SIMULATION OF THE ASYMPTOTIC CONSTANT IN SOME FLUID MODELS

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ABSTRACT

Let $Z(t)$ be a stationary centered Gaussian process with a Markovian structure. In some fluid models, the stationary buffer content V can be expressed as $\sup_{t\geq 0}(\int_0^t Z(s)\,ds - ct)$ and $\mathbb{P}(V>u) = Ce^{-\gamma u}(1+o(1))$. The asymptotic constant C can be expressed by the so called generalized Pickands constants $\mathcal{H}$. In most cases no formula or approximation for C are known. In this paper we show a method of simulation of C by the use of change of measure technique. The method is applicable when $Z(t)$ is a stationary Ornstein-Uhlenbeck process or $Z(t) = \sum_{j=1}^n X_j(t)$, where $(X_1(t),\ldots,X_n(t))$ is a Gauss-Markov process. Two examples of simulations are included. Moreover we give a formula for a lower bound for generalized Pickands constants.

Keywords: asymptotic constant, simulation, change of measure, Gauss-Markov process, importance sampling, generalized Pickands constant, Ornstein-Uhlenbeck process.

*Research supported by KBN under grant 5 P03A 021 20 (2001–2003).
†Research supported by KBN under grant 1 H02B 002 18 (2000–2001).
‡Research supported in part by KBN under grant 2 P03A 049 15 (1998–2001).

1 INTRODUCTION

Over the past few decades fluid models have found to be successfully used in modeling and analysis of high-speed telecommunication networks. In the classical setting, the dynamics of the buffer content process $\{V(t),\ t \geq 0\}$ in fluid models has the following form

$$\frac{dV(t)}{dt} = \begin{cases} Z(t) - ct, & \text{if } V(t) > 0 \\ (Z(t) - ct)_+, & \text{if } V(t) = 0, \end{cases} \tag{1}$$

where process $Z(t)$ can be interpreted as the input rate at time t into the buffer and $c > 0$ as the constant output rate.

In the seminal paper of Anick, Mitra & Sondhi [1] it was assumed that $Z(t) \equiv \sum_{i=1}^{N} Z_i(t)$, where $Z_1(t), ..., Z_N(t)$ are independent stationary homogeneous Markovian *on-off* processes. Subsequent work extended AMS model, allowing the superposition of heterogeneous sources (e.g. [13]) or more general *on-off* sources (e.g. [18]).

Gaussian fluid models appear as an approximation of appropriately scaled fluid models with the input rate process being the superposition of large number of independent *on-off* processes. For example Kulkarni and Rolski [14] showed that the fluid model with $Z(t)$ being a stationary Ornstein-Uhlenbeck process appears as the limit of AMS fluid model under heavy traffic environment parameterization. This result was generalized by Dębicki and Palmowski [8] to the fluid models where $Z(t)$ is a Gaussian process being the limit of suitable scaled superposition of N general *on-off* processes, as $N \to \infty$. This justified the interest in analyzing of (1) for $Z(t)$ being a general centered stationary Gaussian process. Note that in this context constant c in (1) looses the output rate interpretation.

In this paper we focus on the study of some Gaussian fluid models. Thus in the sequel we assume that $Z(t)$ in (1) is a centered stationary Gaussian process with continuous covariance function $R(t) = \mathbf{Cov}(Z(s+t), Z(s))$. In particular we analyze the following special cases:

OU $Z(t)$ is an Ornstein-Uhlenbeck process, that is a centered stationary Gaussian process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$ for $r, \alpha > 0$. Equivalently $Z(t)$ is $(-\alpha x, 2\alpha r^2)$-diffusion with $Z(0) \sim \mathcal{N}(0, r^2)$;

GM $Z(t) = \sum_{j=1}^{n} X_j(t)$, where $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))^T$ is a stationary Gauss-Markov process, that is a centered Gaussian process with covariance $\boldsymbol{\Sigma}\, e^{\boldsymbol{A}\, t}$, where all eigenvalues have strictly negative real parts. Recall that $\boldsymbol{X}(t)$ is the solution of the stochastic differential equation

$$d\boldsymbol{X}(t) = \boldsymbol{A}\, \boldsymbol{X}(t)\, dt + \boldsymbol{B}\, d\boldsymbol{W}(t), \qquad t \geq 0,$$

where components of $\boldsymbol{W}(t)$ are independent Brownian motions, $\boldsymbol{A}\,\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\,\boldsymbol{A}^T = -\boldsymbol{B}\,\boldsymbol{B}^T$ and $\boldsymbol{\Sigma}$ is the covariance matrix of $\boldsymbol{X}(0)$; see [11], Chapter 5.6.

One of the most important quantities in performance evaluation is the probability $\psi(u) = \mathbb{P}(V > u)$ that the stationary buffer content V exceeds given threshold u. It was demonstrated (see e.g. [14]) that the stationary buffer content equals in distribution to

$$V =_{\mathrm{d}} \sup_{t\geq 0}(\eta(t) - ct),$$

where

$$\eta(t) = \int_0^t Z(s)ds$$

and has the interpretation as the accumulated input to the buffer over the time interval $[0,t]$. Note that $\eta(t)$ is a centered Gaussian process with stationary increments.

The stationary buffer overflow probability $\psi(u)$ for Gaussian fluid models was studied in a sequence of papers; e.g. Kulkarni and Rolski [14], Simonian [22], Kobayashi and Takahashi [12], Dębicki and Rolski [6], Dębicki, Michna and Rolski [7]. One of the standard approaches is to use the theory of large deviations which provides us with the *logarithmic* asymptotic, suggesting an approximation of type $\psi(x) \approx \exp(-\gamma x)$, where γ is given explicitly. However in view of some experiments, large deviation asymptotics may not be effective; see e.g. Choudry, Lucantoni & Whitt [4], McDonald and Qian [15]. In particular, in the AMS model, it is well known that $\psi(x) = C\exp(-\gamma x)(1+o(1))$, and McDonald and Qian [15] noticed that frequently C is very small. Therefore, in such cases, for approximation of $\psi(x)$ we need to know C besides γ. Unfortunately for this and many other models recent theories seem not to provide even numerical procedures for computing C.

Recently Dębicki [5] proved that for a quite wide class of Gaussian fluid models

$$\psi(u) = \mathbb{P}(\sup_{t\geq 0}(\eta(t) - ct) > u) = \frac{\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta}}{\gamma c} e^{-\gamma^2\beta} e^{-\gamma u}(1+o(1)), \tag{2}$$

as $u\to\infty$, where $\gamma = c/\int_0^\infty R(t)\,dt$, $\beta = \int_0^\infty tR(t)\,dt$ and $\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta}$ is the *generalized Pickands constant.* Formally generalized Pickands constants are defined via the following limit

$$\mathcal{H}_\mu = \lim_{T\to\infty}\frac{\mathcal{H}_\mu(T)}{T}\ , \tag{3}$$

where

$$\mathcal{H}_\mu(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}\mu(t) - \sigma_\mu^2(t)\right)\right) \tag{4}$$

and $\mu(t)$ is a given centered Gaussian process with stationary increments and variance function $\sigma_\mu^2(t) = \mathbf{Var}(\mu(t))$ that satisfies some regularity conditions (see assumptions **C1-C2** in Appendix). Thus in (2) $\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta} = \mathcal{H}_\mu$ for $\mu(t) = \frac{\gamma}{\sqrt{2}}\eta(t) = \frac{\gamma}{\sqrt{2}}\int_0^t Z(s)ds$. Note that in the case of $\mu(t) = B_H(t)$ being a fractional Brownian motion with Hurst parameter $H\in(0,1]$, $\mathcal{H}_\mu = \mathcal{H}_{B_H}$ are known as the *classical* Pickands constants; e.g. Piterbarg [20].

Since the exact value of $\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta}$ is not known, (2) only ensures that

$$\psi(u) = Ce^{-\gamma u}(1+o(1)) \tag{5}$$

for some positive constant C. Following Choudry *et al* [4], we call C the *asymptotic constant* and γ the *asymptotic decay rate.* Note that from (2) we have

$$C = \frac{\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta}}{\gamma c} e^{-\gamma^2\beta}. \tag{6}$$

Hence, for the case we study, it is equivalent to know, either the generalized Pickands constant or the corresponding asymptotic constant. In this paper we will use exchangeable these two notions. Although for the fluid models formula (5) is more interesting, sometimes the form like (2) is more convenient for considerations.

The main goal of this paper is to give a reliable method of simulation of C defined in (6) for Gaussian fluid models with **OU** and **GM** input process. One of the possible ways to simulate C is by direct simulation of $\psi(u)$ by Monte Carlo method. Unfortunately this method is not efficient; since for large u we are in the so called *rare event* regime and the relative error of the simulation goes to ∞ as $u \to \infty$. The detailed discussion on the weakness of this method is given in Chapter 4 of Asmussen [2]. Also the straightforward application of the definition (3) for $\mathcal{H}_{\frac{\gamma}{\sqrt{2}}\eta}$ in (6) produces instable algorithms (see Michna [16]).

Michna [16] (see also Burnecki & Michna [3]) proposed a new method of simulation of classical Pickands constants $\mathcal{H}_\mu$; that is for $\mu(t) = B_H(t)$ being fractional Brownian motion with Hurst parameter $H \in (1/2, 1]$. His idea is based on an application of the Girsanov type theorem for computing $\mathbb{P}(\sup_{t\geq 0} B_H(t) - ct > u)$ and comparing it with the theoretical result of Hüsler & Piterbarg [10] (see also Narayan [17]). The relevance of this method is based on the fact that the use of change of measure technique (by appropriate application of a Girsanov type theorem) produces algorithms with bounded relative error (see e.g. Theorem 2.1 in [2]). For other applications of the change of measure technique in queues with fractional Brownian motion traffic we refer to Huand, Devetsikiotis, Lambadaris and Kaye [9]. In this paper we follow the idea of Michna [16] to compute the asymptotic constant C for Gaussian fluid models with **OU** and **GM** structure of process $Z(t)$. The application of Girsanov type theorem for the input process $Z(t)$ implies that a Markovian structure of $Z(t)$ is essential for our considerations.

Since no exact values for C are available, then we propose to compare the simulated results with lower bound given in Proposition 14 (see Appendix) and upper bound given in Corollary 3.1 of Dębicki [5].

2 IMPORTANCE SAMPLING FOR GAUSS-MARKOV MODELS

Before we present the main result of this section, we need some notation and results that are also of independent interest. For simplicity we start with the analysis of the following Ornstein-Uhlenbeck case.

Let $Z(t)$ be the **OU** process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, by $\mathcal{F}_t$ we denote the σ-field generated by the process $Z(t)$ up to t. Put

$$\zeta(t) = Z(t) - c \tag{7}$$

and note that the process $\zeta(t)$ under $\mathbb{P}$ is $(-\alpha x - \alpha c, 2\alpha r^2)$-diffusion. Using the results from Kulkarni and Rolski [14] we have that the process

$$M(t) = \exp\left\{\frac{c}{r^2}(\zeta(t) - \zeta(0)) + \frac{c}{r^2}\alpha \int_0^t \zeta(s)\, ds\right\},\ t \geq 0 \tag{8}$$

is a $\mathbb{P}$-martingale. For each $t \geq 0$ we define the probability measure $\tilde{\mathbb{P}}_t$ on $\mathcal{F}_t$ by

$$\tilde{\mathbb{P}}_t(B) = \int_B M(t)\, d\mathbb{P}_t \qquad B \in \mathcal{F}_t, \tag{9}$$

where the subscript t denotes its restriction to $\mathcal{F}_t$. By $\tilde{\mathbb{P}}$ we denote a probability measure on Ω such that its restriction to $\mathcal{F}_t$ is $\tilde{\mathbb{P}}_t$.

Lemma 1 *The process $\{\zeta(t),\ t \geq 0\}$ on $(\Omega, \mathcal{F}_t, \mathcal{F}, \tilde{\mathbb{P}})$ is a $(-\alpha x + \alpha c, 2\alpha r^2)$-diffusion with $\zeta(0)$ distributed as $\zeta(0)$ under $\mathbb{P}$.*

Proof. See Kulkarni and Rolski [14], Lemma 3.1.

□

Corollary 2 *Under the measure $\tilde{\mathbb{P}}$ the process $\zeta(t)$ can be written as*

$$\zeta(t) = Z(t) + c - 2ce^{-\alpha t} \tag{10}$$

where Z is the stationary Ornstein-Uhlenbeck process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$.

Proof. Lemma 1 yields that the process $\zeta(t)$ under the measure $\tilde{\mathbb{P}}$ satisfies the following stochastic differential equation

$$d\zeta(t) = (-\alpha\zeta(t) + \alpha c)dt + r\sqrt{2\alpha}\, dW(t)$$

with the initial condition $\zeta(0) =_{\mathrm{d}} Z(0) - c$, where $W(t)$ is the standard Brownian motion. Applying Itô formula to the process $\zeta(t)$ we obtain

$$\zeta(t) = e^{-\alpha t}\zeta(0) + c(1 - e^{-\alpha t}) + r\sqrt{2\alpha}e^{-\alpha t}\int_0^t e^{\alpha s}\, dW(s). \tag{11}$$

Recall that $\tilde{\mathbb{E}}\zeta(0) = -c$ and $\tilde{\mathbb{E}}(\zeta(0) - \tilde{\mathbb{E}}\zeta(0))^2 = r^2$ which together with (11) give that $\zeta(t)$ under $\tilde{\mathbb{P}}$ is a Gaussian process with mean $c - 2ce^{-\alpha t}$ and the covariance function $r^2 e^{-\alpha|t|}$. Hence ζ under $\tilde{\mathbb{P}}$ is

$$\zeta(t) = Z(t) + c - 2ce^{-\alpha t}, \tag{12}$$

where $Z(t)$ is a stationary Ornstein-Uhlenbeck process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$.

□

In the following proposition we apply Corollary 2 in order to get an importance sampling method to compute $\psi(u)$.

Proposition 3 *If $Z(s)$ is the Ornstein-Uhlenbeck process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$ for $r, \alpha > 0$, then for each $u > 0$*

$$\begin{aligned} \psi(u) = \mathbb{P}\left(\sup_{0 \leq t < \infty}\left(\int_0^t Z(s)ds - ct\right) > u\right) &= \\ = \quad \mathbb{E}\exp\left(\frac{c}{r^2}\left(Z(0) - Z(\tilde{\tau}(u)) + 2ce^{-\alpha\tilde{\tau}(u)}\right)\right) e^{-2\frac{c^2}{r^2} - \frac{c\alpha}{r^2}u}&, \end{aligned}$$

where

$$\tilde{\tau}(u) = \inf\{t > 0 : \int_0^t (Z(s) + c - 2ce^{-\alpha s})\, ds > u\}. \tag{13}$$

Proof: Define the following stopping time

$$\tau(u) = \inf\{t > 0 : \int_0^t \zeta(s)\, ds > u\}.$$

It follows from Corollary 2 that $\tau(u)$ under the measure $\tilde{\mathbb{P}}$ has the same distribution as $\tilde{\tau}(u)$ defined by (13). Note that $\tilde{\tau}(u) < \infty$ a.s.. In order to simplify the notation let us introduce for all $t \geq 0$

$$N(t) = \frac{d\mathbb{P}_t}{d\tilde{\mathbb{P}}_t}.$$

Recall (see eq. (8) and (9)) that

$$N(t) = \frac{d\mathbb{P}_t}{d\tilde{\mathbb{P}}_t} = \exp\left(-\frac{c}{r^2}(\zeta(t) - \zeta(0)) - \frac{c}{r^2}\alpha \int_0^t \zeta(s)\, ds\right).$$

Thus we get

$$\begin{aligned}
&\mathbb{P}(\sup_{0\leq t<\infty}(\int_0^t Z(s)ds - ct) > u) = \mathbb{P}(\tau(u) < \infty) = \\
&= \lim_{n\to\infty} \mathbb{P}\{\tau(u) \leq n\} = \lim_{n\to\infty} \tilde{\mathbb{E}}N(n)\mathbf{I}\{\tau(u) \leq n\} \\
&= \lim_{n\to\infty} \mathbb{E}N(n)\mathbf{I}\{\tilde{\tau}(u) \leq n\} = \lim_{n\to\infty} \mathbb{E}(\mathbb{E}[N(n)|\mathcal{F}_{\tilde{\tau}(u)\wedge n}]\mathbf{I}\{\tilde{\tau}(u) \leq n\}) \\
&= \lim_{n\to\infty} \mathbb{E}N(\tilde{\tau}(u) \wedge n)\mathbf{I}\{\tilde{\tau}(u) \leq n\}) = \lim_{n\to\infty} \mathbb{E}N(\tilde{\tau}(u))\mathbf{I}\{\tilde{\tau}(u) \leq n\}) \\
&= \mathbb{E}N(\tilde{\tau}(u)).
\end{aligned}$$

□

Remark 4 The fact that $\tilde{\tau} < \infty$ a.s. enables us to simulate the distribution of $\sup_{0\leq t<\infty}(\int_0^t Z(s)ds - ct)$ using the formula given in Proposition 3.

In the following Theorem we extend the result of Proposition 3 to the process $Z(t) = \sum_{j=1}^n X_j(t)$, where $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))^T$ is a **GM** process.

Theorem 5 *Let* $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$ *be a stationary Gauss-Markov process with* $\mathbb{E}\boldsymbol{X}(t) = 0$, *covariance* $\boldsymbol{R}(t) = \boldsymbol{\Sigma} e^{t\boldsymbol{A}}$ *and* $Z(t) = \sum_{j=1}^n X_j(t)$. *Put* $\boldsymbol{B}\boldsymbol{B}^T = -(\boldsymbol{A}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{A})$, $\boldsymbol{e} = (1, 1, \ldots, 1)^T$, $\boldsymbol{\theta}_0^T = -\gamma \boldsymbol{e}^T \boldsymbol{A}^{-1}$ *and*

$$\gamma = \frac{2c}{\boldsymbol{e}^T \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{B}^T (\boldsymbol{A}^T)^{-1} \boldsymbol{e}}$$

where $c > 0$. *Then for each* $u \geq 0$

$$\begin{aligned}
\psi(u) &= \mathbb{P}(\sup_{t\geq 0}(\int_0^t Z(s)\, ds - ct) > u) = \\
&= \exp(-\gamma u)\mathbb{E}\exp\left(\boldsymbol{\theta}_0^T(\boldsymbol{X}(0) - \boldsymbol{X}(\tilde{\tau}(u)) - \gamma(\boldsymbol{I} - e^{\tilde{\tau}(u)\boldsymbol{A}})\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e})\right),
\end{aligned}$$

where

$$\tilde{\tau}(u) = \inf\{t > 0 : \int_0^t (Z(s) - \gamma \boldsymbol{e}^T \exp(s\boldsymbol{A})\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e})\, ds + ct > u\}.$$

Proof: From Stroock and Varadhan [23] (see also Palmowski and Rolski [19]), the process

$$M_{\boldsymbol{\theta}}(t) = \exp\left\{ \boldsymbol{\theta}^T(\boldsymbol{X}(t) - \boldsymbol{X}(0)) - \boldsymbol{\theta}^T \boldsymbol{A} \int_0^t \boldsymbol{X}(s)\, ds - (1/2)t\, \boldsymbol{\theta}^T \boldsymbol{B}\, \boldsymbol{B}^T \boldsymbol{\theta} \right\}$$

is a martingale for all $\boldsymbol{\theta} \in \mathbb{R}^n$. Hence choosing $\boldsymbol{\theta}_0^T = -\gamma\, \boldsymbol{e}^T \boldsymbol{A}^{-1}$ we get that

$$M(t) = \exp\left\{ \boldsymbol{\theta}_0^T(\boldsymbol{X}(t) - \boldsymbol{X}(0)) + \gamma \int_0^t (Z(s) - c)\, ds \right\}$$

is a martingale. We define $\tilde{P}$ by (9). ¿From Example 3.2 from Palmowski and Rolski [19] the process $\boldsymbol{X}(t)$ under $\tilde{P}$ is a Gauss-Markov process with the infinitesimal drift $\boldsymbol{A}\,\boldsymbol{x} - \gamma \boldsymbol{B}\, \boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}$ and the infinitesimal variance $\boldsymbol{B}$. Then $\boldsymbol{X}(t)$ under $\tilde{P}$ is a Gaussian process with mean

$$\tilde{\mathbb{E}}\, \boldsymbol{X}(t) = \gamma(\boldsymbol{I} - e^{\boldsymbol{A}t})\, \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}$$

and covariance $\boldsymbol{R}(t)$. Note that

$$\tilde{\mathbb{E}} Z(t) - c = c - \gamma\, \boldsymbol{e}^T e^{\boldsymbol{A}t}\, \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e} \to c$$

as $t \to \infty$ and so $\mathbb{P}(\tilde{\tau}(u) < \infty) = 1$. The rest of the proof is analogous to the proof of Corollary 2.

□

Theorem 5 enables us to derive a formula for the generalized Pickands constant $\mathcal{H}_\eta$ for $\eta(t) = \int_0^t \sum_{i=1}^n X_i(s)\, ds$, where $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$ is a stationary **GM** process.

Theorem 6 *Let* $\eta(t) = \int_0^t \sum_{i=1}^n X_i(s)\, ds$, *where* $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$ *is a stationary Gauss-Markov process with* $\mathbb{E}\,\boldsymbol{X}(t) = 0$ *and covariance* $\boldsymbol{R}(t) = \boldsymbol{\Sigma}\, e^{tA}$. *If* $\boldsymbol{e}^T \boldsymbol{R}(t)\, \boldsymbol{e} > 0$ *for each* $t > 0$, *then*

$$\mathcal{H}_\eta = \frac{2e^{2\beta}}{G} \lim_{u\to\infty} \mathbb{E} \exp\left(\boldsymbol{\theta}_0^T(\boldsymbol{X}(0) - \boldsymbol{X}(\tilde{\tau}(u)) - \gamma(\boldsymbol{I} - e^{\tilde{\tau}(u)\boldsymbol{A}})\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}) \right),$$

where $\tilde{\tau}(u)$ *is defined with* $c = \sqrt{2}/G$ *and*

$$G = \frac{2}{\boldsymbol{e}^T \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}}, \qquad \beta = \boldsymbol{e}^T \boldsymbol{\Sigma}\,(\boldsymbol{A}^{-2})\,\boldsymbol{e}.$$

Proof: Following Theorem 5.1 in Dębicki [5] we have

$$\mathbb{P}(\sup_{0\le t<\infty} (\eta(t) - ct) > u) \;=\; \mathcal{H}_{\frac{cG}{\sqrt{2}}\eta} \frac{1}{Gc^2} \exp(-c^2G^2\beta) e^{-cGu}(1 + o(1)), \tag{14}$$

as $u \to \infty$, where

$$G = \frac{1}{\int_0^\infty \mathbf{Cov}(Z(v), Z(0))dv} = \frac{2}{\boldsymbol{e}^T \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}}$$

and

$$\beta = \int_0^\infty v\mathbf{Cov}(Z(v), Z(0))dv = \boldsymbol{e}^T \boldsymbol{\Sigma}\,(\boldsymbol{A}^{-2})\,\boldsymbol{e}.$$

Thus for $c = \frac{\sqrt{2}}{G}$ we have

$$\mathcal{H}_\eta = \frac{2e^{2\beta}}{G} e^{\sqrt{2}u} \mathbb{P}(\sup_{0 \le t < \infty} (\eta(t) - ct) > u)(1 + o(1))$$

as $u \to \infty$. Now, using Theorem 5, we obtain

$$\begin{aligned} \mathcal{H}_\eta &= \\ &= \frac{2e^{2\beta}}{G} \mathbb{E} \exp \left(\boldsymbol{\theta}_0^T (\boldsymbol{X}(0) - \boldsymbol{X}(\tilde{\tau}(u)) - \gamma(\boldsymbol{I} - e^{\tilde{\tau}(u)\boldsymbol{A}}) \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{B}^T (\boldsymbol{A}^T)^{-1} \boldsymbol{e}) \right) (1 + o(1)) \end{aligned}$$

as $u \to \infty$.

□

Remark 7 Note that the combination of Theorem 5 with (14) implies that the *asymptotic constant* C introduced in (5) is given by the following limit

$$C = \lim_{u \to \infty} \mathbb{E} \exp \left(\boldsymbol{\theta}_0^T (\boldsymbol{X}(0) - \boldsymbol{X}(\tilde{\tau}(u)) - \gamma(\boldsymbol{I} - e^{\tilde{\tau}(u)\boldsymbol{A}}) \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{B}^T (\boldsymbol{A}^T)^{-1} \boldsymbol{e}) \right),$$

and that $0 < C < \infty$.

As a corollary from Theorem 6 we obtain the following result for Ornstein-Uhlenbeck case.

Corollary 8 *If* $\eta(t) = \int_0^t Z(s)ds$, *where* $Z(s)$ *is the Ornstein-Uhlenbeck process with covariance function* $R(t) = r^2 \exp(-\alpha|t|)$ *for* $r, \alpha > 0$, *and* $\tilde{\tau}(u)$ *is defined in* (13) *with* $c = (\sqrt{2}r^2)/\alpha$, *then*

$$\mathcal{H}_\eta = \frac{2r^2}{\alpha} e^{-2\frac{r^2}{\alpha^2}} \lim_{u \to \infty} \mathbb{E} \exp \left(\frac{\sqrt{2}}{\alpha} \left(Z(0) - Z(\tilde{\tau}(u)) + 2\frac{\sqrt{2}r^2}{\alpha} e^{-\alpha\tilde{\tau}(u)} \right) \right).$$

Proof: The proof is an immediate application of Theorem 6 to the Ornstein-Uhlenbeck process.

□

3 SIMULATION OF ASYMPTOTIC CONSTANT

In this section we present numerical simulations of the asymptotic constant C and related Pickands constant $\mathcal{H}_\eta$, where $\eta = \int_0^t Z(s)ds$, both for **OU** and **GM** cases.

3.1 Ornstein-Uhlenbeck Case

Let $Z(t)$ be **OU** process with covariance function $R(t) = e^{-t}$. Following Corollary 8, for $c = \sqrt{2}$ and $\eta = \int_0^t Z(s)ds$, we have $\mathcal{H}_\eta = \lim_{u \to \infty} I_{\mathrm{OU}}(u)$, where

$$I_{\mathrm{OU}}(u) = 2e^{-2} \mathbb{E} \exp \left\{ \sqrt{2} \left(Z(0) - Z(\tilde{\tau}(u)) + 2\sqrt{2}e^{-\tilde{\tau}(u)} \right) \right\} \tag{15}$$

In order to estimate $I_{\mathrm{OU}}(u)$ we propose to use the following Monte Carlo method. Namely we simulate

$$\chi_{\mathbf{OU}}(u) = 2e^{-2} \exp \left\{ \sqrt{2} \left(Z(0) - Z(\tilde{\tau}(u)) + 2\sqrt{2}e^{-\tilde{\tau}(u)} \right) \right\},$$

where (see Proposition 3)

$$\tilde{\tau}(u) = \min\{t \geq 0 : \int_0^t (Z(s) + \sqrt{2} - 2\sqrt{2}e^{-s})\, ds = u\}. \tag{16}$$

We use the following natural algorithm for generation of sample paths of the process $Z(t)$ on the grid $d = 0.01$:

$$Z(t+d) := e^{-d} Z(t) + \sqrt{1 - e^{-2d}} * \mathbf{randn},$$

where **randn** is the standard normal random number. Then, using the Trapezoid Integration Method, we compute the integral in (16). For each value $u = 1, 2, ..., 30$, we run $N = 100000$ sample paths and compute its mean. The results are presented on Figure 1. We can observe a quite fast convergence to $\mathcal{H}_\eta$ with respect of u.

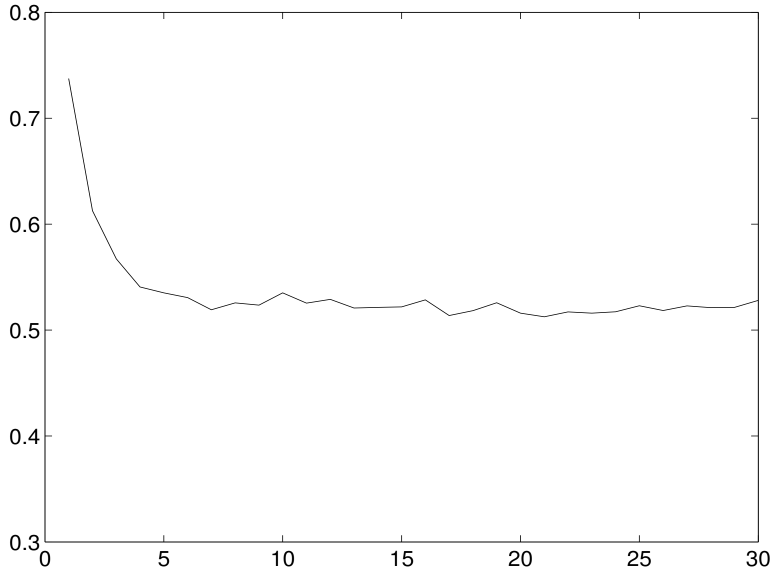

Figure 1: Simulation of $I_{\mathrm{OU}}(u)$ for $u = 1, 2, ..., 30$; Ornstein-Uhlenbeck case.

In particular, for $u = 30$ we get

$$\mathcal{H}_\eta \simeq 0.528$$

with the asymptotic 95% confidence interval $(0.514, 0.541)$. The half-width ϵ of the confidence interval was computed from the formula $\epsilon = (1.96\sigma)/\sqrt{N}$, with the sample standard deviation $\sigma = 2.2350$.

Remark 9 Using Corollary 3.1 from Dębicki [5] we have $\mathcal{H}_\eta \leq \sqrt{R(0)/\pi} = \sqrt{1/\pi} \simeq 0.5642$. Moreover from Proposition 14 from the Appendix we have $\mathcal{H}_\eta \geq 2\mathrm{e}\Psi(\sqrt{2}) \simeq 0.4393$.

Remark 10 Combining (6) with the above simulation we conclude that for the fluid model with input rate process $Z(t)$ and output rate $c = \sqrt{2}$ the *asymptotic constant* $C \simeq 0,0357$.

Remark 11 The stability and exactness of the above simulation follows from the fact that the variance of $\chi_{\mathbf{OU}}(u)$ is finite. In fact not only the variance, but all the moments of $\chi_{\mathbf{OU}}(u)$ are bounded. Namely if $Z(s)$ is an Ornstein-Uhlenbeck process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$ where $r, \alpha > 0$, and $\tilde{\tau}(u)$ is defined in (13) with $c = (\sqrt{2}r^2)/\alpha$, then for each $u > 0$

$$\begin{aligned}
\mathbb{E}\chi^n_{\mathbf{OU}}(u) &= \mathbb{E}\exp\left(\frac{n\sqrt{2}}{\alpha}\left(Z(0) - Z(\tilde{\tau}(u)) + 2\frac{\sqrt{2}r^2}{\alpha}e^{-\alpha\tilde{\tau}(u)}\right)\right) \leq \qquad (17)\\
&\leq \mathbb{E}\exp\left(\frac{n\sqrt{2}}{\alpha}(Z(0) + \frac{\sqrt{2}r^2}{\alpha})\right)\\
&= \exp\left(\frac{r^2}{\alpha^2}(n^2+2n)\right). \qquad (18)
\end{aligned}$$

Inequality (17) follows from the fact that $Z(\tilde{\tau}(u)) + \frac{\sqrt{2}r^2}{\alpha} - 2\frac{\sqrt{2}r^2}{\alpha}e^{-\alpha\tilde{\tau}(u)} > 0$.

3.2 Gauss-Markov Case

Let $Z(t) = X_1(t) + X_2(t)$, where $X_1(t)$ and $X_2(t)$ are independent Ornstein-Uhlenbeck processes with covariance functions $R_1(t) = \frac{1}{16}e^{-t}$ and $R_2(t) = \frac{1}{80}e^{-\frac{1}{5}t}$ respectively. In order to simulate $\mathcal{H}_\eta$, for $\eta(t) = \int_0^t Z(s)ds$, we use Theorem 6 with

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{5} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{\sqrt{2}}{20} \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{80} \end{pmatrix}$$

and $c = \frac{\sqrt{2}}{2}$. Hence $\mathcal{H}_\eta = \lim_{u\to\infty} I_{\mathrm{GM}}(u)$ where

$$I_{\mathrm{GM}}(u) = \frac{1}{4}e^{\frac{3}{4}}\mathbb{E}\ \exp\{\sqrt{2}(X_1(0)-X_1(\tilde{\tau}(u)))+5\sqrt{2}(X_2(0)-X_2(\tilde{\tau}(u)))+\frac{1}{4}(e^{-\tilde{\tau}(u)}+5e^{-\frac{1}{5}\tilde{\tau}(u)}-6)\} \qquad (19)$$

We estimate $I_{\mathrm{GM}}(u)$ using the following Monte Carlo method. Namely we simulate

$$\begin{aligned}
&\chi_{\mathbf{GM}}(u) = \\
&= \frac{1}{4}e^{\frac{3}{4}}\exp\{\sqrt{2}(X_1(0) - X_1(\tilde{\tau}(u))) + 5\sqrt{2}(X_2(0) - X_2(\tilde{\tau}(u))) + \frac{1}{4}(e^{-\tilde{\tau}(u)} + 5e^{-\frac{1}{5}\tilde{\tau}(u)} - 6)\},
\end{aligned}$$

where

$$\tilde{\tau}(u) = \inf\{t \geq 0 :\ 4\int_0^t (X_1(s) + X_2(s))\,ds + \frac{\sqrt{2}}{2}e^{-t} + \frac{5\sqrt{2}}{2}e^{-\frac{1}{5}t} - 3\sqrt{2} + \frac{\sqrt{2}}{2}t = u\}.$$

For each value $u = 1, 2, ..., 30$, we run 100000 sample paths with the discretization grid 0.01. The results of simulations are presented on Figure 2. In particular, for $u = 30$ we get

$$\mathcal{H}_\eta \simeq 0.1177$$

with the asymptotic 95% confidence interval $(0.1175, 0.1179)$. For the latter result we use that the sample standard deviation $\sigma = 0.0357$.

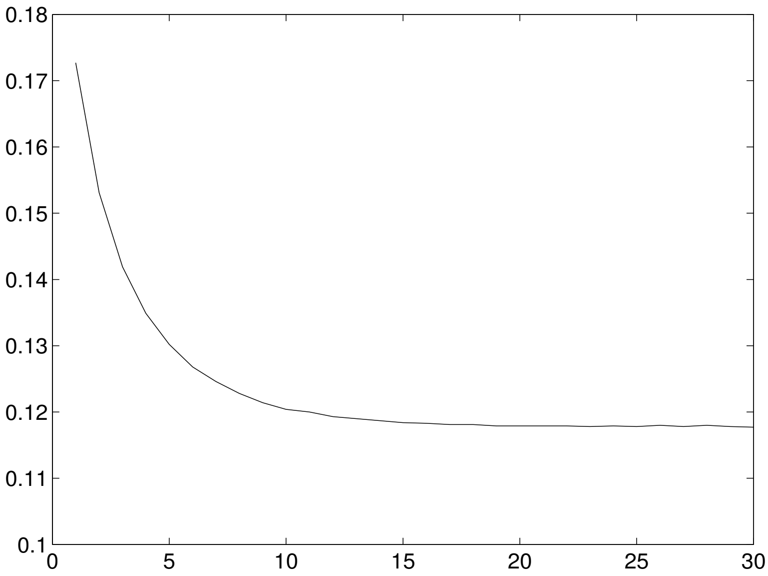

Figure 2: Simulation of $I_{\mathrm{GM}}(u)$ for $u = 1, 2, ..., 30$; Gauss-Markov case.

Remark 12 Using Corollary 3.1 from Dębicki [5] we have

$$\mathcal{H}_\eta \le \sqrt{R(0)/\pi} = \sqrt{\frac{3}{40\pi}} \simeq 0.1545.$$

Moreover, since $\sigma^2_\eta(t) \ge \sigma^2_{\eta_1}(t)$ for $\eta_1(t) = \sqrt{\frac{6}{5}} \int_0^t X_1(s)ds$, then using Theorem 3.2 from Dębicki [5] $\mathcal{H}_\eta \ge \mathcal{H}_{\eta_1}$. Combining it with Proposition 14 we have

$$\mathcal{H}_\eta \ge \mathcal{H}_{\eta_1} \ge \frac{3}{20} \mathrm{e}^{\frac{3}{40}} \Psi \left(\sqrt{\frac{3}{20}} \right) \simeq 0.05686.$$

Remark 13 Combining (6) with the above simulation for **GM** case we conclude that for the fluid model with input rate process $Z(t) = X_1(t) + X_2(t)$ and output rate $c = \frac{\sqrt{2}}{2}$ the *asymptotic constant* $C \simeq 0,2224$.

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APPENDIX

Using Theorem 4.3 of Kulkarni & Rolski [14] for $\eta(t) = \int_0^t Z(s)ds$, where $Z(t)$ is an **OU** process, we can get the following lower bound for $\mathcal{H}_\eta$.

Proposition 14 *If $\eta(t) = \int_0^t Z(s)ds$, where $Z(s)$ is a stationary Ornstein-Uhlenbeck process with covariance function $R(t) = r^2 \exp(-\alpha|t|)$ for $r, \alpha > 0$, then*

$$\mathcal{H}_\eta \geq \frac{2r^2}{\alpha} e^{\frac{r^2}{\alpha^2}} \Psi\left(\frac{\sqrt{2}r}{\alpha}\right),$$

where $\Psi(x)$ is the tail of standard normal distribution.

Proof: Let $\eta(t) = \int_0^t Z(s)ds$. From Kulkarni & Rolski [14], Theorem 4.3,

$$\mathbb{P}\left(\sup_{t\geq 0}(\eta(t) - ct) > u\right) \geq e^{-\frac{c^2}{2r^2} - \frac{c\alpha}{r^2}u} \Psi(\frac{c}{r}) + o(e^{-\frac{c\alpha}{r^2}u}) \tag{20}$$

as $u \to \infty$. Thus the lemma follows immediately from comparison of (20) with exact asymptotic of $\mathbb{P}(\sup_{t\geq 0}(\eta(t) - ct) > u)$ given by Dębicki [5], Theorem 5.1 (see also (14)) for $c = \frac{\sqrt{2}r^2}{\alpha}$.

□

In the remaining part of this appendix we give a lower bounds for $\mathcal{H}_\mu$, which is valid, in contrast to Proposition 14, for a fairly large class of Gaussian processes. However we do not use this lower bound in the paper because for special cases considered here we could use a more tight bound from Proposition 14. Thus let $\mu(t)$ be a centered Gaussian process with stationary increments, continuous sample paths a.s., $\mu(0) = 0$ and the variance function $\mathbf{Var}(\mu(t)) = \sigma_\mu^2(t)$ satisfying

C1 $\sigma_\mu^2(t) \in C^1([0, \infty))$ is strictly increasing and there exists $\epsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{t \cdot \left(\sigma_\mu^2\right)'(t)}{\sigma_\mu^2(t)} \leq \epsilon$$

C2 $\sigma_\mu^2(t)$ is regularly varying at 0 with index $\alpha_0 \in (0, 2]$ and $\sigma_\mu^2(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0, 2)$.

Under assumptions **C1-C2** Dębicki [5] proved that $\mathcal{H}_\mu$ are well defined.

Theorem 15 *If $\mu(t)$ is a centered Gaussian process with stationary increments that satisfies* **C1-C2**, *then*

$$\mathcal{H}_\mu \geq \frac{1}{8\int_0^\infty \exp\left(-\frac{\sigma_\mu^2(t)}{4}\right) dt}. \tag{21}$$

In order to prove the above theorem we need the following lemma which is an extension of Lemma 7 from Shao [21].

Lemma 16 *If $\mu(t)$ is a centered Gaussian process with stationary increments that satisfies* **C1-C2** *then for each $a>0$*

$$\mathcal{H}_\mu \geq \frac{1}{a}\left(1-2\sum_{k=1}^{\infty}\exp\left(-\frac{\sigma_\mu^2(ak)}{4}\right)\right).$$

Proof: Let $a>0$ be given. The proof is based on the observation that

$$\begin{aligned}
&\mathbb{E}\exp\left(\sup_{t\in[0,aN]}\sqrt{2}\mu(t)-\sigma_\mu^2(t)\right)\\
&\geq \mathbb{E}\exp\left(\sup_{1\leq i\leq N}\left(\sqrt{2}\mu(ai)-\sigma_\mu^2(ai)\right)\right)\\
&\geq \left(\sum_{i=1}^{N}\mathbb{E}\exp(\sqrt{2}\mu(ai)-\sigma_\mu^2(ai))\right)\\
&- 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathbb{E}\exp\left(\left(\sqrt{2}(\mu(ai)+\mu(aj))-\sigma_\mu^2(ai)-\sigma_\mu^2(aj)\right)/2\right),
\end{aligned}$$

where the last inequality follows from the fact that due to Bonferroni inequality

$$\begin{aligned}
\mathbb{P}(\sup_{1\leq i\leq N}(\sqrt{2}\mu(ai)-\sigma_\mu^2(ai))>x) &\geq \sum_{i=1}^{N}\mathbb{P}(\sqrt{2}\mu(ai)-\sigma_\mu^2(ai)>x)\\
&- 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathbb{P}(\sqrt{2}(\mu(ai)+\mu(aj))-\sigma_\mu^2(ai)-\sigma_\mu^2(aj)>2x)
\end{aligned}$$

for each $x\geq 0$. Using that $\exp(\mu(ai))$ is lognormal we have

$$\begin{aligned}
\mathbb{E}\exp(\sqrt{2}\mu(ai)-\sigma_\mu^2(ai)) &= 1\\
\mathbb{E}\exp\left(\left(\sqrt{2}(\mu(ai)+\mu(aj))-\sigma_\mu^2(ai)-\sigma_\mu^2(aj)\right)/2\right) &= \exp(-\sigma_\mu^2(a(j-i))/4).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\exp\left(\sup_{t\in[0,aN]}\left(\sqrt{2}\mu(t)-\sigma_\mu^2(t)\right)\right) &\geq N-2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\exp(-\sigma_\mu^2(a(j-i))/4)\\
&\geq N\left(1-2\sum_{k=1}^{N}\exp\left(-\frac{\sigma_\mu^2(ak)}{4}\right)\right). \qquad (22)
\end{aligned}$$

Now from the definition of the generalized Pickands constants and (22) we get

$$\mathcal{H}_\mu \geq \frac{1}{a}\left(1-2\sum_{k=1}^{\infty}\exp\left(-\frac{\sigma_\mu^2(ak)}{4}\right)\right).$$

□

Proof of Theorem 15:

Note that for each $a>0$

$$\begin{aligned}
\sum_{k=1}^{\infty}\exp\left(-\frac{\sigma_\mu^2(ak)}{4}\right) &\leq \int_0^\infty \exp\left(-\frac{\sigma_\mu^2(at)}{4}\right)dt\\
&= \frac{1}{a}\int_0^\infty \exp\left(-\frac{\sigma_\mu^2(t)}{4}\right)dt.
\end{aligned}$$

Thus using Lemma 16

$$\mathcal{H}_\mu \geq \frac{1}{a}\left(1 - \frac{2}{a}\int_0^\infty \exp\left(-\frac{\sigma_\mu^2(t)}{4}\right)dt\right)$$

for each $a > 0$. The maximum over a of the right hand side of the above inequality is attained at $a = 4\int_0^\infty \exp(-\sigma_\mu^2(t)/4)dt$. Hence

$$\mathcal{H}_\mu \geq \frac{1}{8\int_0^\infty \exp\left(-\frac{\sigma_\mu^2(t)}{4}\right)dt}.$$

□

Finally, let us remark on a lower bound, which can be drawn for the classical Pickands constant $\mathcal{H}_{B_H}$, where $B_H(t)$ is the fractional Brownian motion with Hurst parameter H, that is a centered Gaussian process with stationary increments and variance function $\mathbf{Var}(B_H(t)) = t^{2H}$, $H \in (0,1]$.

Corollary 17 *If* $H \in (0,1]$, *then* $\mathcal{H}_{B_H} \geq \frac{H}{2^{2+\frac{1}{H}}\Gamma(\frac{1}{2H})}$.

Proof:

Using Theorem 15 for $\sigma_\mu^2(t) = t^{2H}$, we have

$$\begin{aligned}\mathcal{H}_{B_H} \geq \frac{1}{8\int_0^\infty \exp\left(-\frac{t^{2H}}{4}\right)dt} = \frac{1}{4^{1/2H}8\int_0^\infty \exp\left(-t^{2H}\right)dt} &= \frac{1}{\frac{4^{1/2H}8}{2H}\Gamma(\frac{1}{2H})} \\ &= \frac{H}{2^{2+1/H}\Gamma(\frac{1}{2H})}.\end{aligned}$$

□