Exercises - Combinatorial Group Theory
List $2\frac{1}{2}$

Applications of Nielsen theory (and of its consequences)

1. Prove that the set a^2, ab, ba freely generates a subgroup in the free group $F_{\{a,b\}}$. Hint: check wether this tuple of elements is N-reduced (reduced in the sense of Nielsen), and if not, modify it by means of elementary Nielsen transformations.
2. Let $h : F_n \to F_m$ ba any **surjective** homomorphism between free groups of finite rank. Show that there exists a basis Y in F_n, being the disjoint union $Y = Y_1 \cup Y_2$, such that:
   (i) h restricted to the subgroup generated by Y_1 is an isomorphism onto F_m, and
   (ii) h restricted to the subgroup generated by Y_2 is trivial.
   Hint: for any basis X in F_n consider the tuple $h(X)$ in F_m Nielsen transformations which modify this tuple into a tuple $(u_1, \ldots, u_n)$, in which the subtuple $(u_1, \ldots, u_m)$ is a basis of F_m, and where $u_j = 1$ for $j > m$.
3. Around 1930 Hopf has posed the problem of existence of a finitely generated group G whose proper quotient G/H is isomorphic to G. Groups G with no proper quotient isomorphic to G are called since then *hopfian.* Prove that free groups of finite rank are hopfian. Hint: show that each surjective homomorphism $h : F_n \to F_n$ is an isomorphism; to do it, consider the tuple $U = h(X)$ consisting of the images of elements from the standard basis X of F_n.

   COMMENT: the first *non-hopfian* groups have been discovered in the 1950-ties by Higman i Neumann.
4. Consider the free group F_S, $S = \{a, b\}$, and let $c = [a, b]$ be the commutator of its standard generators. Prove that any automorphism of the group F_S maps c either to c or to c^{-1} or to a conjugate of one of these two elements. Hint: use the generators of of the automorphism group $\mathrm{Aut}(F_S)$ and cyclically reduced representatives of the conjugacy classes.
5. Let a, b be such elements of a free group F, that some of their powers a^m, b^n, for $m, n \neq 0$, commute. Prove that then a and b are powers of the same element $c \in F$. Hint: use the fact that subgroups of a free group are free.