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The superposition of alternating on-off flows and a fluid model

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Abstract

An *on-off* process is a $0-1$ process ξ_t in which consecutive 0-periods $\{T_{0,n}\}$ alternate with 1-periods $\{T_{1,n}\}$ $(n = 1, 2, \ldots)$. The *on* and *off* time sequences are independent, each consisting of i.i.d. r.v.s. By the superposed flow we mean the process $Z_t = \sum_{\ell=1}^{N} r^\ell \xi_t^\ell$, where $r^\ell > 0$ and $\{\xi_t^1\}, \ldots, \{\xi_t^N\}$ are independent *on-off* flows. The process ξ_t^ℓ is not Markovian, however with the age component η_t^ℓ the process $w_t^\ell = (\xi_t^\ell, \eta_t^\ell)$ is a Piecewise Deterministic Markov process. In this paper we study the buffer content process for which the tail of its steady state distribution $\Psi(b)$ fulfills inequality $C_- e^{-\gamma b} \le \Psi(b) \le C_+ e^{-\gamma b}$, where $\gamma > 0$ is the solution of some basic non-linear system of equations.

Keywords: queueing fluid model, exponential bound, *on-off* flow, superposition of *on-off* flows, generator.

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1 Introduction

Fluid models with exponential *on* and *off* times were intensively studied by many authors beginning from a pioneering paper of Anick, Mitra & Sondhi (1982) in the homogeneous case and Stern & Elwalid (1991) for a nonhomogeneous case; see also a survey by Kulkarni (1995). In this paper we study fluid models with general *on* and *off* times and, under some assumptions, we derive exponential lower and upper bounds for the tail of the steady state distribution of the buffer content. Bensaou *et al* (1994) and Guibert (1992) studied fluid models using Beneš-Borovkov equation. However their results for nonexponential *on* and *off* times are computationally quite complex. Simonian & Guibert (1995) found approximations for the tail of the steady state distribution of the buffer content using large deviations techniques, however the form of the rate function derived in this paper can be only utilized numerically. Other references on related topics are Palmowski & Rolski (1996), Kulkarni (1994), Asmussen & Rubinstein (1995), Whitt (1993). Nonexponential estimates for the tail of the buffer content distribution were studied for instance by Heath *et al* (1996) and Jelenkovič & Lazar (1996).

By an *on-off* flow we mean a $0-1$ process ξ_t, in which consecutive 0-periods $\{T_{0,n}\}$ alternate with 1-periods $\{T_{1,n}\}$ $(n = 1, 2, \ldots)$. Random variables $\{T_{0,n}\}$ are *off* times and $\{T_{1,n}\}$ are *on* times. We assume that both sequences are independent, each consisting of i.i.d. random variables. In the paper we consider N flows $\xi_t^1, \ldots, \xi_t^N$. The flows are not supposed to be identically distributed and we denote the generic *on* and *off* times of the ℓth flow by T_1^ℓ and T_0^ℓ respectively $(\ell = 1, \ldots, N)$. If $\mathbb{E}(T_0^\ell + T_1^\ell) < \infty$, then there exists the (time) stationary *on-off* flow ξ_t^ℓ.

In the paper we consider the fluid model with input rate $Z_t = \sum_{\ell=1}^N r^\ell \xi_t^\ell$ and the constant output rate c, wherein the *buffer content process* X_t is governed by the equation

$$\frac{dX_t}{dt} = \begin{cases} Z_t - c, & \text{for } X_t > 0 \\ (Z_t - c)_+, & \text{for } X_t = 0\ . \end{cases}$$

It is known (see e.g. Kulkarni & Rolski (1994)) that if the stability condition

$$\sum_{\ell=1}^{N} p^\ell r^\ell < c \tag{1.1}$$

holds, where

$$p^\ell = \frac{\mathbb{E}T_1^\ell}{\mathbb{E}T_0^\ell + \mathbb{E}T_1^\ell}\ ,$$

then there exists $\Psi(b) = \lim_{t\to\infty} \mathbb{P}(X_t \geq b)$ and

$$\begin{aligned} \Psi(b) &= \mathbb{P}(\sup_{t\leq 0} \int_t^0 (Z_s - c)\, ds \geq b) \\ &= \mathbb{P}(\sup_{t\geq 0} \int_0^t (Z_s - c)\, ds \geq b). \end{aligned}$$

In this case $\lim_{b\to\infty} \Psi(b) = 0$ and we study how fast is the convergence of $\Psi(b)$ to 0. We also suppose that

$$c < \sum_{\ell=1}^{N} r^\ell, \tag{1.2}$$

otherwise the buffer is empty in the steady-state. Notice that p^ℓ is the steady-state probability that the ℓth flow is in the state *on* . The main result of this paper are two-sided exponential bounds for $\Psi(x)$. For the ℓth source ($\ell = 1, \ldots, N$), let F_1^ℓ denotes the distribution of T_1^ℓ and F_0^ℓ the distribution of T_0^ℓ and the corresponding moment generating functions are $\hat{F}_1^\ell$ and $\hat{F}_0^\ell$ respectively. We assume that the distributions are absolutely continuous with densities f_1^ℓ and f_0^ℓ and the corresponding hazard rate functions

$$r_1^\ell(x) = \frac{f_1^\ell(x)}{1 - F_1^\ell(x)}, \qquad r_0^\ell(x) = \frac{f_0^\ell(x)}{1 - F_0^\ell(x)}$$

respectively. If flows are identically distributed, then the superscript ℓ is omitted. Such flows are called homogeneous flows, in contrast with more general case of heterogeneous flows.

Throughout the paper we make the following assumption about the existence of the solution of a system of equations, which we call Basic System of Nonlinear Equations (BSNL). There exist $\gamma > 0$ and $c^\ell > p^\ell r^\ell$ ($\ell = 1, \ldots, N$) fulfilling the following system of equations:

$$\hat{F}_1^\ell(-\gamma(c^\ell - r^\ell))\hat{F}_0^\ell(-\gamma c^\ell) = 1, \qquad \ell = 1, \ldots, N \tag{1.3}$$

$$\sum_{\ell=1}^{N} c^\ell = c. \tag{1.4}$$

We call γ the adjustment coefficient and we discuss this system later in Section 5. It is clear that F_1^ℓ cannot be heavy-tailed and in Theorem 1.1 we impose even stronger conditions. For the homogeneous case it suffices to assume that there exists $\gamma > 0$ such that

$$\hat{F}_1(-\gamma(\frac{c}{N} - r))\hat{F}_0(-\gamma\frac{c}{N}) = 1. \tag{1.5}$$

The main result of the paper is the following theorem.

Theorem 1.1 *Suppose that*

$$\inf_{y\to\infty} \mathbb{E}\left[e^{-\gamma c^\ell (T_0^\ell - y)} | T_0^\ell > y\right] > 0 \tag{1.6}$$

and

$$\sup_{y\to\infty} \mathbb{E}\left[e^{\gamma(r^\ell-c^\ell)(T_1^\ell-y)}|T_1^\ell>y\right]<\infty \tag{1.7}$$

for $\ell = 1,\dots,N$. *Then for some constants* $0 < C_- \le C_+$

$$C_- e^{-\gamma x} \le \Psi(x) \le C_+ e^{-\gamma x}, \qquad x \ge 0.$$

A sufficient condition for (1.6) is

$$\liminf_{x\to\infty} r_0^\ell(x) = \rho_0^\ell > 0 \tag{1.8}$$

and for (1.7) is

$$\limsup_{x\to\infty} r_1^\ell(x) = \rho_1^\ell > 0 \qquad \text{and} \qquad \rho_1^\ell > \gamma(r^\ell - c^\ell). \tag{1.9}$$

The proof is given in Section 3, wherein we also give forms for C_- and C_+ in formulas (3.12) and (3.13) respectively. We give a comment on conditions (1.8) and (1.9) in Section 5.

For the proof of Theorem 1.1 we use a Markovian theory of *on-off* flows. Process ξ_t is not Markovian alone but it is Markovian with the supplementary age component η_t. Thus the process $w_t = (\xi_t, \eta_t)$ is a Markov process, which is a Piecewise Deterministic (PD) Markov process, a class introduced by Davis (1984), (1993). Moreover it is known the (extended) generator $\boldsymbol{Q}$ of the Markov process w_t. We recall needed concepts and results from PD processes in Section 2. In the proof we use a form of the exponential martingales and a perturbation theorem from the Appendix to define a new underlying probability measure. The change of measure techniques, called in large deviation theory twisting (see Schwartz & Weiss (1995)) is standard for such purposes (see e.g. Palmowski & Rolski (1996), Asmussen (1994)).

2 Process w_t

Formally we define the process $w_t = (\xi_t, \eta_t)$ as follows. Let $\tau_n,\ n = 1, 2, \ldots$ be the sequence of switch-over epochs. It is defined regarding the initial condition $w_0 = (\xi_0, \eta_0)$. If $\xi_0 = 0$ and $\eta_0 = y$, then τ_1 has distribution $F_0(dt + y)/\bar{F}_0(y)$ and it is independent of $\{T_{0,n}\}$, $\{T_{1,n}\}$. Then we define recursively

$$\tau_{n+1} = \begin{cases} \tau_n + T_{0,n/2} & \text{if } n \text{ is even} \\ \tau_n + T_{1,\lceil n/2 \rceil} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\xi_t = \begin{cases} 0 & \text{if } \tau_n \le t < \tau_{n+1} \text{ and } n \text{ is even} \\ 1 & \text{if } \tau_n \le t < \tau_{n+1} \text{ and } n \text{ is odd .} \end{cases}$$

Similarly if $\xi_0 = 1$ and $\eta_0 = y$, then τ_1 has distribution $F_1(dt + y)/\bar{F}_1(y)$ and it is independent of $\{T_{0,n}\}$, $\{T_{1,n}\}$ and we define recursively

$$\tau_{n+1} = \begin{cases} \tau_n + T_{1,n/2} & \text{if } n \text{ is even} \\ \tau_n + T_{0,\lceil n/2 \rceil} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\xi_t = \begin{cases} 1 & \text{if } \tau_n \le t < \tau_{n+1} \text{ and } n \text{ is even} \\ 0 & \text{if } \tau_n \le t < \tau_{n+1} \text{ and } n \text{ is odd .} \end{cases}$$

We define the supplementary age process by $\eta_t = t - \tau_n$ if $\tau_n \le t < \tau_{n+1}$. For all initial conditions $i = 0, 1$ and $y \ge 0$, the process w_t is Markov and we denote the underlying probability measure by $\mathbb{P}^{(i,y)}$. If $\mu(di, dy)$ is a probability on $\{0,1\} \times \mathbb{R}_+$, then we denote

$$\mathbb{P}_\mu = \int \mathbb{P}^{(i,y)} \mu(di, dy). \tag{2.1}$$

The process w_t is stationary if we choose

$$\begin{aligned} \mu(i, dy) &= \pi(i, dy) & (2.2)\\ &= \frac{1}{\mathbb{E}(T_0 + T_1)} \bar{F}_i(y)\, dy. & (2.3) \end{aligned}$$

Generator $\boldsymbol{Q}$ of w_t and its domain $\mathcal{D}(\boldsymbol{Q})$ are defined as follows. We denote by $C_b(\{0,1\} \times \mathbb{R}_+)$ all continuous and bounded functions $g : \{0,1\} \times \mathbb{R}_+ \to \mathbb{R}$. For $g, g^* \in C_b(\{0,1\} \times \mathbb{R}_+)$ we denote by

$$M_{g,g^*}(w_t) = g(\xi_t, \eta_t) - g(i, y) - \int_0^t g^*(\xi_s, \eta_s)\, ds, \qquad t \geq 0 \ .$$

We look for all functions $g, g^* \in C_b(\{0,1\} \times \mathbb{R}_+)$ such that $M_{g,g^*}(w_t)$, $t \geq 0$ is a $\mathbb{P}^{(i,y)}$-martingale for all (i, y) and then we denote this family of g by $\mathcal{D}(\boldsymbol{Q})$ and the mapping $\boldsymbol{Q} : g \to g^*$ we call a (full) generator. The result of Theorem 26.14 from Davies (1993) adapted to the process w_t says that for $i = 0, 1$:

$$(\boldsymbol{Q}g)(i, x) = \frac{\partial}{\partial x} g(i, x) + r_i(x)\left(g(1 - i, 0) - g(i, x)\right) \tag{2.4}$$

and $\mathcal{D}(\boldsymbol{Q})$ consists of all functions $g(i, y) \in C_b(\{0,1\} \times \mathbb{R}_+)$, such that $g(i, y)$ are absolute continuous on $[0, s_i^*)$, where $s_i^* = \inf\{t : F_i(t) = 1\}$.

Remark 2.1 Following Davis (1993), Remark 26.16 we can consider unbounded functions g. Then we have that $\mathcal{D}(\boldsymbol{Q})$ consist of all functions $g(i, y)$, such that $g(i, y)$ are absolute continuous on $[0, s_i^*)$, where $s_i^* = \inf\{t : F_i(t) = 1\}$ and $\mathbb{E}|g(i, T_i^\ell| < \infty$ ($\ell = 1, \ldots, N$; $i = 0, 1$).

3 Proof of the Theorem

In this section we show the proof of Theorem 1.1. We first prove it for $N = 1$ and then we show how to adapt the proof for the general case. In this case we do not write superscript ℓ. In the first lemma we consider the following system of differential equations:

$$(\boldsymbol{Q}h)(i, y) = \beta(ri - c)h(i, y), \qquad i = 0, 1. \tag{3.1}$$

Lemma 3.1 *The smallest $\beta < 0$ such that there exists $h \in \mathcal{D}(\boldsymbol{Q})$ and $\inf_{i,y} h(i, y) > 0$ fulfilling (3.1) is $\beta = -\gamma$. Then*

$$\begin{aligned} h(1, y) &= \frac{e^{-\gamma(r-c)y}}{\bar{F}_1(y)} \int_y^\infty e^{\gamma(r-c)z} f_1(z)\, dz \\ &= \mathbb{E}\left[e^{\gamma(r-c)(T_1 - y)} | T_1 > y\right] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} h(0, y) &= \hat{F}_1(-\gamma(c - r)) \frac{e^{\gamma c y}}{\bar{F}_0(y)} \int_y^\infty e^{-\gamma c z} f_0(z)\, dz \\ &= \hat{F}_1(-\gamma(c - r)) \mathbb{E}\left[e^{-\gamma c(T_0 - y)} | T_0 > y\right]. \end{aligned} \tag{3.3}$$

Proof. Solving system (3.1) with initial conditions we get

$$h(i, y) = \frac{e^{\beta(ir-c)y}}{\bar{F}_i(y)} \left[h(i, 0) - h(1 - i, 0) \int_0^y e^{-\beta(ir-c)z} f_i(z)\, dz\right], \qquad i = 0, 1, y \geq 0.$$

Since we want $\inf_{i,y} h(i,y) > 0$, therefore for all $i = 0, 1$ and $y \geq 0$

$$h(i,0) - h(1-i,0) \int_0^y e^{-\beta(ir-c)z} f_i(z)\, dz > 0 \ ,$$

from which, passing with $y \to \infty$, we obtain

$$h(i,0) - h(1-i,0) \int_0^\infty e^{-\beta(ir-c)z} f_i(z)\, dz \geq 0 \ .$$

Choosing $h(0,0) = 1$ we write

$$\begin{aligned} 1 - h(1,0)\hat{F}_0(\beta c) &\geq 0 \\ h(1,0) - \hat{F}_1(\beta(c-r)) &\geq 0 \end{aligned}$$

or

$$\hat{F}_1(\beta(c-r)) \leq h(1,0) \leq \hat{F}_0^{-1}(\beta c).$$

Therefore we look for the smallest negative β for which

$$\hat{F}_1(\beta(c-r)) \leq \hat{F}_0^{-1}(\beta c)$$

or $H(\beta) \leq 1$, where

$$H(x) = \hat{F}_1(x(c-r))\hat{F}_0(xc). \tag{3.4}$$

The function H is strictly logconvex as $\log H$ is a sum of two strictly convex functions $\log \hat{F}_i$, $(i = 0, 1)$ (see Kingman (1961)). Moreover it is equal to 1 at $x = 0$ and in view of assumption (1.1) its derivative at zero is positive. Therefore the function is strictly less than 1 in the interval $(-\gamma, 0)$ and bigger than 1 in $(-\infty, -\gamma)$. Thus

$h(1,0) = \hat{F}_1(-\gamma(c-r)) = \hat{F}_0^{-1}(-\gamma c)$ and hence

$$\begin{aligned} h(1,y) &= \frac{e^{-\gamma(r-c)y}}{\bar{F}_1(y)} \int_y^\infty e^{\gamma(r-c)z} f_1(z)\, dz\ , \\ h(0,y) &= \hat{F}_1(-\gamma(c-r)) \frac{e^{\gamma c y}}{\bar{F}_0(y)} \int_y^\infty e^{-\gamma c z} f_0(z)\, dz\ . \end{aligned}$$

We now show that $h \in \mathcal{D}(\boldsymbol{Q})$ and $\inf_{i,y} h(i,y) > 0$. Clearly functions $h(i,y)$ $(i=1,2)$ are absolute continuous and so it suffices to show that they are bounded away from 0 and ∞. Thus

$$\begin{aligned} h(1,y) &= \int_y^\infty \frac{e^{\gamma(r-c)(z-y)}}{\bar{F}_1(y)} f_1(z)\, dz \geq \int_y^\infty \frac{f_1(z)}{\bar{F}_1(y)}\, dz = 1 \\ h(0,y) &\leq \hat{F}_1(-\gamma(c-r)). \end{aligned}$$

Moreover $h(1,y)$ is bounded above by (1.7) and $h(0,y)$ is cut off zero by (1.6).

$\square$

Using (3.1) we have

$$(\boldsymbol{Q}h)\,(\xi_s,\eta_s) = -\gamma(r\xi_s - c)h(\xi_s,\eta_s).$$

From Proposition 3.2 of Ethier & Kurtz (1986) (recalled in the Appendix in Proposition A.1) we have that

$$M_t = \frac{h(w_t)}{h(w_0)} e^{-\int_0^t \frac{\boldsymbol{Q}h(w_s)}{h(w_s)}\, ds} = \frac{h(w_t)}{h(w_0)} e^{\gamma \int_0^t (Z_s - c)\, ds} \tag{3.5}$$

is a martingale. Define now a family of probability measures on $(\Omega, \mathcal{F}_t^w)$ by $d\tilde{\mathbb{P}}_t^{w_0} = M_t d\mathbb{P}_t^{w_0}$, where $\mathbb{P}_t^{w_0} = \mathbb{P}^{w_0}_{|\mathcal{F}_t^w}$ $(t \geq 0)$. Since $\{M_t, t \geq 0\}$ is a multiplicative functional of the process w_t from Kunita (1976) we get that this family defines exactly one probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \bigvee_{t\geq 0}\mathcal{F}^w_t)$, where $\bigvee_{t\geq 0}\mathcal{F}^w_t$ is the smallest σ-field generated by all elements of $\mathcal{F}_t$ $(t \geq 0)$ and w_t is also Markov process. By $\tilde{\mathbb{E}}$ we denote the corresponding to $\tilde{\mathbb{P}}$ expetation.

Lemma 3.2 *The process w_t on $(\Omega, \bigvee_{t\geq 0}\mathcal{F}^w_t, \tilde{\mathbb{P}})$ is an alternating on-off flow with on-times and off-times distributions $\tilde{F}_1$ and $\tilde{F}_0$ respectively with densities*

$$\tilde{f}_0(x) = \frac{e^{-\gamma cx}}{\hat{F}_0(-\gamma c)} f_0(x), \qquad \tilde{f}_1(x) = \frac{e^{-\gamma(c-r)x}}{\hat{F}_0(-\gamma(c-r))} f_1(x) \tag{3.6}$$

and

$$r\frac{\tilde{\mathbb{E}}T_1}{\tilde{\mathbb{E}}(T_1+T_0)} - c > 0. \tag{3.7}$$

Proof. From the perturbation theorem from the Appendix

$$\begin{aligned}
(\tilde{\boldsymbol{Q}}g)(i,y) &= \\
&= \frac{\partial g(i,y)}{\partial y} + \frac{g(i,y)}{h(i,y)}\frac{\partial h(i,y)}{\partial y} + \gamma(ri-c)g(i,y) + r_i(y)\left[\frac{h(1-i,0)}{h(i,y)}g(1-i,0) - g(i,y)\right] \\
&= \frac{\partial g(i,y)}{\partial y} + \tilde{r}_i(y)\left[g(1-i,0) - g(i,y)\right],
\end{aligned}$$

where

$$\tilde{r}_0(y) = \frac{e^{-\gamma cy} f_0(y)}{\int_y^\infty e^{-\gamma cz} f_0(z)\,dz}, \qquad \tilde{r}_1(y) = \frac{e^{-\gamma(c-r)y} f_1(y)}{\int_y^\infty e^{-\gamma(c-r)z} f_1(z)\,dz}. \tag{3.8}$$

□

From (2.4) we get (3.6). We now compute that

$$\tilde{\mathbb{E}}T_1 = \frac{\int_0^\infty z e^{-\gamma(c-r)z} f_1(z)\,dz}{\hat{F}_1(-\gamma(c-r))} = \frac{\hat{F}_1'(-\gamma(c-r))}{\hat{F}_1(-\gamma(c-r))},$$

$$\tilde{\mathbb{E}}T_0 = \frac{\int_0^\infty z e^{-\gamma c z} f_0(z)\, dz}{\hat{F}_0(-\gamma c)} = \frac{\hat{F}_0'(-\gamma c))}{\hat{F}_0(-\gamma c)}.$$

To demonstrate that the new drift is positive we compute

$$\begin{aligned} r\frac{\tilde{\mathbb{E}}T_1}{\tilde{\mathbb{E}}T_1+\tilde{\mathbb{E}}T_0} - c &= r\frac{\hat{F}_1'(-\gamma(c-r))\hat{F}_0(-\gamma c)}{\hat{F}_1'(-\gamma(c-r))\hat{F}_0(-\gamma c)+\hat{F}_0'(-\gamma c)\hat{F}_1(-\gamma(r-c))} - c \\ &= -\frac{H'(-\gamma)}{\hat{F}_1'(-\gamma(c-r))\hat{F}_0(-\gamma c)+\hat{F}_0'(-\gamma c)\hat{F}_1(-\gamma(r-c))}. \end{aligned}$$

The derivative of function H defined in (3.4) at $x = -\gamma$ is clearly negative while the denominator is positive. Therefore the above expression is positive.

$\square$

We are now ready to give the proof of Theorem 1.1 for $N = 1$. Let $\tau(b) = \min\{t \geq 0 : \int_0^t (r\xi_s - c)\, ds \geq b\}$. Thus from $d\mathbb{P}_t = M_t^{-1} d\tilde{\mathbb{P}}_t$, following the idea from Asmussen (1994), using the fact that by Lemma 3.2 stopping time $\tau(b)$ is $\tilde{\mathbb{P}}^{(i,y)}$-a.s. finite and the strong Markov property we can write

$$\begin{aligned} \Psi_{(i,y)}(b) &= \mathbb{P}^{(i,y)}(\tau(b) < \infty) \\ &= \tilde{\mathbb{E}}_{(i,y)}\left[\frac{h(i,y)}{h(w_{\tau(b)})} e^{\int_0^{\tau(b)} \frac{\boldsymbol{Q}h(w_s)}{h(w_s)}\, ds}; \tau(b) < \infty\right] \\ &= \tilde{\mathbb{E}}_{(i,y)}\left[\frac{h(i,y)}{h(w_{\tau(b)})}\right] e^{-\gamma b}. \end{aligned} \tag{3.9}$$

Hence, by (2.1)

$$\Psi(b) = e^{-\gamma b}\frac{1}{\mathbb{E}(T_0+T_1)} \sum_{i=0}^{1} \int_0^\infty \bar{F}_i(y) h(i,y) \tilde{\mathbb{E}}_{(i,y)}\left[\frac{1}{h(\xi_{\tau(b)}, \eta_{\tau(b)})}\right] dy. \tag{3.10}$$

Lemma 3.3 *For $N=1$ constants C_+ and C_- from Theorem 1.1 are*

$$C_+ = \frac{1}{\mathbb{E}(T_0+T_1)}\frac{1}{\inf_{i,y} h(i,y)}\frac{r}{c}\frac{\hat{F}_1(\gamma(r-c))-1}{\gamma(r-c)}$$

and

$$C_- = \frac{1}{\mathbb{E}(T_0+T_1)}\frac{1}{\sup_{i,y} h(i,y)}\frac{r}{c}\frac{\hat{F}_1(\gamma(r-c))-1}{\gamma(r-c)}.$$

Proof. For deriving C_+ we write

$$\begin{aligned}
\Psi(b) &= e^{-\gamma b}\sum_{i=0}^{1}\frac{\mathbb{E}T_i}{\mathbb{E}(T_0+\mathbb{E}T_1)}\int_0^\infty \frac{\bar{F}_i(y)\,dy}{\mathbb{E}T_i}h(i,y)\tilde{\mathbb{E}}_{(i,y)}\left[\frac{1}{h(\xi_{\tau(b)},\eta_{\tau(b)})}\right]\\
&\le e^{-\gamma b}\frac{1}{\mathbb{E}(T_0+T_1)}\frac{1}{\inf_{i,y}h(i,y)}\times\\
&\times\left(\int_0^\infty e^{-\gamma(r-c)y}\,dy\int_y^\infty e^{\gamma(r-c)z}f_1(z)\,dz+\hat{F}_1(\gamma(r-c))\int_0^\infty e^{\gamma cy}\,dy\int_y^\infty e^{-\gamma cz}f_0(z)\,dz\right)\\
&= e^{-\gamma b}\frac{1}{\mathbb{E}(T_0+T_1)}\frac{1}{\inf_{i,y}h(i,y)}\left(\frac{\hat{F}_1(\gamma(r-c))-1}{\gamma(r-c)}+\frac{\hat{F}_1(\gamma(r-c))(1-\hat{F}_0(-\gamma c))}{\gamma c}\right)\\
&= e^{-\gamma b}\frac{1}{\mathbb{E}(T_0+T_1)}\frac{1}{\inf_{i,y}h(i,y)}\left(\frac{\hat{F}_1(\gamma(r-c))-1}{\gamma}\right)\left(\frac{1}{r-c}+\frac{1}{c}\right).
\end{aligned}$$

Similar derivations prove C_-.

□

We now consider general N. Consider Markov process $W_t=(\boldsymbol{\xi}_t,\boldsymbol{\eta}_t)$. We make it stationary if the initial distribution is

$$\pi(\boldsymbol{i},d\boldsymbol{y})=\prod_{\ell=1}^{N}\pi^\ell(i^\ell,dy^\ell)\ ,$$

where π^ℓ is defined in (2.3) for the ℓ-th flow. Under probability measure $\mathbb{P}_\pi=\int\mathbb{P}^{(\boldsymbol{i},\boldsymbol{y})}\,d\pi$ the process $Z_t=\sum_{\ell=1}^N r^\ell\xi_t^\ell$ is the input rate and Z_t-c is the net rate. Let $\tau(b)=\min\{t\ge 0:\int_0^t(Z_s-c)\,ds\ge 0\}$ and $\boldsymbol{h}(\boldsymbol{i},\boldsymbol{y})=\prod_{\ell=1}^N h^\ell(i^\ell,y^\ell)$. Following (3.5) each process

$$M_t^\ell(c^\ell) = \frac{h^\ell(w_t^\ell)}{h^\ell(w_0^\ell)} e^{\gamma \int_0^t (r^\ell \xi_s^\ell - c^\ell)\, ds}, \qquad t \geq 0$$

is a $(\mathbb{P}^{(i^\ell, y^\ell)}, \mathcal{F}_t^{w^\ell})$-martingale provided the ℓth equation of BSNL (1.3) holds. Consider now

$$M_t = \prod_{\ell=1}^N M_t^\ell(c^\ell), \qquad t \geq 0 \ .$$

Since $\sum_{\ell=1}^N c^\ell = c$

$$M_t = \frac{\boldsymbol{h}(W_t)}{\boldsymbol{h}(\boldsymbol{i}, \boldsymbol{y})} e^{\gamma \int_0^t (Z_s - c)\, ds}$$

and M_t is a $(\mathbb{P}^{(\boldsymbol{i}, \boldsymbol{y})}, \mathcal{F}_t^W)$-martingale. Similarly as for $N = 1$ we define new probability measure $\tilde{\mathbb{P}}^{(\boldsymbol{i}, \boldsymbol{y})}$ by $d\tilde{\mathbb{P}}_t^{(\boldsymbol{i}, \boldsymbol{y})} = M_t d\mathbb{P}_t^{(\boldsymbol{i}, \boldsymbol{y})}$ on $(\Omega, \mathcal{F}_t^W)$, where now $\mathcal{F}_t^W$ is the history of W_t up to t. The extension of (3.10) for general N is

$$\Psi(b) = e^{-\gamma b} \prod_{\ell=1}^N \frac{1}{\mathbb{E}(T_0^\ell + T_1^\ell)} \sum_{\boldsymbol{i} \in 2^{\{0,1\}}} \int_{\mathbb{R}_+^N} \prod_{\ell=1}^N \bar{F}_{i^\ell}(y^\ell) h^\ell(i^\ell, y^\ell) \tilde{\mathbb{E}}_{(\boldsymbol{i}, \boldsymbol{y})} \left[\frac{1}{\boldsymbol{h}(\boldsymbol{\xi}_{\tau(b)}, \boldsymbol{\eta}_{\tau(b)})} \right] d\boldsymbol{y} \tag{3.11}$$

and hence for the upper bound we have

$$\Psi(b) \leq \prod_{\ell=1}^N \frac{1}{\mathbb{E}(T_0^\ell + T_1^\ell)} \left(\sum_{i^\ell=0}^1 \int_0^\infty \bar{F}_{i^\ell}(y^\ell) h^\ell(i^\ell, y^\ell)\, dy^\ell \right) \frac{1}{\inf_{i,y} h^\ell(i, y)}.$$

The proof of the lower bound is similar. Hence we obtain

$$\prod_{\ell=1}^N C_-^\ell e^{-\gamma b} \leq \Psi(b) \leq \prod_{\ell=1}^N C_+^\ell e^{-\gamma b}, \qquad b \geq 0 \ ,$$

where

$$C_+^\ell = \frac{1}{\inf_{i,y} h^\ell(i, y)} \left(\frac{1}{\mathbb{E}(T_0^\ell + T_1^\ell)} \frac{r^\ell}{c^\ell} \frac{\hat{F}_1^\ell(\gamma(r^\ell - c^\ell)) - 1}{\gamma(r^\ell - c^\ell)} \right) \tag{3.12}$$

and

$$C_-^\ell = \frac{1}{\sup_{i,y} h^\ell(i,y)} \left(\frac{1}{\mathbb{E}(T_0^\ell + T_1^\ell)} \frac{r^\ell}{c^\ell} \frac{\hat{F}_1^\ell(\gamma(r^\ell - c^\ell)) - 1}{\gamma(r^\ell - c^\ell)} \right) . \tag{3.13}$$

4 Gaussian asymptotics

In this Section we find the asymptotics for γ_N in the homogeneous *on-off* model parametrized by the number of sources N with F_0 and F_1 fixed, input rate r_N and output rate c_N. Let for some $c, r > 0$

$$r_N = \frac{r}{\sqrt{N}} , \qquad c_N = c + pr\sqrt{N} . \tag{4.1}$$

Under some mild conditions (see Szczotka (1980)) the sequence of processes

$$\tilde{Z}_N(t) = \frac{\Sigma_{i=1}^N \xi_i(t) - Np}{\sqrt{N}}$$

considered as random elements on $D[0,\infty)$ converges in distribution to a Gaussian process $\{\tilde{Z}(t)\}$ with mean 0 and covariance function $R(t)$ of the process $\{\xi_i(t)\}$. It can be proved (see Kulkarni & Rolski (1994)) that in this case under conditions (4.1)

$$X_N \stackrel{\mathcal{D}}{\to} X,$$

where X_N, X are respectively the steady-state buffer contents in the Nth *on-off* model and in the fluid model driven by the Gaussian process $\{r\tilde{Z}(t)\}$. For the Gaussian fluid model with net rate $\{r\tilde{Z}(t) - c\}$ Dębicki & Rolski (1995) show

$$\mathbb{P}(X > b) \leq Ce^{-\gamma b} + o(\exp(-\gamma x)) ,$$

where constant C is given explicitly and

$$\gamma = \frac{c}{r^2 \int_0^\infty R(t)\, dt} \ . \tag{4.2}$$

It can be demonstrated (see e.g. Kopociński (1973), p. 294 or Kopociński (1967)) that for $\{\xi_i(t)\}$ and hence also for $\{\tilde{Z}(t)\}$

$$\int_0^\infty R(t)\, dt = \mu_{01}^2(\mu_{12} - \mu_{11}^2) + \mu_{11}^2(\mu_{02} - \mu_{01}^2) \ , \tag{4.3}$$

where

$$\mu_{0i} = \int_0^\infty x^i\, dF_0(x) \ , \qquad \mu_{1i} = \int_0^\infty x^i\, dF_1(x) \ .$$

In the following proposition we demonstrate that from Theorem 1.1 straightforward computations lead us to exactly the same limit.

Proposition 4.1 *Suppose that the third moments of T_0 and T_1 are finite. If γ_N and γ are given by (1.5) and (4.2) respectively and (4.1) holds, then $\gamma_N \to \gamma$ as $N \to \infty$.*

Proof. Applying Taylor formula to

$$\hat{F}_1(-\gamma_N(\frac{c_N}{N} - r_N))\hat{F}_0(-\gamma_N\frac{c_N}{N}) = 1$$

we get

$$\left(1 + a_1(N)\gamma_N + \frac{1}{2}a_2(N)\gamma_N^2 + o(N^{-1})\right)\left(1 + b_1(N)\gamma_N + \frac{1}{2}b_2(N)\gamma_N^2 + o(N^{-1})\right) = 1 \ ,$$

where

$$a_i(N) = \mu_{0i}\left(\frac{c_N}{N}\right)^i \ , \qquad b_i(N) = \mu_{1i}\left(\frac{c_N}{N} - r_N\right)^i$$

which is equivalent by (4.1) to

$$(\mu_{01}+\mu_{11})\gamma_N c - \frac{\gamma_N^2 r^2}{2(\mu_{01}+\mu_{11})^2}(\mu_{01}^2\mu_{12}+\mu_{02}\mu_{11}^2-2\mu_{01}^2\mu_{11}^2)+o(N^{-1})=0\ .$$

Hence by (4.3)

$$\gamma_N \to \frac{c}{r^2}\Big[\mu_{01}^2(\mu_{12}-\mu_{11}^2)+\mu_{11}^2(\mu_{02}-\mu_{01}^2)\Big]^{-1} = \frac{c}{r^2\int_0^\infty R(t)\,dt}\ .$$

□

5 System of nonlinear equations

In this Section we consider the BSNL in the form: for some $c^1,\ldots,c^N>0$ such that $\sum_{\ell=1}^N c^\ell = c$ and $\gamma>0$

$$\hat F_1^\ell(-\gamma(c^\ell - r^\ell))\hat F_0^\ell(-\gamma c^\ell)=1, \qquad \ell=1,\ldots,N. \tag{5.1}$$

Since $c^\ell < r^\ell$, from Jensen inequality we have

$$e^{-\gamma((c^\ell-r^\ell)\mathbf{E}T_1+c\mathbf{E}T_0)} < \mathbb{E}e^{-\gamma((c^\ell-r^\ell)T_1+c^\ell T_0)}=1\ .$$

Hence $-\gamma((c^\ell-r^\ell)\mathbb{E}T_1+c^\ell\mathbb{E}T_0)<0$, which yields $c^\ell > p^\ell r^\ell$, explaining the assumption for c^ℓ made in Section 1.

Consider first the homogeneous case with single equation

$$H(-\gamma)=\hat F_1(-\gamma(c-r))\hat F_0(-\gamma c)=1. \tag{5.2}$$

This equation appeared in Asmussen & Rubinstein (1995) in the context of optimal change of measure. Clearly, if $\lim_{s\to s^*} \hat{F}_1(s) = \infty$ for some $0 \le s^* \le \infty$, then (5.2) has the unique solution, because in this case for continuous, convex function H we have $\lim_{x\to x_0} H(x) = \infty$ for some $-\infty \le x_0 \le 0$. We point out that this is fulfilled for the large class of phase-type distributions (for the definition of phase-type distributions we refer to Neuts (1981)). Indeed, let T_1 be phase-type with representation $(\boldsymbol{\alpha}, \boldsymbol{T})$, where $\boldsymbol{\alpha}$ is a probability vector and $\boldsymbol{T}$ is a transient intensity matrix. Then

$$\bar{F}_1(x) = \boldsymbol{\alpha} \exp(x\boldsymbol{T})\boldsymbol{e} \ , \tag{5.3}$$

$$\hat{F}_1(s) = -\boldsymbol{\alpha}(s\boldsymbol{I} + \boldsymbol{T})^{-1}\boldsymbol{t}^{\circ} \ , \tag{5.4}$$

$$f_1(x) = \boldsymbol{\alpha} \exp(x\boldsymbol{T})\boldsymbol{t}^{\circ} \ , \tag{5.5}$$

where $\boldsymbol{e}$ is a column vector consisting of 1's and $\boldsymbol{t}^{\circ} = -\boldsymbol{T}\boldsymbol{e}$. We assume that $\boldsymbol{T}$ is a subintensity matrix (off-diagonal entries are nonnegative and at least one row sums up to a strictly negative number) and hence the Perron-Frobenius eigenvalue λ_1 is negative and for all remaining eigenvalues $\Re(\lambda_j) < \Re(\lambda_1)$ $(j = 2, \ldots, n)$. Hence from (5.4) we get that $\lim_{s\to -\lambda_1} \hat{F}_1(s) = \infty$. Moreover, for phase-type distributions conditions (1.8) and (1.9) are also fulfilled. In fact, let F_1 be phase-type. Then from equation (1.3) and (5.4) we get

$$\gamma\,(c - r) > \lambda_1 \ . \tag{5.6}$$

Suppose that $\boldsymbol{\phi}_i$ and $\boldsymbol{\xi}_i$ are the left and right eigenvector corresponding to λ_i respectively. If moreover the eigenvectors $\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n\}$ are independent (for instance in the case when eigenvalues of $\boldsymbol{T}$ are distinct), then

$$\exp(x\boldsymbol{T}) = \sum_{i=1}^{n} e^{\lambda_i x} \boldsymbol{\xi}_i \boldsymbol{\phi}_i \ .$$

Using this representation, inequalities (5.3), (5.5) and definition of vector $\boldsymbol{t}^{\circ}$

$$\lim_{x \to \infty} r_1(x) = \frac{\boldsymbol{\alpha \xi}_1 \boldsymbol{\phi}_0 \boldsymbol{t}^{\circ}}{\boldsymbol{\alpha \xi}_1 \boldsymbol{\phi}_0 \boldsymbol{e}} = -\lambda_1 \ .$$

Hence condition (1.9) follows from (5.6). Similarly we can show that for phase-type distribution F_0 condition (1.8) is fulfilled. Thus all assumptions of Theorem 1.1 are fulfilled if *on* and *off* times are phase-type.

Sufficient conditions for the unique existence of the general BSNL are given in the following proposition:

Proposition 5.1 *Let* $\lim_{s \to s_\ell^*} \hat{F}_1^\ell = \infty$ *for some* $0 \le s_\ell^* \le \infty$ *(* $\ell = 1, \ldots, N$*). Then there exist the unique* $\gamma > 0, c^1 > p^1 r^1, \ldots, c^N > p^N r^N$ *solving BSNL (1.3).*

Proof. We first recall that $H^\ell(x, c^\ell) = \hat{F}_1^\ell(x(c^\ell - r^\ell))\hat{F}_0(xc^\ell)$ $(\ell = 1, \ldots, N)$ are continuous and strictly convex functions and define function $\kappa^\ell(c^\ell)$ by equation $H^\ell(-\kappa^\ell(c^\ell), c^\ell) = 1$. Each function $H^\ell(x, c^\ell)$ is decreasing with respect c^ℓ. Hence $\kappa^\ell(c^\ell)$ is a continuous increasing function. Moreover, the following argument shows that $\kappa^\ell(c^\ell) \to 0$ as $c^\ell \searrow p^\ell r^\ell$. If $c^\ell \searrow p^\ell r^\ell$, then $\frac{d}{dx} H^\ell(x, c^\ell)|_{x=0} \to 0$. Supposing that for some $\kappa_0^\ell > 0$ there is $\kappa^\ell(c^\ell) \to \kappa_0^\ell$ as $c^\ell \searrow p^\ell r^\ell$, we would have $H^\ell(x, p^\ell r^\ell) = 1$ for $-\kappa_0^\ell < x \le 0$, which is impossible, because each $H^\ell(x, c^\ell)$ is strictly convex. Therefore the graph of function H^ℓ moves at level 1 from 0 to certain constant as c^ℓ move from $p^\ell r^\ell$ to c. Thus there exist constants $c^1 > p^1 r^1, \ldots, c^N > p^N r^N$ and $\sum_{\ell=1}^{N} c^\ell = c$ such that these graphs have one common point $-\gamma$ at level 1, which is the solution of BSNL.

$\square$

Similarly as for the one source case we get the following corollary:

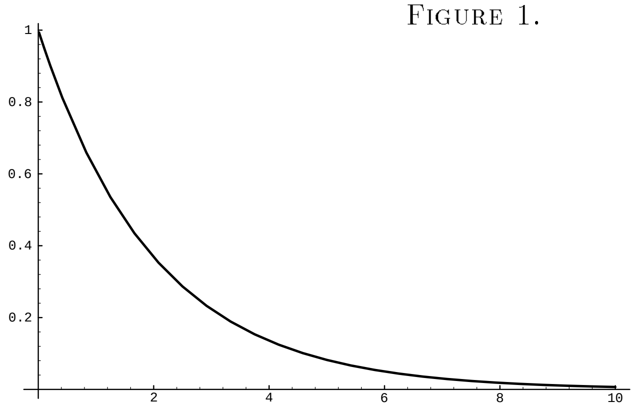
Corollary 5.1 *If F_0^ℓ and F_1^ℓ ($\ell = 1, \ldots, N$) are phase-type distributions, then there exist the unique solution of BSNL. Moreover, conditions (1.8) and (1.9) hold.*

However, there are cases when the system has no solutions as the following example shows. The simple case is when F_0 and F_1 are degenerated at $\mathbb{E}T_0$ and $\mathbb{E}T_1$ respectively. Then $e^{-\gamma((c-r)\mathbf{E}T_1 + c\mathbf{E}T_0)} = 1$ yields $-\gamma((c-r)\mathbb{E}T_1 + c\mathbb{E}T_0) = 0$, which is possible only if $c = rp$. However this is impossible because $c > rp$. The explanation of this fact is that for the periodic *on-off* stream the buffer content process is clearly bounded and therefore exponential bounds are too strong. The above example can be easily extended to nondegenerated random variables taking for the F_1 and F_0 uniform distribution over $(\mathbb{E}T_i - \epsilon, \mathbb{E}T_i + \epsilon)$ $(i = 0, 1)$ for sufficiently small $\epsilon > 0$. Then equation (5.2) reads

$$H_\epsilon(-\gamma) = e^{-\gamma((c-r)\mathbf{E}T_1 + c\mathbf{E}T_0)} = \frac{\sinh(-\gamma\epsilon(c-r))}{-\gamma\epsilon(c-r)} \frac{\sinh(-\gamma\epsilon c)}{-\gamma\epsilon c} = 1.$$

However the left hand side is strictly less than 1 for all positive $\gamma > 0$; see Fig.1.

FIGURE 1.



Consider now the case when there exist groups of size $N_1, \ldots, N_m$. Within groups the input rate is $r_i > 0$ and $F_{1,i}$, $F_{0,i}$ are *on* and *off* time distributions respectively. Namely

$$\begin{aligned} r^1 &= \ldots = r^{N_1} = r_1, \\ r^{N_1+1} &= \ldots = r^{N_1+N_2} = r_2, \\ \vdots &= \vdots \\ r^{N_1+\ldots+N_{m-1}+1} &= \ldots = r^{N_1+\ldots+N_m} = r_m \end{aligned}$$

and for $i = 0, 1$

$$\begin{aligned} F_i^1 &= \ldots = F_i^{N_1} = F_{i,1}, \\ F_i^{N_1+1} &= \ldots = F_i^{N_1+N_2} = F_{i,2}, \\ \vdots &= \vdots \\ F_i^{N_1+\ldots+N_{m-1}+1} &= \ldots = F_i^{N_1+\ldots+N_m} = F_{i,m} \ . \end{aligned}$$

Then the BSNL is reduced to the following system: there exists for some $c_1, \ldots, c_m > 0$ such that $\sum_{j=1}^{m} N_j c_j = c$ and $\gamma > 0$

$$\hat{F}_{1,j}(-\gamma(\frac{c_j}{N_j} - r_j))\hat{F}_{0,j}(-\gamma\frac{c_j}{N_j}) = 1, \qquad j = 1, \ldots, m. \tag{5.7}$$

To compare different fluid models we made some numerical experiments. To simplify calculations we consider only the homogeneous case. Following Anick, Mitra & Sondhi (1982) we take $r_N = 1$ and $T_0 \sim Exp(0.4)$ and $c_N = 16.666$ for $N = 30, 50$, $c_N = 33.333$ for $N = 85, 100$, $c_N = 66.666$ for $N = 150, 200$. In the first case T_1 has exponential distribution; $T_1 \sim Exp(1)$, in the second case T_1 has Erlang distribution Erl(2,2) and in the third case T_1 has hyperexponential distribution $pExp(2) + (1-p)Exp(\frac{2-2p}{2-p})$, where $p = 0.1$. In all cases $ET_1 = 1$. It is easy to verify that in the model considered by Anick, Mitra & Sondhi (both T_0 and T_1 have exponential distribution) equation (1.5) has the same solution as it was given by Anick, Mitra & Sondhi (1982) or recently by Palmowski & Rolski (1996).

NUMERICAL COMPARISONS OF γ.

	$c_N = 16.666$		$c_N = 33.333$		$c_N = 66.666$	
T_1	N= 30	N=50	N=85	N=100	N=150	N=200
Exp	1.5299	0.2999	0.6251	0.3	0.9	0.3
Erl	2.304	0.41	0.8786	0.41	1.294	0.41
Hyperexp	1.4677	0.2909	0.6043	0.291	0.8677	0.291

We can justify the order $\gamma_{\text{Hyper}} < \gamma_{\text{Exp}} < \gamma_{\text{Erl}}$ as follows. Let $F_1^* \sim \text{Erl}(2,2)$, $F_1 \sim \text{Exp}(1)$ and $F_0 \sim \text{Exp}(0.4)$. Since $\text{Erl}(2,2)$ is IFR (see Szekli (1995), p.17,), so $\mathrm{Erl}(2,2) \prec_{icx} \mathrm{Exp}(1)$. Thus

$$\hat{F}_1^*(-\gamma_{\mathrm{Exp}}(\frac{c}{N}-r))\hat{F}_0(-\gamma_{\mathrm{Exp}}\frac{c}{N}) \leq \hat{F}_1(-\gamma_{\mathrm{Exp}}(\frac{c}{N}-r))\hat{F}_0(-\gamma_{\mathrm{Exp}}\frac{c}{N}) = 1$$

and hence $\gamma_{\mathrm{Erl}} \geq \gamma_{\mathrm{Exp}}$. Similar argument together with Theorem E from Szekli (1994), p.17 proves the first inequality.

A Appendix

We now recall a theorem from theory of Markov processes. The result was used by Ethier & Kurtz (1986) and Fukushima & Stroock (1986). For the proof we refer also to Palmowski (1996). Let $\{X_t,\ t \geq 0\}$ be a E-valued Markov process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in\mathbb{R}})$, where E is a Polish space. Here $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration and let $\mathcal{F} = \bigvee_{t\geq 0}\mathcal{F}_t$. We assume that realization of X are cádlág. By $\mathcal{C}_b$ we denote the class of bounded and continuous functions $g : E \to \mathbb{R}$. We consider now the class $\mathcal{D}(\boldsymbol{A})$ of all functions $g \in \mathcal{C}_b$ such that for some function $g^* \in \mathcal{C}_b$ the process

$$M_g(t) = g(X_t) - g(x) - \int_0^t g^*(X_s)\,ds, \qquad t \geq 0$$

is a $\mathbb{P}_x$-martingale. The mapping $g \to g^*$ defines full generator of the Markov process X_t and we denote it by $g^* = \boldsymbol{A}g$. The class $\mathcal{D}(\boldsymbol{A})$ is called the domain of the full generator $\boldsymbol{A}$. For $g_1, g_2 \in \mathcal{D}(\boldsymbol{A})$ denote

$$< g_1, g_2 >_A (x) = (\boldsymbol{A}g_1g_2)(x) - g_1(x)(\boldsymbol{A}g_2)(x) - g_2(x)(\boldsymbol{A}g_1)(x) \ ,$$

provided $g_1 g_2 \in D(\mathcal{A})$. The following result was proved in Ethier & Kurtz (1986), page 175.

Proposition A.1 *If* $h \in \mathcal{D}(\boldsymbol{A})$ *and* $\inf_x h(x) > 0$, *then*

$$M_t = \frac{h(X_t)}{h(x)} e^{-\int_0^t \frac{\boldsymbol{A}h(X_s)}{h(X_s)}\,ds}, \qquad t \geq 0$$

is a $\mathbb{P}_x$*-martingale.*

Define now a new probability on $(\Omega, \mathcal{F})$ as follows. Let $\mathbb{P}^t$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_t$ and define on $\mathcal{F}_t$ probability $\tilde{\mathbb{P}}_x^t = M_t d\mathbb{P}_x^t$ for all $t \geq 0$. Since M_t is a martingale the family of probabilities $\{\tilde{\mathbb{P}}_x^t,\ t \geq 0\}$ is consistent. From Daniel-Kolmogorov theorem we get that there exists the unique probability $\tilde{\mathbb{P}}_x$ such that for each $t \geq 0$ the restriction of $\tilde{\mathbb{P}}_x$ to $\mathcal{F}_t$ is $\tilde{\mathbb{P}}_x^t$, $x \in \mathbb{R}$. Moreover, from Kunita (1976), the process $X(t)$ is a Markov process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\tilde{\mathbb{P}}_x\}_{x\in E})$.

Proposition A.2 *Let* $h \in \mathcal{D}(\boldsymbol{A})$ *be such that* $gh \in \mathcal{D}(\boldsymbol{A})$ *for all* $g \in \mathcal{D}(\boldsymbol{A})$ *and* $\inf_x h(x) > 0$. *Then the process* X *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\tilde{\mathbb{P}}_x\}_{x\in E})$ *is a Markov family with generator*

$$\tilde{\boldsymbol{A}}g(x) = \boldsymbol{A}g(x) + \frac{< h, g >_A (x)}{h(x)}$$

and the domain $\mathcal{D}(\tilde{\boldsymbol{A}}) = \mathcal{D}(\boldsymbol{A})$.

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