**Exercises - Combinatorial Group Theory**
**List 5+. HNN-extensions.**

1. Show that when $A = B = G$ then for any isomorphism $\varphi : A \to B$ the HNN-extension $G*_\varphi$ coincides with a semi-direct product of $G$ and the infinite cyclic group $Z$. HINT: find normal forms of elements and the rule for their product.
2. Verify that $G$, viewed naturally as a subgroup in its any HNN-extension $G*_\varphi$, in general is NOT a normal subgroup. Find a necessary and sufficient condition for the subgroups $A, B$ under which $G$ is normal in $G*_\varphi$.
3. Define a suitable notion of cyclically $t$-reduced words for HNN extensions. Show that every element is conjugate to a cyclically $t$-reduced word.
4. Show that in an HNN extension $G*_\varphi$, any element of finite order is conjugate to an element of G.
5. (0) Show that the surface group

$$\Sigma_g = \langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \rangle$$

   is naturally an HNN-extension of the free group $F = \langle b_1, a_2, b_2, \ldots, a_g, b_g | \emptyset \rangle$.

   (1) Verify that the element $a_1 a_2 a_1^{-1} a_2^{-1} \in \Sigma_2$ is nontrivial, and that it is distinct from the element $b_1 b_2 b_1^{-1} b_2^{-1}$.

   (2) Derive an algorithm which solves the word problem in the surface group $\Sigma_g$, by viewing this group as an HNN-extension, as above.
6. Verify Exercise 10(c) from List 3 by referring to $p$-reduced forms in HNN-extensions.
7. Given an HNN-extension

$$G*_\varphi = \langle S, t | R, t^{-1} a t = \varphi(a) : a \in A \rangle,$$

   consider the subgroup $L < G*_\varphi$ generated by the union of the subgroups $G \cup tGt^{-1}$. Prove that $L$ is naturally isomorhic with the amalgamated free product $G *_A tGt^{-1}$.