniew Palmowski[1,2] Tomasz Rolski[1,3]

October 19, 2004

Abstract

In this paper we study Markov processes never exiting (NE) a subspace A of the state space E or in another words, Markov processes conditioned to stay in the subspace A. We show how the knowledge of the exact asymptotics of the tail distribution of the exit time helps to find the suitable exponential martingale, which in turn, serves for the change of measure. Under the new probability measure the process is the sought for never exiting one the subspace A. We also find its extended generator and study relationships between the invariant measure (INE) and the quasi-stationary distribution (QS). We analyze in detail the PDMP processes.

Keywords: Markov process, extended generator, exponential change of measure, piecewise deterministic Markov process, workload conditioned to stay positive, NE process, INE measure, quasi-stationary distribution.

AMS 1991 Subject Classification: Primary: 60J25, 60J35.

[1]Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

[2]Mathematical Institute, Utrecht University, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands

[3]This work was partially supported by KBN Grant No 2 P03A 020 23 (2002–2004).

1 Introduction

In this paper we study Markov processes never exiting a subspace A of the state space E or in other words, Markov processes conditioned to stay always in A. Classical examples concern real valued Markov processes conditioned to stay positive. We use the abbreviation NE for never exiting. Typically, the event that the process never exit the subspace has probability zero and thus we cannot define NE processes by the conditional probability. In this paper we propose to define NE processes either by a limiting probability or by the change of probability measure. First studies of such the concept, under the name of the taboo process, were done by Knight (1969). He showed, that the Brownian process NE the positive axis is the Bessel^3 process. Knight also studied Brownian processes NE an interval. Diffusion processes NE a bounded open subset of R^d were studied by Pinsky (1985). The case of spectrally negative Levy processes conditioned to stay positive was considered by Bertoin (1996) and Chaumont (1996); see also Lambert (2000) for the Levy process conditioned to stay in an interval. Another example of such processes are so-called non-colliding Brownian motions, which are k-dimensional Brownian motions never exiting the Weyl chamber $\{\boldsymbol{x} : x_1 \leq x_2 \leq \ldots < x_k\}$. A limit of such the process turns out to be the eigenvalue process associated with the Hermitian Brownian motion; see Grabiner (1999), König *et al* (2002). Jacka and Roberts (1995) study the notion of NE processes for continuous time Markov chains (CTMC).

In this paper, in contrast to Pinsky (1985), we present the theory for general Markov processes and we allow unbounded subspace A. We present a detailed account of the exponential change of measure approach to define NE processes. Among others we show how to adapt some results from the paper of Palmowski and Rolski (2002) to make it possible to study NE processes. In particular, we show how to choose a *good function* (the terminology of Palmowski and Rolski (2002)) to define the NE process via change of measure. It turns out that this good function can be learned from the asymptotic behavior of the tail probability of the exit time from the subspace A. This part complements the study of Glynn and Thorisson (2001, 2002) explaining the role of the change of measure technique. The case of piecewise deterministic Markov processes (PDMP) is worked out. We also give a relationship between invariant measures of NE Markov process (provided they exist), abbreviated here by INE measures, and quasi-stationary (QS) distributions. Note that if the NE process is recurrent, then INE measure is just its stationary measure. The concept of quasi-stationary distributions attracted attention in many papers, in particular see Kyprianou (1971), Keener (1992), Nair and Pollett (1993), Ferrari *et al* (1995) and references therein.

2 Finite CTMC prototype

Consider a CTMC $\{X(t)\}$ with $E = \{0, 1, \ldots, k\}$ and $A = \{1, 2, \ldots, k\}$ with transition matrix $\boldsymbol{Q} = (q_{ij})_{i,j=0,1,\ldots,k}$. We assume that realizations are right continuous and with left hand limits. Let $\tau = \min\{t \geq 0 : X(t) = 0\}$ be the exit time from A and

$$X^\dagger(t) = \begin{cases} X(t), & t < \tau \\ \dagger, & t \geq \tau \end{cases} \tag{1}$$

be the process $\{X(t)\}$ killed at the exit from A. The transition matrix of the process $\{X^\dagger(t)\}$ on $\{1, \ldots, k\}$ is as follows. Take the subintensity matrix $\boldsymbol{Q}^\dagger = (q_{ij})_{i,j=1,\ldots,k}$ of the intensity matrix of the process $\{X(t)\}$ and set $\boldsymbol{P}^\dagger(t) = (p^\dagger_{ij}(t))_{i,j=1,\ldots,k} = e^{\boldsymbol{Q}^\dagger t}$. Then $\mathbb{P}_i(X^\dagger(t) = j) = p^\dagger_{ij}(t)$ for $i, j, = 1, \ldots, k$. Notice that $\sum_{j=1}^k p^\dagger_{ij}(t) \leq 1$, $i = 1, \ldots, k$. We also remark that operator $\boldsymbol{A}^\dagger f = \boldsymbol{Q}^\dagger f'$, working on vectors $f = (f_1, \ldots, f_k, f_\dagger)$ such that $f_\dagger = 0$ and $f_i = f'_i$ $(i \neq \dagger)$, is a full generator of $\{X^\dagger(t)\}$. Assume the irreducibility on the subspace $1, \ldots, k$ and let $-\gamma$ be the Perron-Frobenius eigenvalue of $\boldsymbol{Q}^\dagger$ with μ and h to be the left (row vector normalized to have sum 1) or the right (column vector) eigenvectors (respectively). Without loss of generality we may assume that $\mu h = 1$. We assume $\gamma > 0$, which means that τ is proper a.s. (we have always $\gamma \geq 0$). Thus

$$\boldsymbol{P}^\dagger(t) h = e^{-\gamma t} h, \qquad \mu \boldsymbol{P}^\dagger(t) = e^{-\gamma t} \mu,$$

for all $t \geq 0$ or equivalently

$$\boldsymbol{Q}^\dagger h = -\gamma h, \qquad \mu \boldsymbol{Q}^\dagger = -\gamma \mu.$$

We will call μ the quasi-stationary (QS) distribution of $\{X(t)\}$ on A. From Perron–Frobienius theorem we know that for $i, j = 1, \ldots, k$

$$\mathbb{P}_i(X^\dagger(t) = j) = h_i \mu_j e^{-\gamma t} + o(e^{-\gamma t}),$$

as $t \to \infty$. Conversely, the knowledge of the limit

$$\lim_{t\to\infty} \frac{\mathbb{P}_i(\tau > t)}{\mathbb{P}_j(\tau > s+t)}$$

gives us the needed spectral information. Namely, we have as $s \to \infty$,

$$\begin{aligned} \frac{\mathbb{P}_i(\tau > s)}{\mathbb{P}_j(\tau > s+t)} &= \frac{\sum_{\ell=1}^k \mathbb{P}_i(\tau > s, X(s) = \ell)}{\sum_{\ell=1}^k \mathbb{P}_j(\tau > s+t, X(t+s) = \ell)} \\ \to \frac{\sum_{\ell=1}^k e^{-\gamma s} h_i \mu_\ell}{\sum_{\ell=1}^k e^{-\gamma(t+s)} h_j \mu_\ell} &= e^{\gamma t} \frac{h_i}{h_j}. \end{aligned}$$

Similarly we can show that for $i, j = 1, \ldots, k$ and $t > 0$ the limit

$$p_{ij}^{\uparrow}(t) = \lim_{s\to\infty} \mathbb{P}_i(X^{\dagger}(t) = j | \tau > t+s)$$

exists and defines a probability transition function for a CTMC on $\{1, \ldots, k\}$. Markov processes like $\{X^{\uparrow}(t)\}$ with transition function $\boldsymbol{P}^{\uparrow}(t) = (p_{ij}^{\uparrow}(t))_{i,j=1,\ldots,k}$ are to be studied in this paper. Can we find its generator ? The starting point is the observation that h is a $(-\gamma)$–harmonic function on $\{1, \ldots, k\}$, thus $\{\mathcal{E}^{\uparrow}(t) = e^{\gamma t} h_{X^{\dagger}(t)} \mathbb{I}(t < \tau)\}$ is an $\mathcal{F}_t$ – martingale. As we show later, this is an exponential martingale, and we may use the theory from Palmowski and Rolski (2002) to define Markov process $\{X^{\uparrow}(t)\}$ on $\{1, \ldots, k\}$ via exponential change of measure using the density process $\{\mathcal{E}^{\uparrow}(t)\}$. We shall call $\{X^{\uparrow}(t)\}$ as the process $X(t)$ conditioned to never exit (NE) set A. This new process is a true Markov process with intensity matrix (check that rows sum to 0):

$$\boldsymbol{Q}^{\uparrow} = (\boldsymbol{\Delta}^{-1} \boldsymbol{Q}^{\dagger} \boldsymbol{\Delta} + \gamma \boldsymbol{I}),$$

where $\boldsymbol{\Delta} = \mathrm{diag}(h)$. Thus indeed the process lives on $A = \{1, \ldots, k\}$. We may compute the stationary distribution π for $\boldsymbol{Q}^{\uparrow}$, that is the distribution fulfilling $\pi \boldsymbol{Q}^{\uparrow} = 0$, which is the INE distribution. Now writing $\pi \boldsymbol{\Delta}^{-1} \boldsymbol{Q}^{\dagger} \boldsymbol{\Delta} = -\gamma \pi$, it yields $(\pi \boldsymbol{\Delta}^{-1}) \boldsymbol{Q}^{\dagger} = -\gamma(\pi \boldsymbol{\Delta}^{-1})$. This means that $\mu = b \pi \boldsymbol{\Delta}^{-1}$ is QS, for some normalizing constant b. Our goal is to generalize above considerations for more general Markov processes on general state spaces.

3 General framework

Let a state space E be Borel and consider a continuous time Markov process $\{X(t)\}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_x)$, $x \in E$. We assume that all considered processes are cádlág. Thus without loss of generality we may suppose that $\Omega \subset E^{[0,\infty)}$ is the space $D_E[0,\infty)$ of cádlág functions from $[0,\infty)$ into E (in some cases we can also use $\mathcal{C}_E[0,\infty)$, the space of continuous functions into E) and consider the canonical process $\{X(t)\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_x)$ defined by $X(\omega, t) = \omega(t)$. We denote the natural filtration $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma\{X(s), s \le t\}$ and $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$.

Let $A \subset E$ be an open subset of E and $\tau = \min\{t \ge 0 : X(t) \notin A\}$ be an exit time. Note that τ is a Markov time. Throughout this note, A always denotes the subspace and we assume that

$$\mathbb{P}_x(\tau < \infty) = 1 \quad \text{for all} \quad x \in A.$$

We also need the killed process $\{X^\dagger(t)\}$ defined in (1).

3.1 Never exiting process

The concept of *Never Exiting* $A \subset E$ by a CTMP $\{X(t)\}$ (or in short (NE)) can be defined as follows. For $t \ge 0$ and $x \in A$ define

$$\lim_{s\to\infty} \mathbb{P}_x(\Lambda | \tau > t+s) = P^{\uparrow}_{x,t}(\Lambda), \qquad \Lambda \in \mathcal{F}_t \tag{2}$$

provided the limits exist for all t, Λ and they define the proper probability measures. Let $x \in A$. Then the family $\{P^{\uparrow}_{x,t}(\cdot),\ t \ge 0\}$ defines one probability measure $\mathbb{P}^{\uparrow}_x$ on $D_E[0,\infty)$ such that $P^{\uparrow}_{x,t}(\Lambda) = \mathbb{P}^{\uparrow}_x(\Lambda)$ for $\Lambda \in \mathcal{F}_t$; see e.g. for more details Palmowski and Rolski (2002). Thus provided these limits exist we may consider the process $\{X(t)\}$ on the space $D_E[0,\infty)$ (and in view of Proposition 3.3 on $D_A[0,\infty)$) under $P^{\uparrow}_x$, which we call by NE the subspace A. With this approach we face the following problems. The first question is to determine when the limit has sense and the second one, is whether we can define $\mathbb{P}^{\uparrow}_x$ from $\mathbb{P}_x$ by the exponential change of probability measure argument. Then it turns out that the process $X(t)$ under $\mathbb{P}^{\uparrow}_x$ is Markovian.

The following general scheme can be found in some papers, like Lambert (2000) and Jacka and Roberts (1995).

Lemma 3.1 *For all $s, t \ge 0$ and $x \in A$*

$$\mathbb{P}_x(\Lambda | \tau > t+s) = \mathbb{E}_x D(X^\dagger(t), x, t, s); \Lambda], \qquad \Lambda \in \mathcal{F}_t,$$

where

$$D(y,x,t,s) = \frac{\mathbb{P}_y(\tau > s)}{\mathbb{P}_x(\tau > t+s)}.$$

Proof Let $\Lambda \in \mathcal{F}_t$. We write

$$\begin{aligned}
\mathbb{P}_x(\Lambda|\tau > t+s) &= \frac{\mathbb{P}_x(\Lambda, \tau > t+s)}{\mathbb{P}_x(\tau > t+s)} \\
&= \frac{\mathbb{E}_x(\mathbb{P}_x(\Lambda, \tau > t, \tau > t+s|\mathcal{F}_t))}{\mathbb{P}_x(\tau > t+s)} = \frac{\mathbb{E}_x[\mathbb{P}_{X(t)}(\Lambda, \tau > s)\mathbb{I}(\tau > t)]}{\mathbb{P}_x(\tau > t+s)} \\
&= \frac{\mathbb{E}_x[\mathbb{P}_{X(t)}(\Lambda, \tau > s)]}{\mathbb{P}_x(\tau > t+s)} = \mathbb{E}_x\left[\frac{\mathbb{P}_{X(t)}(\tau > s)\mathbb{I}(\tau > t)}{\mathbb{P}_x(\tau > t+s)}; \Lambda\right] \\
&= \mathbb{E}_x[D(X(t), x, t, s)\mathbb{I}(\tau > t); \Lambda].
\end{aligned}$$

□

We now look for the limiting function $h^*(x,t)$ such that

$$\lim_{s\to\infty} D(y,x,t,s) = \frac{h^*(y,t)}{h^*(x,0)}. \tag{3}$$

Let τ be the exit time. We now introduce a hypothesis, which yields the required spectral information.

(PF) We say that the Perron-Frobenius hypothesis is fulfilled if the limit

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(\tau > t)}{\mathbb{P}_y(\tau > t)} = z(x,y) \tag{4}$$

exists for all $x, y \in A$, $0 < z(x,y) < \infty$ and

$$\lim_{s\to\infty} \mathbb{E}_x\left[\frac{\mathbb{P}_{X(t)}(\tau > s)}{\mathbb{P}_y(\tau > s)}\mathbb{I}(t < \tau)\right] = \mathbb{E}_y z(X(t), y)\mathbb{I}(t < \tau), \tag{5}$$

for all $y \in A$.

Conditions under which (PF) holds in the case of $E = \{0, 1, \ldots\}$ and $A = \{1, 2, \ldots, \}$ are given e.g. in Kesten (1995) and references therein.

Proposition 3.1 *If (PF) hypothesis is fulfilled, then there exists a function $h : A \cup \{\dagger\} \to \mathbb{R}$ such that $h(x) > 0$, $h(\dagger) = 0$ and a constant $\gamma' \geq 0$ such that*

$$\lim_{s\to\infty} \frac{\mathbb{P}_x(\tau > s)}{\mathbb{P}_y(\tau > s+t)} = e^{\gamma' t}\frac{h(x)}{h(y)}$$

for all $x, y \in A$.

Proof Fix $x_0 \in A$ and define

$$h(x) = \begin{cases} \lim_{s\to\infty} \frac{\mathbb{P}_x(\tau>s)}{\mathbb{P}_{x_0}(\tau>s)}, & x \in A \\ 0, & x = \dagger. \end{cases}$$

Then

$$\lim_{s\to\infty} \frac{\mathbb{P}_y(\tau > s)}{\mathbb{P}_x(\tau > s)} = \frac{h(y)}{h(x)}, \qquad x, y \in A.$$

Since

$$\mathbb{P}_y(\tau > t + s) = \mathbb{E}_y[\mathbb{P}_{X^\dagger(t)}(\tau > s); t < \tau]$$

we have by (5) that

$$\begin{aligned} \lim_{s\to\infty} \frac{\mathbb{P}_y(\tau > t+s)}{\mathbb{P}_x(\tau > s)} &= \lim_{s\to\infty} \frac{\mathbb{E}_y[\mathbb{P}_{X^\dagger(t)}(\tau > s); t < \tau]}{\mathbb{P}_x(\tau > s)} \\ &= \mathbb{E}_y\left[\frac{h(X^\dagger(t))}{h(x)}; t < \tau\right] \\ &= \frac{h(y)}{h(x)}\mathbb{E}_y\left[\frac{h(X^\dagger(t))}{h(y)}; t < \tau\right]. \end{aligned}$$

Define now

$$f(t) = \mathbb{E}_x\left[\frac{h(X^\dagger(t))}{h(x)}; t < \tau\right].$$

We first show that the above definition does not depend on $x \in A$. We write

$$\begin{aligned} \frac{\mathbb{E}_x[h(X^\dagger(t))/h(x); t < \tau]}{\mathbb{E}_y[h(X^\dagger(t))/h(y); t < \tau]} &= \frac{\lim_{s\to\infty} \mathbb{P}_x(\tau > s+t)/\mathbb{P}_x(\tau > s)}{\lim_{s\to\infty} \mathbb{P}_y(\tau > s+t)/\mathbb{P}_y(\tau > s)} \\ &= \frac{\lim_{s\to\infty} \mathbb{P}_x(\tau > s+t)/\mathbb{P}_y(\tau > s+t)}{\lim_{s\to\infty} \mathbb{P}_y(\tau > s)/\mathbb{P}_x(\tau > s)} = 1. \end{aligned}$$

Recall that $X^\dagger(t) = \dagger$ for $t \geq \tau$. We now have by the Markov property

$$\begin{aligned} f(t+t') &= \mathbb{E}_x\left[\frac{h(X^\dagger(t))}{h(x)}\frac{h(X^\dagger(t+t'))}{h(X^\dagger(t))}; t+t' < \tau\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left[\frac{h(X^\dagger(t))}{h(x)}\frac{h(X^\dagger(t+t'))}{h(X^\dagger(t))}\mathbb{I}(\tau > t+t')|\mathcal{F}_t\right]\right] \\ &= \mathbb{E}_x\left[\frac{h(X^\dagger(t))}{h(x)}\mathbb{E}_{X^\dagger(t)}\left[\frac{h(X^\dagger(t'))}{h(X^\dagger(t))}; t' < \tau\right]; t < \tau\right]. \end{aligned}$$

Since $\mathbb{E}_x[h(X^\dagger(t'))]/h(x)$ does not depend on $x \in A$, we have that the above equals to $f(t+t') = f(t)f(t')$. Clearly, the function $f(t)$ is nonincreasing, so it must be of the form $f(t) = \exp(-\gamma' t)$ for some $\gamma' \geq 0$. $\square$

The following case will be assumed from now on. Suppose that for all $x \in A$ we have

$$\mathbb{P}_x(\tau > t) \sim Ch(x)e^{-\gamma t}L(t), \tag{6}$$

for $t \to \infty$, where $\gamma \geq 0$, $C > 0$ and $0 < h(x) < \infty$ and $L(t)$ belongs to class $\mathcal{L}$ of functions such that

$$\frac{L(s+t)}{L(s)} \to 1, \qquad s \to \infty$$

and that (5) holds. Then for $s \to \infty$

$$D(y,x,t,s) \to \frac{e^{\gamma t}h(y)}{h(x)}, \tag{7}$$

for all $t \geq 0$, $x, y \in A$. Hence $h^*(x,t) = e^{\gamma t}h(x)$, where $h^*(x,t)$ was defined in (3). This means that $\gamma = \gamma'$ from Proposition 3.1.

We have the following proposition.

Proposition 3.2 *Suppose that (7) holds for some positive function $h(x)$ on A and $\gamma \geq 0$, and*

$$\mathbb{E}_x[h(X^\dagger(t)); t < \tau] = e^{-\gamma t}h(x), \qquad t \geq 0, x \in A. \tag{8}$$

Then for all $x \in A$, $t \geq 0$

$$\lim_{s\to\infty} \mathbb{P}_x(\Lambda | \tau > t+s) = \mathbb{E}_x\left[e^{\gamma t}\frac{h(X^\dagger(t))}{h(x)}\mathbb{I}(\tau > t); \Lambda\right], \qquad \Lambda \in \mathcal{F}_t\,. \tag{9}$$

Proof Thus under assumption (8) applying the Markov property, we have that the right-hand side of (9)

$$\mathbb{P}^\uparrow_{x,t}(\Lambda) = \mathbb{E}_x\left[e^{\gamma t}\frac{h(X^\dagger(t))}{h(x)}\mathbb{I}(\tau > t); \Lambda\right], \qquad \Lambda \in \mathcal{F}_t. \tag{10}$$

defines a consistent family of distributions. By the Kolmogorov consistency theorem, for each $x \in A$, (10) defines the unique probability measure $\mathbb{P}^\uparrow_x$ on $(D_E[0,\infty), \mathcal{F}, \{\mathcal{F}_t\})$ such that its restriction to $\mathcal{F}_t$ is $\mathbb{P}^\uparrow_{x,t}$, for all $t \geq 0$. The proof will follow the proof of Theorem 3.1 in Lambert (2000) and the proof of Thorem 1 in Bertoin and Doney (1994). Suppose that $0 \leq f \leq 1$ for f living on $D[0,t]$. It follows from Lemma 3.1 and Fatou's Lemma that

$$\liminf_{s\to\infty} \mathbb{E}_x\left[f(X) | \tau > t+s\right] \geq \mathbb{E}^\uparrow_x f(X)\,. \tag{11}$$

Replacing f by $1-f$, we get

$$
\begin{aligned}
&\limsup_{s\to\infty} \mathbb{E}_x\left[f(X)|\tau > t+s\right] \\
&\quad = 1 - \liminf_{s\to\infty} \mathbb{E}_x\left[(1-f)(X)|\tau > t+s\right] \\
&\quad \le 1 - \mathbb{E}_x^{\uparrow}\left[(1-f)(X)\right] = \mathbb{E}_x^{\uparrow} f(X) \ ,
\end{aligned}
$$

where in the last equality comes from the fact that $\mathbb{P}^{\uparrow}$ is conservative (see equation (8)) and would be false otherwise. □

Note that if (5) and (7) are stisfied, then condition (8) holds. In other words, by Proposition 3.1, if condition (PF) holds, then assumptions of Proposition 3.2 are fulfilled.

Proposition 3.3 *For all* $x \in A$

$$\mathbb{P}_x^{\uparrow}(\tau < \infty) = 0 \quad \text{and} \quad \mathbb{P}_x^{\uparrow}(D_A[0,\infty)) = 1.$$

Proof For each $t \ge 0$

$$\mathbb{P}_x^{\uparrow}(\tau < \infty) = e^{\gamma t}\mathbb{E}_x\left[\frac{h(X(t))}{h(x)}\mathbb{I}(\tau < t); t \ge \tau\right] = 0.$$

Hence $\mathbb{P}_x^{\uparrow}(\tau < \infty) = \lim_{t\to\infty} \mathbb{P}_x^{\uparrow}(\tau \le t) = 0$. □

We may conclude that the conditioned process $\{X(t)\}$ can be considered on $(D_A[0,\infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_x^{\uparrow})$ and it is a Markov process with the state space A. This Markov process is NE a subspace A process defined by $(h(x), \gamma)$. Note that then

$$\mathcal{E}^{\uparrow}(t) = e^{\gamma t}\frac{h(X^{\dagger}(t))}{h(x)}\mathbb{I}(\tau > t) \tag{12}$$

is a $(\mathbb{P}_x, \mathcal{F}_t)$-martingale and (10) can be read

$$\frac{d\mathbb{P}_{x,t}^{\uparrow}}{d\mathbb{P}_{x,t}} = \mathcal{E}^{\uparrow}(t), \qquad t \ge 0. \tag{13}$$

3.2 Exponential change of measure

Let $\{X(t)\}$ be a Markov process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_x)$ with values in E. We define the *full (extended) generator* $\boldsymbol{A}$ of process $\{X(t)\}$ by

$$\boldsymbol{A} = \{(f, f^*) \in \mathcal{M}(E) \times \mathcal{M}(E) : D_f(t) \quad \text{is a (local) martingale}\} \ ,$$

where

$$D_f(t) = f(X(t)) - \int_0^t f^*(X(s))\,ds \tag{14}$$

is called *Dynkin's (local) martingale* and function $s \to f^*(X(s))$ is integrable on $[0,t]$ a.s. for all $t \geq 0$. Further on we will identify all versions of functions f^* up to the sets of potential zero and we denote all these versions by $\boldsymbol{A}f$ if $(f, f^*) \in \boldsymbol{A}$. The family of such the functions f for which (14) is a (local) martingale forms the domain denoted by $\mathcal{D}(\boldsymbol{A})$. For details see Ethier and Kurtz (1986).

For a function $f^{'} \in \mathcal{D}(\boldsymbol{A})$ such that $f^{'}(x) = 0$, $x \notin A$, define

$$f(x) = \begin{cases} f^{'}(x), & x \in A, \\ 0, & x = \dagger \end{cases}$$

and formally

$$(\boldsymbol{A}^{\dagger} f)(x) = \begin{cases} (\boldsymbol{A}f^{'})(x), & x \in A, \\ 0, & x = \dagger \end{cases}$$

We denote by $\mathcal{D}(\boldsymbol{A}^{\dagger})$ the set of such f functions.

Lemma 3.2 *For $f \in \mathcal{D}(\boldsymbol{A}^{\dagger})$*

$$f(X^{\dagger}(t)) - \int_0^t (\boldsymbol{A}^{\dagger} f)(X^{\dagger}(s))\,ds$$

is a local $\mathcal{F}_t$-martingale.

Proof Let $f^{'} \in \mathcal{D}(\boldsymbol{A})$. Thus $D_{f'}(t)$ and $D_{f'}(t \wedge \tau)$ are local $\mathcal{F}_t$-martingales. Notice however that

$$f^{'}(X(t \wedge \tau)) - \int_0^{t \wedge \tau} (\boldsymbol{A}f^{'})(X(s))\,ds = f(X^{\dagger}(t)) - \int_0^t (\boldsymbol{A}^{\dagger} f)(X^{\dagger}(s))\,ds,$$

which completes the proof. □

From the above lemma we see that $\boldsymbol{A}^{\dagger}$ is an extended generator of the Markov process $\{X^{\dagger}(t)\}$ and $\mathcal{D}(\boldsymbol{A}^{\dagger})$ is the domain of $\boldsymbol{A}^{\dagger}$.

By $f \to g \cdot f$ we mean the multiplication operator by function g. If $g(x) \neq 0$, then $f \to g^{-1} \cdot f$ denotes multiplication by $g^{-1}(x)$. Define now formally the operator $\boldsymbol{A}^{\uparrow}$ by

$$\begin{aligned}(\boldsymbol{A}^{\uparrow} f)(x) &= \frac{(\boldsymbol{A}^{\dagger} f \cdot h)(x) + \gamma f \cdot h(x)}{h(x)} \\ &= h^{-1}(x)(\boldsymbol{A}^{\dagger} f \cdot h)(x) + \gamma f(x), \quad x \in A,\end{aligned}$$

where function h and constant γ are given in Proposition 3.2.

Consider the exponential martingale (12). We set $h(\dagger) = 0$ and then $\mathcal{E}^{\dagger}(t) = e^{\gamma t} h(X^{\dagger}(t))/h(x)$. is a $(\mathbb{P}_x, \mathcal{F}_t)$–martingale for all $x \in A$. Repeating the proof of Lemma 3.1 from Palmowski and Rolski (2002) for $\{\mathcal{E}^{\dagger}(t)\}$ we obtain that

$$h(X^{\dagger}(t)) + \gamma \int_0^t h(X^{\dagger}(s))\, ds$$

is a local martingale. Thus directly from definition (14) of extended generator we can conclude $h \in \mathcal{D}(\boldsymbol{A}^{\dagger})$ and $\boldsymbol{A}^{\dagger} h = -\gamma h$. Therefore $h : A \cup \{\dagger\} \to R$ is a good function according to the terminology of Palmowski and Rolski (2002) and $\{\mathcal{E}^{\dagger}(t)\}$ is an exponential martingale. Since the NE process is obtained by the exponential change of measure with respect to exponential martingale $\{\mathcal{E}^{\dagger}(t)\}$, the NE is a *Markov process* with extended generator $\boldsymbol{A}^{\uparrow}$ (see Theorem 4.2 of Palmowski and Rolski (2002)). We summarize the above considerations in the following proposition.

Proposition 3.4 *We have $h \in \mathcal{D}(\boldsymbol{A}^{\dagger})$ and $\boldsymbol{A}^{\dagger} h = -\gamma h$, and hence the NE process defined by $(h(x), \gamma)$ is a Markov process with extended generator $\boldsymbol{A}^{\uparrow}$.*

Example 3.0 Let $X(t) = x + B(t) - \theta t$ be a Brownian motion with drift and $A = (0, \infty)$. We have to assume that $\theta \geq 0$. Following Borodin and Salminen (1996), for $x > 0$

$$\mathbb{P}_x(\tau > s) = \frac{x}{\sqrt{2\pi}} \int_s^{\infty} t^{-3/2} \exp\left(-\frac{(x - \theta t)^2}{2t} \right) dt.$$

Hence

$$\lim_{s \to \infty} \frac{\mathbb{P}_y(\tau > s)}{\mathbb{P}_x(\tau > s + t)} = \frac{y e^{\theta y} e^{\frac{\theta^2 t}{2}}}{x e^{\theta x}}.$$

Therefore $h^*(x,t) = x e^{\theta x} e^{\frac{\theta^2 t}{2}}$ and $h(x) = x e^{\theta x}$. For $\theta = 0$, the NE process is the Bessel[3] process, which is known to be transient (see Knight (1969)), having invariant measure $\pi(dx) = 2x^2 dx$ (see Karatzas and Shreve (1991), p. 362). Moreover, for $\theta > 0$ we have $\int_0^\infty \frac{1}{h(x)} \pi(dx) = 2 \int_0^\infty x e^{-\theta x} dx <$ ∞ and there exists QS distribution $\mu(dx) = \frac{\theta^2}{2h(x)} \pi(dx) = \theta^2 x e^{-\theta x}$ (see Martinez and Martin (1994) and Proposition 4.1 for general Markov process). Consider now k independent Brownian motions $X_i(t) = x_i + B_i(t)$, where $x_1 < x_2 < \ldots < x_k$, running until the collision, that is up to time $\tau = \min\{t \geq 0 : \prod_{i=1}^{k-1} (X_{i+1}(t) - X_i(t)) = 0\}$. In this case $E = R^k$ and $A = \{\boldsymbol{x} \in R^k :\ x_1 < x_2 <$

$\ldots < x_k\}$. Then $\mathbb{P}_{\boldsymbol{x}}(\tau < \infty) = 1$, for $\boldsymbol{x} \in A$. Using the asymptotics of the tail of the exit time we have

$$\lim_{s\to\infty} \frac{\mathbb{P}_{\boldsymbol{x}}(\tau > s)}{\mathbb{P}_{\boldsymbol{y}}(\tau > s+t)} = \frac{h(\boldsymbol{x})}{h(\boldsymbol{y})}, \qquad \boldsymbol{x}, \boldsymbol{y} \in A,$$

where

$$h(\boldsymbol{x}) = \prod_{i<j} (x_j - x_i)$$

is the Vandermond determinant (see Grabiner (1999)). In this case the NE process defined by $h(\boldsymbol{x})$ is the so-called non-colliding Brownian motion; see also Kőnig *et al* (2002).

3.3 Application to PDMPs

We now show how Palmowski and Rolski (2002) result for PDMPs can be adapted for the case of NE PDMPs. We refer to Palmowski and Rolski (2002), based on Davis (1993), for the description of PDMP $\{X(t)\}$. We recall only that E is a state space consisting of pairs $x = (\nu, z)$, where ν assumes a finite number of values from a finite set $\mathcal{I}$ and z belongs to an open subset $\mathcal{O}_\nu$ of $\mathbb{R}^{d(\nu)}$ (note that E is a Borel space). Moreover, the PDMP process $\{X(t)\}$ is determined by a differential operator $\mathcal{X}$ describing deterministic vector field between jumps, a jump intensity $\lambda(x)$ and a transition kernel $Q(x, dy)$. Let $\{T_n\}$ be jump epochs of $\{X(t)\}$. We assume that $T_n \to \infty$ a.s.

Consider now an open subspace $A \subset E$ consisting of all pairs $x = (\nu, z)$, where ν assumes a finite number of values from a set $\mathcal{I}^A$, z belongs to an open subset $\mathcal{O}_\nu^A \subset \mathcal{O}_\nu$ of $\mathbb{R}^{d(\nu)}$ and τ is the exit time of the PDMP $\{X(t)\}$ from A. We denote by Γ_A the active boundary defined for the subspace A and $t_A^*(\nu, z)$ the time needed to reach the boundary from (ν, z). We assume that $\phi_\nu(t_A^*(\nu, z), z) \in \Gamma_A$ if $t_A^*(\nu, z) < \infty$, where $\phi_\nu(t, z)$ is integral curve defined by $\mathcal{X}$. Then $\{X^\dagger(t)\}$ is also PDMP with the parameters $(\mathcal{X}^\dagger, \lambda^\dagger, Q^\dagger) = (\mathcal{X}, \lambda, Q)$ on A and on the cemetery state $\lambda^\dagger(\dagger) = \mathcal{X}^\dagger f(\dagger) = 0$, $Q^\dagger(x, \dagger) = Q(x, A^c)$ for $x \in A$. We have the following lemma.

Lemma 3.3 *The formula for the extended generator is*

$$\boldsymbol{A}^\dagger f(x) = \mathcal{X} f(x) + \lambda(x) \int_A (f(y) - f(x)) Q(x, dy) \ , \tag{15}$$

where $x \in A$ and $\boldsymbol{A}^{\dagger} f(\dagger) = 0$. The domain $\mathcal{D}(\boldsymbol{A}^{\dagger})$ consists of every function f, that is the restriction to A of a measurable function $\overline{f} : A \cup \Gamma_A \cup \{\dagger\} \to \mathbb{R}$, such that $\overline{f}(\dagger) = 0$, and satisfying the following three conditions:

(i) *for each $(\nu, z) \in E$ the function $t \to \overline{f}(\nu, \phi_\nu(t, z))$ is absolutely continuous on $(0, t_A^*(\nu, z))$,*

(ii) *for each $x \in \Gamma_A$,*

$$\overline{f}(x) = \int_A \overline{f}(y) Q(x, dy) \ , \tag{16}$$

(iii) *for $n = 1, 2, \ldots,$*

$$\mathbb{E}\left(\sum_{i=1}^{n} |\overline{f}(X(\tau \wedge T_i)) - \overline{f}(\tau \wedge X(T_i -))| \right) < \infty \ . \tag{17}$$

If moreover, for all $t \geq 0$,

$$\mathbb{E}\left(\sum_{i \leq t} |\overline{f}(X(\tau \wedge T_i)) - \overline{f}(X(\tau \wedge T_i -))| \right) < \infty, \tag{18}$$

then f is in the domain of the full generator.

Following Theorem 5.3 of Palmowski and Rolski (2002), we now compute the extended generator of the NE process defined by $(h(x), \gamma)$.

Theorem 3.1 *Assume that $h(x)$ is such that $h \in \mathcal{D}(\boldsymbol{A}^{\dagger})$ and $H(x) = \int_A h(y) Q(x, dy) < \infty$ for all $x \in A$. Then on the new probability space $(\mathcal{D}_A[0, \infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_x^{\uparrow})$, the process $X(t)$ is a PDMP with the unchanged differential operator $\mathcal{X}$ and the following jump intensity and transition kernel:*

$$\lambda^{\uparrow}(x) = \frac{\lambda(x) H(x)}{h(x)} \ , \qquad Q^{\uparrow}(x, dy) = \frac{h(y)}{H(x)} Q(x, dy) \ . \tag{19}$$

4 INE and QS distributions

Define for $B \in \mathcal{B}(A)$

$$p_t^{\uparrow}(x, B) = \mathbb{P}_x^{\uparrow}(X^{\dagger}(t) \in B) = \mathbb{E}_x \left[\frac{h(X^{\dagger}(t)) e^{\gamma t}}{h(x)} \mathbb{I}(\tau > t); X^{\dagger}(t) \in B \right] . \tag{20}$$

The family $\{p_t^{\uparrow}(x, \cdot)\}$ is a family of Markov transition function on A. The Markov process with the family of transition function $\{p_t^{\uparrow}(x, B)\}$ is NE A. If there exists a invariant measure π on A for

$p_t^{\uparrow}(x,B)$, that is fulfilling $\pi(B)=\int_A \pi(dx)p^{\uparrow}(x,B)$ for all B, we call it by *Invariant Never Exiting* measure (INE). If additionaly NE process $X^{\uparrow}(t)$ is reccurent then π is a stationary measure.

In the literature there is another related concept, which we now recall. We say that a distribution μ on $A\subset E$ is *quasi-stationary* (QS) if

$$\mathbb{P}_{\mu}(X(t)\in B|\tau>t)=\mu(B), \qquad B\in\mathcal{B}(E). \tag{21}$$

We have immediately from definition that $\mathbb{P}_{\mu}(\tau>t+s)=\mathbb{P}_{\mu}(\tau>s)\mathbb{P}_{\mu}(\tau>t)$, which yields $\mathbb{P}_{\mu}(\tau>t)=e^{-\gamma' t}$ for some $\gamma'>0$. A vast number of results exist giving conditions for the existence of QS distributions. A special case with $E=R_+$ or $E=Z_+$ and $A=E\backslash\{0\}$ attracted the special attention; see e.g. Ferrari *et al* (1995) and references therein.

The question is whether INE and QS distributions are related. We saw the answer in the introduction for the case of a finite state space. The result can be generalized as follows.

Proposition 4.1 *Suppose that for $\{X(t)\}$ there exists NE process defined by $(h(x),\gamma)$. If there exists INE a σ-finite measure π (not necessarily finite) such that*

$$\int_A \frac{1}{h(x)}\,\pi(dx)<\infty, \tag{22}$$

then the probability measure μ defined by

$$\mu(dx)=\frac{b}{h(x)}\pi(dx),$$

where b is a normalizing constant, is a QS distribution, and $\mathbb{P}_{\mu}(\tau>t)=\exp(-\gamma t)$. Conversely, suppose that for $\{X(t)\}$ there exists a QS distribution μ with $\gamma'>0$ and there exits NE process defined by $(h(x),\gamma)$. Then π defined by

$$\pi(dx)=h(x)\mu(dx)$$

is a INE measure, and $\gamma=\gamma'$.

Proof Suppose there exists the NE process defined by $(h(x),\gamma)$ and π is an INE measure fulfilling (22). It means that

$$\pi(B)=\int_A \pi(dx)h^{-1}(x)e^{\gamma t}\mathbb{E}_x[h(X(t));\tau>t,X(t)\in B]$$

for all $B \subset A$ and $t \geq 0$. Hence

$$e^{-\gamma t} \int_B h^{-1}(y) d\pi(dy) = \int_A \pi(dx) h^{-1}(x) \int_B \mathbb{E}_x[h(X(t)); \tau > t, X(t) \in dy] h^{-1}(y)$$

and

$$e^{-\gamma t} \int_B h^{-1}(y) d\pi(dy) = \int_A \pi(dx) h^{-1}(x) \mathbb{P}_x(X(t) \in B, \tau > t).$$

Now define $\mu(dy) = h^{-1}(y)\pi(dy)$ and suppose it is a probability measure (otherwise we may make normalizations by b). Then we see first that subsituting $A = B$ we have

$$e^{-\gamma t} = \mathbb{P}_\mu(\tau > t)$$

and hence μ is QS. For the converse implication, suppose there exists QS distribution μ with $\gamma^{'} > 0$ and NE is defined by $(h(x), \gamma)$. Then

$$\begin{aligned} e^{-\gamma^{'} t} \int_B h(x)\mu(dx) &= \int_A \mu(dx) \int_B \mathbb{P}_x(X(t) \in dy, \tau > t) h(y) \\ &= \int_A \mu(dx) h(x) \frac{\mathbb{E}_x[h(X(t)), \tau > t, X(t) \in B]}{h(x)}. \end{aligned}$$

Hence $\gamma = \gamma^{'}$ and $\pi(dx) = h(x)\mu(dx)$ is INE. $\square$

We remark that in view of Example 3.0, condition (22) is essential.

Remark 4.1 The same assertion under stronger assumption of γ-reccurence one can find in Theorem 7 in Touminen and Tweedie (1979) and Theorem 7 in Arjas *et al* (1979) (in fact they additionaly also prove the existence of so-called limiting quasi-stationary distribution $\mu(B) = \lim_{t\to\infty} \mathbb{P}_x(X(t) \in B | \tau > t)$ that implies (21)).

Acknowledgment

Both authors appreciate discussions with David McDonald on the previous version of this paper. The second author is grateful to Masakiyo Miyazawa for attracting his attention to the concept of quasi-stationary distributions. Part of this collaborative work was carried out when the first author was a researcher at EURANDOM and the second author was visiting EURANDOM. Both authors would like to express respective thanks to this institution for its hospitality and support. In addition, first author gratefully acknowledges grant nr. 613.000.310 from Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

References

[1] Arjas, E., Nummelin, E. and Tweedie, R.L. (1980) Semi-Markov processes ona general state space:α-theory and quasi-stationarity. *J. Austral. Math. Soc. Ser. A* **30**, 187–200.

[2] Asmussen, S.(1982) Conditioned limit theorems relating a random walk to its associate, with applications to risk reserve process and the GI/G/1 queue. *Adv. Appl. Probab.* **14**, 143–170.

[3] Asmussen, S. (2000) *Ruin Probabilities.* World Scientific, Singapore.

[4] Asmussen, S. (2003) *Applied Probability and Queues.* Second Ed., Springer, New York.

[5] Bertoin, J. (1996) *Lévy Processes.* Cambridge University Press, Cambridge.

[6] Bertoin, J. and Doney, R.A. (1994) On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22**, 2152–2167.

[7] Bingham, J., Goldie, C.M. and Teugels, J.L. (1987) *Regular Variation.* Cambridge University Press, Cambridge.

[8] Borodin, A.N. & Salminen, P. (1996) *Handbook of Brownian Motion – Facts and Formulae.* Birkhäuser, Basel.

[9] Davis, M.H.A. (1993) *Markov Models and Optimization.* Chapman and Hall, London.

[10] Doney, R.A. (1989) On the asymptotics of first passage times for transient random walks *Z. Wahrsch. Verw. Geb.* **81**, 239–246.

[11] Ethier, H.J. and Kurtz, T.G. (1986) *Markov processes. Characterization and Convergence.* John and Wiley and Sons, Inc., New York.

[12] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications.* Wiley, New York.

[13] Ferrari, P.A., Kesten, H., Martinez, S. and Picco, P. (1995) Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.* **23**, 501–521.

[14] Glynn, P. and Thorisson, H. (2001) Two-sided taboo limits for Markov processes and associated perfect simulation. *Stoch. Proc. Appl.* **91**, 1–20.

[15] Glynn, P. and Thorisson, H. (2002) Structural characterization of taboo-stationarity for general processes in two-sided time. *Stoch. Proc. Appl.* **102**, 311–318.

[16] Grabiner, D.J. (1999) Brownian motion in a Wayl chamber, non-colliding particles, and random matrices. *Ann. Inst. Henri Poincaré* **35**, 177–204.

[17] Jacka, S.D. and Roberts, G.O. (1995) Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.* **32**, 902–916.

[18] Karatzas, I. and Shreve, E.S. (1991) *Brownian motion and stochastic calculus.* Second Ed., Springer, New York.

[19] Keener, R.W. (1992) Limit theorems for randim walks conditioned to stay positive. *Ann. Probab.* **20**, 801–824.

[20] Kesten, H. (1995) A ratio limit theorem for (sub)Markov chains on $\{1, 2, \ldots\}$ with bounded jumps. *Adv. Appl. Prob.* **27**, 652–691.

[21] Knight, F.B. (1969) Brownian local times and taboo processes. *Trans. Amer. Math. Soc.* **143**, 173–185.

[22] König, W., O'Connell, N. and Roch, S. (2002) Non-colliding random walks, tandem queues and discrete ensembles. *Elect. J. Probab.* **7**, 1–24.

[23] Kyprianou, E.K. (1971) On the quasi-stationary distribution of he virtual waiting time in queues with Poisson arrivals. *J. Appl. Probab.* **8**, 494–507.

[24] Lambert, A. (2000) Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. H. Poincaré. Prob. et Stat.* **36**, 251–274.

[25] Martinez, S. and Martin, J.S. (1994) Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.* **31**, 911–920.

[26] Nair, M.G. and Pollett, P.K. (1993) On the relationship between μ-invariant measures and quasi-stationary distributions for contonuous-time Markov chains. *Adv. Appl. Probab.* **25(1)**, 82–102.

[27] Palmowski, Z. and Rolski, T. (2002) A technique for exponential change of measure for Markov processes. *Bernoulli* **8**, 767–785 .

[28] Pinsky, R.G. (1985) On the convergence of diffusion processes conditioned to remain in a bounded region for a large time to limiting positive recurrent diffusion processes. *Ann. Probab.* **13**, 363–378.

[29] Prabhu, N.U. (1980) *Stochastic Storage Processes.* Springer, New York.

[30] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999) *Stochastic Processes for Insurance and Finance.* Wiley, Chichester.

[31] Touminen, P. and Tweedie, R.L. (1979) Exponential decay and ergodicity of general Markov processes and their discrete skeletons. *Adv. Appl. Prob.* **11**, 784–803.