A DICHOTOMY FOR BOREL FUNCTIONS

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Abstract. The dichotomy discovered by Solecki in [3] states that any Baire class 1 function is either \( \sigma \)-continuous or “includes” the Pawlikowski function \( P \). The aim of this paper is to give an argument which is simpler than the original proof of Solecki and gives a stronger statement: a dichotomy for all Borel functions.

1. Introduction

An old question of Lusin asked whether there exists a Borel function which cannot be decomposed into countably many continuous functions. By now several examples have been given, by Keldiš, Adyan and Novikov among others. A particularly simple example, the function \( P : (\omega + 1)\omega \rightarrow \omega^\omega \), has been found by Pawlikowski (cf. [1]). By definition,

\[
P(x)(n) = \begin{cases} 
  x(n) + 1 & \text{if } x(n) < \omega, \\
  0 & \text{if } x(n) = \omega.
\end{cases}
\]

It is proved in [1] that if \( A \subseteq (\omega + 1)^\omega \) is such that \( P|A \) is continuous then \( P[A] \subseteq \omega^\omega \) is nowhere dense. Since \( P \) is a surjection, it is not \( \sigma \)-continuous.

In [3] Solecki showed that the above function is, in a sense, the only such example, at least among Baire class 1 functions (in other words, it is the initial object in a certain category).

Theorem 1 (Solecki, [3]). For any Baire class 1 function \( f : X \rightarrow Y \), where \( X, Y \) are Polish spaces, either \( f \) is \( \sigma \)-continuous or there exist topological embeddings \( \varphi \) and \( \psi \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\omega^\omega & \xrightarrow{\psi} & Y \\
\uparrow_P & \quad & \uparrow_f \\
(\omega + 1)^\omega & \xrightarrow{\varphi} & X
\end{array}
\]

In [4] Zapletal generalized Solecki’s dichotomy to all Borel functions by proving the following theorem.
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**Theorem 2** (Zapletal, [4]). If \( f : X \rightarrow Y \) is a Borel function which is not \( \sigma \)-continuous then there is a compact set \( C \subseteq X \) such that \( f \restriction C \) is not \( \sigma \)-continuous and of Baire class 1.

In this paper we give a new proof of the above dichotomy for all Borel functions, which is direct, shorter and more general than the original proof from [3].

2. **Notation**

We say that a Borel function \( f : X \rightarrow Y \), where \( X, Y \) are Polish spaces, is \( \sigma \)-continuous if there exist a countable cover of the space \( X = \bigcup_n X_n \) (with arbitrary sets \( X_n \)) such that \( f \restriction X_n \) is continuous for each \( n \). It follows from the Kuratowski extension theorem that we may require that the sets \( X_n \) be Borel. If \( f \) is a Borel function which is not \( \sigma \)-continuous then the family of sets on which it is \( \sigma \)-continuous is a proper \( \sigma \)-ideal in \( X \). We denote this \( \sigma \)-ideal by \( I_f \).

In a metric space \((X,d)\) for \( A,B \subseteq X \) let us denote by \( h(A,B) \) the Hausdorff distance between \( A \) and \( B \).

The spaces \((\omega + 1)^\omega\) and \((\omega + 1)^n\) for \( n < \omega \) are endowed with the product topology of order topologies on \( \omega + 1 \).

3. **The Zapletal’s game**

In [4] Zapletal introduced a two-player game, which turns out to be very useful in examining \( \sigma \)-continuity of Borel functions. Let \( B \subseteq \omega^\omega \) be a Borel set and \( f : B \rightarrow 2^\omega \) be a Borel function. Let \( \rho : \omega \rightarrow \omega \times 2^{<\omega} \times \omega \) be a bijection. The game \( G_f(B) \) is played by Adam and Eve. They take turns playing natural numbers. In his \( n \)-th move, Adam picks \( x_n \in \omega \). In her \( n \)-th move, Eve chooses \( y_n \in 2 \). At the end of the game we have \( x \in \omega^\omega \) and \( y \in 2^\omega \) formed by the numbers picked by Adam and Eve, respectively. Next, \( y \in 2^\omega \) is used to define a sequence of partial continuous functions (with domains of type \( G_\delta \) in \( \omega^\omega \)) in the following way. For \( n < \omega \) let \( f_n \) be a partial function from \( \omega^\omega \) to \( 2^\omega \) such that for \( t \in \omega^\omega \) and \( \sigma \in 2^{<\omega} \)

\[
f_n(t) \supseteq \sigma \quad \text{iff} \quad \exists k \in \omega \ y(\rho(n,\sigma,k)) = 1
\]

and \( \text{dom}(f_n) = \{ t \in \omega^\omega : \forall n < \omega \exists ! \sigma \in 2^n \ f_n(t) \supseteq \sigma \} \). Eve wins the game \( G_f(B) \) if \( x \not\in B \) or \( \exists n f(x) = f_n(x) \). Otherwise Adam wins the game.

It is easy to see that if \( f \) is a Borel function then \( G_f \) is a Borel game. The key feature of the game \( G_f \) is that it detects \( \sigma \)-continuity of the function \( f \).
Theorem 3 (Zapletal,[4]). For $B \subseteq \omega^\omega$ and $f : B \to 2^\omega$ Eve has a winning strategy in the game $G_f(B)$ if and only if $f$ is $\sigma$-continuous on $B$.

Note that if Adam has a winning strategy then the image of his strategy (treated as a continuous function from $2^\omega$ to $B$) is a compact set on which $f$ is also not $\sigma$-continuous. This observation and the Borel determinacy gives the following corollary.

Corollary 1 (Zapletal,[4]). If $B$ is a Borel set and $f : B \to 2^\omega$ is a Borel function which is not $\sigma$-continuous then there is a compact set $C \subseteq B$ such that $f|C$ is also not $\sigma$-continuous.

4. Proof of the Dichotomy

In the statement of Theorem 1 both functions $\varphi$ and $\psi$ are to be topological embeddings. However, as we will see below, for the dichotomy it is enough that they both are injective, $\varphi$ continuous and $\psi$ open. We are going to prove first this version of the dichotomy.

Theorem 4. Let $X$ be a Polish space and $f : X \to 2^\omega$ be a Borel function. Then precisely one of the following conditions holds:

1. either $f$ is $\sigma$-continuous
2. or there are an open injection $\psi$ and a continuous injection $\varphi$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\omega^\omega & \xrightarrow{\psi} & 2^\omega \\
\uparrow P & & \uparrow f \\
(\omega + 1)^\omega & \xrightarrow{\varphi} & X
\end{array}
$$

Notice that compactness of $(\omega + 1)^\omega$ implies that the $\psi$ above must be a topological embedding.

Proof. It is straightforward that (2) implies that $f$ is not $\sigma$-continuous. Let us assume that $f$ is not $\sigma$-continuous and prove that (2) holds. By Corollary 1 we may assume that $X$ is compact.

Notation. First we introduce some notation. For a fixed $n$ and $0 \leq k \leq n$ let $S_n^k$ be the set of points in $(\omega + 1)^n$ of Cantor-Bendixson rank $\geq n - k$. For each $n < \omega$ and $1 \leq k \leq n$ let us pick a function $\pi_n^k : S_n^k \to S_n^{k-1}$ such that

- on $S_{k-1}$ $\pi_n^k$ is the identity,
• if \( \tau \in S_k^m \setminus S_{k-1}^m \) then we pick one \( i \in n \) such that \( \tau(i) < \omega \) and \( \tau(i) \) is maximal such and define

\[
\pi^m_k(\tau)(i) = \omega, \quad \pi^m_k(\tau)(j) = \tau(j) \quad \text{for} \quad j \neq i.
\]

This definition clearly depends on the choice of the index \( i \) above. Note, however, that we may pick the functions \( \pi^m_k \) so that they are coherent, in the sense that for \( \tau \in (\omega + 1)^{n+1} \), unless \( \tau(n) \) is the biggest finite value of \( \tau \), we have \( \pi^m_{k+1}(\tau) = \pi^m_k(\tau \upharpoonright n) \upharpoonright \tau(n) \). In particular \( \pi^m_{k+1}(\sigma \upharpoonright \omega) = \pi^m_k(\sigma) \upharpoonright \omega \) for any \( \sigma \in (\omega + 1)^n \). The functions \( \pi^m_k \) will be called projections.

**Lemma 1.** For each \( n \) and \( 1 \leq k \leq n \) the projection \( \pi^m_k : S_k^n \to S_{k-1}^n \) is continuous.

**Proof.** Note that any point in \( S_k^n \) except \( (\omega, \ldots, \omega) \) \((k \text{ times } \omega)\) has a neighborhood in which projection is unambiguous and hence continuous. But it is easy to see that at the point \( (\omega, \ldots, \omega) \) any projection is continuous. \( \square \)

For each \( n < \omega \) let us also introduce the function \( r_n : (\omega + 1)^n \to (\omega + 1)^n \) defined as \( r_n(\tau \upharpoonright a) = \tau \upharpoonright \omega \).

To make the above notation more readable we will usually drop subscripts and superscripts in \( \pi^m_k \) and \( r_n \).

We pick a well-ordering \( \leq \) of \( (\omega + 1)^{<\omega} \) into type \( \omega \) such that for each point \( \tau \in (\omega + 1)^{<\omega} \) all elements of the transitive closure of \( \tau \) with respect to \( \pi, r \) and restrictions (i.e. functions of the form \( (\omega + 1)^n \ni \tau \mapsto \tau|m \in (\omega + 1)^m \) for \( m < n \)) are \( \leq \tau \).

For a set \( B \in \text{Bor}(X) \setminus I_f \) let \( B^* \) denote the set \( B \) shrunk by all basic clopens \( C \) which have \( I \)-small intersection with \( B \).

**Strategy of the construction.** In order to define functions \( \varphi \) and \( \psi \), we will construct for each \( \tau \in (\omega + 1)^{<\omega} \) a clopen set \( C_\tau \subseteq 2^\omega \) and a compact set \( X_\tau \subseteq X \) such that if \( \sigma \subseteq \tau \) then \( C_\tau \subseteq C_\sigma \) and \( X_\tau \subseteq X_\sigma \).

The sets \( C_\tau \) will be disjoint, which means that for \( \tau \neq \tau' \), \( |\tau| = |\tau'| \), \( C_\tau \cap C_\tau = \emptyset \). We will also need \( X_\tau \subseteq f^{-1}[C_\tau] \) and \( \text{diam}(X_\tau) < 1/|\tau| \).

The construction of the sets \( X_\tau, C_\tau \) will be done by induction along the ordering \( \leq \) on \( (\omega + 1)^{<\omega} \). In fact, we will do something more: at each step \( n \) if \( \tau \) is the \( n \)-th element of \( (\omega + 1)^{<\omega} \) we will construct not only a compact set \( X_\tau \) but also \( I_f \)-positive Borel sets \( X_\sigma^n \) for \( \sigma \leq \tau \) such that:

- \( X_\tau \subseteq X_\tau^{n-1} \tau([|\tau| - 1]), \)
- \( X_\sigma^n \subseteq X_\sigma^{n-1} \) if \( \sigma < \tau \),
- \( X_\sigma^{n-\sigma} \subseteq X_\sigma^n \) if \( \sigma, \sigma \upharpoonright \sigma < \tau \),
- \( X_\sigma^n \cap f^{-1}[C_{\sigma \upharpoonright \sigma}] = \emptyset \) if \( \sigma, \sigma \upharpoonright \sigma \leq \tau \),
\[ X^n_{\sigma \omega} \subseteq \text{cl}(X^n_{\sigma}) \] if \( \sigma, \sigma^\omega \leq \tau \).

The set \( X^n_{\sigma} \) is to be understood as the space for further construction of sets \( X^n_{\rho} \) for \( \rho \supseteq \sigma \) and \( \rho > \tau \), as can be seen in the first condition above. The last condition, as we will see later, will be used to guarantee “continuity” of the family of sets \( X_\tau \). For technical reasons we will also make sure that \( X^n_{\sigma} = (X^n_{\sigma})^* \).

We are going to ensure disjointness of \( C_\tau \)'s by satisfying the following conditions:

- \( C_{\tau-a} \subseteq C_\tau \),
- \( C_{\tau-a} \cap C_{\tau-b} = \emptyset \) for \( a \neq b \).

The fact that \( \text{diam}(X_\tau) < 1/|\tau| \) will follow from the following inductive conditions (recall that \( \pi(\tau) \leq \tau \) for any \( \tau \)):

- \( \text{diam}(X_\tau) < 3 \text{diam}(X_{\pi(\tau)}) \),
- \( \text{diam}(X_{\tau-\omega}) < 1/(3|\tau|+1(|\tau|+1)), \)

because iterating projections in \((\omega+1)^n\) stabilizes before \( n+1 \) steps.

The crucial feature of the sets \( X_\tau \) is that this family should be “continuous”. Namely, we will require that if \( \tau \) and \( \pi(\tau) \) occur by the \( n \)-th step then

\[ h(X^n_\tau, X^n_{\pi(\tau)}) < 3|\tau| \text{d}(\tau, \pi(\tau)) \]

This condition is the most difficult. To fulfill it we will construct yet another kind of objects. Notice first that if \( h(A, B) < \varepsilon \) for two nonempty sets in \( X \) then there are two finite families (we will refer to them as to “anchors”) \( A_i \) and \( B_i \) \( (i \in I_0) \) of subsets of \( A \) and \( B \) respectively such that for any \( A'_i \subseteq A_i, B'_i \subseteq B_i \) still \( h(\bigcup_i A'_i, \bigcup_i B'_i) < \varepsilon \). Similarly, if \( h(A, B) < \varepsilon \) and \( C \subseteq A \) then there exist a finite family \( D_i \) \( (i \in I_0) \) of subsets of \( B \) such that for any \( D'_i \subseteq D_i \) \( h(\bigcup_i D'_i, C) < \varepsilon \).

At each step \( n \) if \( \tau \) is the \( n \)-the element of \((\omega+1)^{<\omega}\) we will additionally construct anchors

- for each pair \( X^n_{\sigma} \) and \( X^n_{\pi(\sigma)} \) such that \( \sigma, \pi(\sigma) \leq \tau \)
- and for each tripple \( X^n_{\sigma}, X^n_{\pi(\sigma)}, X^n_{\sigma-a} \) such that \( a \in \omega+1 \), \( \pi(\sigma-a) \subseteq \pi(\sigma) \) and \( \sigma, \pi(\sigma), \sigma-a \leq \tau \).

**Completing the diagram.** As we now have a clear picture of what should be constructed let us argue that this is enough to finish the proof. For each \( t \in (\omega+1)^{\omega} \) the intersection \( \bigcap_{n} X_{t|n} \) has precisely one point so let us define \( \varphi(t) \) to be this point. The other function, \( \psi \) is defined as \( f \circ \varphi \circ P^{-1} \). Let us check that this works. Both functions \( \psi \) and \( \varphi \) are injective thanks to the disjointness of the sets \( C_\tau \) and to the fact that \( X_\tau \subseteq f^{-1}[C_\tau] \). The function \( \psi \) is open because \( C_\tau \) are clopens.
To see continuity of $\varphi$ notice first that since the sets $X_\tau$ have diameters vanishing to 0, it suffices to check that $\varphi$ is continuous on each $(\omega+1)^n$ (which are treated as subsets of $(\omega+1)^\omega$ via the embedding $e: \tau \mapsto \tau^\omega(\omega,\omega,\ldots)$). Continuity on $(\omega+1)^n$ is checked inductively on the sets $S_k^n$ for $0 \leq k \leq n$.

The set $S_0^n$ consists of one point, so there is nothing to check. Suppose that $\tau_i \to \tau$, $\tau, \tau_i \in S_k^n$, $i \in \omega$. Then either the sequence is eventually constant or $\tau \in S_{k-1}^n$. Let us assume the latter. By the inductive assumption and continuity of projection $\varphi(\pi(\tau_i)) \to \varphi(\tau)$. Now pick any $\varepsilon > 0$. Let $m$ be such that $\text{diam}(X_\sigma) < \varepsilon$ for $\sigma \in (\omega+1)^n$ and $j \in \omega$ such that $d(\tau_j, \pi(\tau_j)) < 3^{-m}\varepsilon$. Let us write $\rho^ω$ for $\rho$ extended by $l$ many $\omega$’s. By (1) and coherence of projections we have

$$h(X_{\pi(\tau)}^\omega \setminus \omega^m, X_{\pi(\tau_j)}^\omega \setminus \omega^m) < \varepsilon,$$

which implies that $\varphi(\tau)$ and $\varphi(\pi(\tau_j))$ are closer than $3\varepsilon$. This shows that $\varphi(\tau_j) \to \varphi(\tau)$ and proves continuity of $\varphi$.

**Key lemma.** Now we state the key lemma, which will be used to guarantee “continuity” of the family of sets $X_\tau$.

**Lemma 2.** Let $X$ be a Borel set, $f: X \to \omega^\omega$ a Borel, not $\sigma$-continuous function. There exist a basic clopen $C_\omega \subseteq f[X]$ and a compact set $X_\omega \subseteq f^{-1}[C_\omega]$ such that

- $f[X_\omega]$ is not $\sigma$-continuous,
- $X_\omega \subseteq \text{cl}(f^{-1}[\omega^\omega \setminus C_\omega]^*)$.

The compact set $X_\omega$ can be chosen of arbitrarily small diameter.

**Proof.** Without loss of generality assume that $f^{-1}[C] = (f^{-1}[C])^*$ for all clopen sets $C \subseteq \omega^\omega$. Let us consider the following tree of open sets, indexed by $\omega^\omega$.

$$U_\tau = \text{int}(f^{-1}[[\tau]])$$

Let $G = \cap_n \cup_{|\tau| = n} U_\tau$ and $Z_\tau = f^{-1}[[\tau]] \setminus U_\tau$. Notice that $f \upharpoonright G$ is continuous and since $X = G \cup \bigcup_\tau Z_\tau$ there is $\tau \in \omega^\omega$ such that $Z_\tau \notin I_f$. Observe that $Z_\tau \subseteq \text{cl}(\bigcup_{\tau' \neq \tau, |\tau'| = |\tau|} f^{-1}[[\tau']])$ because if an open set $U \subseteq f^{-1}[[\tau]]$ is disjoint from $\bigcup_{\tau' \neq \tau, |\tau'| = |\tau|} f^{-1}[[\tau']]$ then $U \subseteq U_\tau$. Now put $C_\omega = [\tau]$ and pick any compact set with small diameter $X_\omega \subseteq Z_\tau$ such that $X_\omega \notin I_f$. 

**The construction.** We begin with $X_\emptyset = X_\emptyset^0 = X$ and $C_\emptyset = \omega^\omega$. Without loss of generality assume that $X = X^\omega$. Suppose we have done $n-1$ steps of the inductive construction up to $\tau \in (\omega+1)^{<\omega}$. Let $|\tau| = l$ and $\sigma = \tau|(l-1)$. There are three cases.
Case 1. The four points $\tau$, $\pi(\tau)$, $r(\tau)$ and $r(\pi(\tau))$ are equal. So $\tau = (\omega, \ldots, \omega)$ and $C_{\tau|n-1}$ and $X_{\omega}^{n-1}$ are already constructed. In this case we use Lemma 2 to find a clopen set $C_{\tau}$ and a compact set $X_{\omega} \subseteq X_{\omega}^{n-1}$ of diameter $< |\tau|/3^{n+1}$ small enough so that no element of the anchors constructed so far is contained in $X_{\tau}$. We put $X_{\tau}^n = X_{\tau}^n$, $X_{\sigma}^n = (X_{\sigma}^{n-1} \setminus f^{-1}[C_{\tau}])^*$ and $X_{\rho}^n = X_{\rho}^{n-1}$ for other $\rho < \tau$. By the assertion of Lemma 2 we still have $X_{\tau}^n \subseteq \text{cl}(X_{\sigma}^n)$. In this case we do not need to construct any new anchors.

Case 2. The two points $\pi(\tau)$ and $r(\tau)$ are equal but distinct from $\tau$. Let $\delta = d(\tau, r(\tau))$. Since $X_{\tau}^{n-1} \subseteq \text{cl}(X_{\omega}^{n-1})$ by the inductive assumption, we may find finitely many sets $B_i \subseteq X_{\omega}^{n-1}$, $i \leq k$ such that

- $h(\bigcup_i B_i, X_{\tau}^{n-1}) < \delta$ for any $B_i \subseteq B_i$,
- $B_i \not\in I_f$.

The second condition follows from $X_{\omega}^{n-1} = (X_{\omega}^{n-1})^*$. We may assume that for each clopen set $C \subseteq 2^\omega$ the set $B_i \cap f^{-1}[C]$ is either empty or outside of the ideal $I_f$.

We are going to find clopens $C_i \subseteq C_{\omega}$, for $i \leq k$ such that $C_i \cap C_{\tau} = \emptyset$ and then put $C_{\tau} = \bigcup_{i \leq k} C_i$, $X_{\omega}^n = (X_{\omega}^{n-1} \setminus \bigcup_i f^{-1}[C_i])^*$ and find $X_{\tau} = \bigcup_{i \leq k} B_i \cap f^{-1}[C_i]$. We will have to carefully define $X_{\tau}^{n-1}$ so that $X_{\tau}^{n-1} \subseteq \text{cl}(X_{\omega}^n)$.

It is easy to see that for any $A \subseteq X_{\omega}^{n-1}$

$$X_{\tau}^{n-1} = X_{\tau}^{n-1} \cap \text{cl}((X_{\omega}^{n-1} \setminus A)^*) \cup X_{\tau}^{n-1} \cap \text{cl}((X_{\omega}^{n-1} \setminus A^c)^*)$$

so (putting $A = f^{-1}[C_{\omega} \cap [(m, 0)]$ for $m < \omega$) we may inductively on $m$ pick binary sequences $\beta_i^m \in 2^m$, $i \leq k$ such that $f^{-1}[[\beta_i^m]] \cap B_i \neq \emptyset$ and

$$X_{\tau}^{n-1} \cap \text{cl}((X_{\omega}^{n-1} \setminus f^{-1}(\bigcup_{i \leq k} [[\beta_i^m]]))^*) \not\in I_f.$$
If \( \delta > \text{diam}(X_{\pi(\tau)}^{n-1}) \) then we can pick one \( X_i \) as \( X_{\tau} \) and then \( h(X_{\tau}, X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}^{n-1}) < 3^{|\tau|} \delta \). Otherwise, let \( X_{\tau} = \bigcup_{i \leq k} X_i \) and then \( \text{diam}(X_{\tau}) < 3 \text{diam}(X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}) \). Define \( X_{\rho}^{n} = X_{\tau}^{n} \).

At this step we create anchors for the pair \( X_{\tau}^{n} \) and \( X_{\rho(\tau)}^{n} \) as well as for the triples \( X_{\tau}^{n}, X_{\rho}^{n}, X_{\tau}^{n} \) for \( \rho < \tau \).

**Case 3.** The two points \( \pi(\tau), r(\tau) \) are distinct. Let \( \delta = d(\tau, \pi(\tau)) \). By coherence of the projections \( \pi(\tau) \supseteq \pi(\sigma) \). By the inductive assumption we have \( h(X_{\pi(\tau)}^{n-1}, X_{\pi(\sigma)}^{n-1}) < 3^{|\sigma|} \delta \). Using the existing anchor for the triple \( X_{\pi(\tau)}^{n-1}, X_{\pi(\sigma)}^{n-1}, X_{\pi(\tau)}^{n-1} \) let us find finitely many sets \( B_i, i \leq k \) in \( X_{\sigma} \) such that

- \( h(\bigcup B_i', X_{\pi(\tau)}) < 3^{|\sigma|} \delta \) for any \( B_i' \subseteq B_i \),
- \( B_i \not\subseteq I_f \).

As before, we assume assume that for each clopen set \( C \subseteq 2^\omega \) if \( B_i \cap f^{-1}[C] \not\subseteq I_f \) then it is empty. We have now two subcases, in analogy to the two previous cases.

**Subcase 3.1.** Suppose \( \tau = r(\tau) \). Similarly as in Case 1, we use Lemma 2 to find \( X_i \subseteq B_i \) and \( C_i \) for \( i \leq k \). Put \( C_{\tau} = \bigcup_{i \leq k} C_i \). If \( \delta > \text{diam}(X_{\pi(\tau)}^{n-1}) \) we can pick one \( X_i \) as \( X_{\tau} \) and then \( h(X_{\tau}, X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}^{n-1}) < 3^{|\tau|} \delta \). Otherwise, let \( X_{\tau} = \bigcup_{i \leq k} X_i \) and then \( \text{diam}(X_{\tau}) < 3 \text{diam}(X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}) \). Again, similarly as in Case 1, we put \( X_{\tau} = (X_{\tau})^*, X_{\sigma} = (X_{\sigma}^{n-1} \setminus f^{-1}[C])^*, X_{\rho} = X_{\rho}^{n-1} \) for other \( \rho < \tau \).

**Subcase 3.2.** Suppose \( \tau \neq r(\tau) \). Similarly as in Case 2, we find clopen \( C_i \) in \( \omega^\omega \) such that \( X_{\pi(\tau)}^{n-1} \cap \text{cl}((f^{-1}[C_{\sigma} \setminus \bigcup_{i \leq k} C_i])^*) \not\subseteq I_f \) and no existing anchor is destroyed when we put \( X_{\rho}^{n} = (X_{\rho}^{n-1} \setminus \bigcup_{i \leq k} f^{-1}[C_i])^* \) for \( \rho < \tau, \rho \not\geq r(\tau) \) and \( X_{\rho}^{n} = X_{\rho}^{n-1} \cap \text{cl}(X_{\pi(\tau)}^{n}) \) for \( \rho < \tau, \rho \geq r(\tau) \).

Next we find \( I_f \)-positive compact sets \( X_i \subseteq B_i \cap f^{-1}[C_i] \) each of diameter \( < 1/(3^{|\tau|+1}|\tau|) \). As previously, if \( \delta > \text{diam}(X_{\pi(\tau)}^{n-1}) \) then we can pick one \( X_i \) as \( X_{\tau} \) and then \( h(X_{\tau}, X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}^{n-1}) < 3^{|\tau|} \delta \). Otherwise, let \( X_{\tau} = \bigcup_{i \leq k} X_i \) and then \( \text{diam}(X_{\tau}) < 3 \text{diam}(X_{\pi(\tau)}^{n-1}) \leq 3 \text{diam}(X_{\pi(\tau)}) \). Again, we put \( X_{\tau} = (X_{\tau})^* \).

In Case 3 we construct the same anchors as in Case 2.

This ends the construction and the entire proof. \( \square \)
Theorem 5. If \( f : X \to \omega^\omega \) is not \( \sigma \)-continuous then there exist topological embeddings \( \varphi \) and \( \psi \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\omega^\omega & \xrightarrow{\psi} & \omega^\omega \\
\uparrow P \vphantom{\psi} & & \uparrow f \\
(\omega + 1)^\omega & \xrightarrow{\varphi} & X
\end{array}
\]

Proof. By Theorem 4 we have \( \psi \) and \( \varphi \) such that \( \psi \) is 1-1 open. But as a Borel function it continuous on a dense \( G_\delta \) set \( G \subseteq \omega^\omega \). On the other hand by the properties of the function \( P X \in I_P \) implies \( P[X] \) is meager. So \( P^{-1}[G] \notin I_P \) and the problem reduces to the restriction of the function \( P \). This is, however, much easier than the general case and a proof can be found in [2] (Corollary 2, Proposition 2). So we get the following diagram:

\[
\begin{array}{ccc}
\omega^\omega & \xrightarrow{\psi'} & G & \xrightarrow{\psi} & \omega^\omega \\
\uparrow P \vphantom{\psi'} & & \uparrow P[G] \vphantom{\psi} & & \uparrow f \\
(\omega + 1)^\omega & \xrightarrow{\varphi'} & P^{-1}[G] & \xrightarrow{\varphi} & X
\end{array}
\]

which ends the proof. \( \square \)

References


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