Exercises - Algebraic Topology 1. List 7

Basics of singular homology. Simplicial homology.

Singular chains and cycles, singular homology

0. Let $\sigma, \sigma' : \Delta^n \to X$ be any two maps whose restrictions to the boundary of Δ^n coincide. Show that $\sigma - \sigma'$ is then an n-cycle in X.
1. Let $s : [0,1] \to X$ be any path in a topological space X. Consider the following two singular 1-simplices $\sigma_i : \Delta^1 \to X$, $i = 1,2$: $\sigma_1((1-t)e_0 + te_1) = s(t)$ and $\sigma_2((1-t)e_0 + te_1) = s(1-t)$.
   (1) Prove that $\sigma_1 + \sigma_2$ is a 1-cycle.
   (2) Prove that $\sigma_1 + \sigma_2$ is null-homologous, by describing an explicit 2-chain $a \in C_2X$ with $\sigma_1 + \sigma_2 = \partial a$.
2. A singular 1-simplex $\sigma : \Delta^1 \to X$ is called a *loop* if $\sigma(e_0) = \sigma(e_1)$.
   (a) Show that each loop is a 1-cycle.

   Two loops σ_0, σ_1 are *freely homotopic* if there is a continuous map $F : \Delta^1 \times [0,1]$ such that
   - for each $x \in \Delta^1$ we have $F(x,0) = \sigma_0(x)$ and $F(x,1) = \sigma_1(x)$,
   - for each $t \in [0,1]$ the map $\sigma_t : \Delta^1 \to X$ given by $\sigma_t(x) := F(x,t)$ is a loop.

   (b) Prove that any two freely homotopic loops are homologous, i.e. they induce the same element in the homology group H_1X.

   A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ such that $\sigma_i(e_0) = \sigma_{i-1}(e_1)$ for each $i \in Z/rZ$ is called an *elementary 1-cycle.*

   (c) Prove that each elementary 1-cycle is a 1-cycle.
   (d) Prove that each elementary 1-cycle is homologous with some loop.
   (e) Show that the elements in the homology group H_1X induced by loops generate this group.
   (f) Prove that if X is path-wise connected then each element in the homology group H_1X is induced by a loop.
   (g) Prove that if X is path-wise connected, and if $\pi_1X = 0$, then $H_1X = 0$.
3. Prove that the homomorphisms $H_kX \to H_kY$, for $k > 0$, induced by the maps $f : X \to Y$ which are constant, are trivial.
4. Let $A \subset X$ be a retract, and let $r : X \to A$ be a retraction map, i.e. a continuous map such that $r(x) = x$ for all $x \in A$. Denote also by $i : A \to X$ the corresponding inclusioon map. For each integer $k \geq 0$, denote by $r_k : H_kX \to H_kA$ and $i_k : H_kA \to H_kX$ the homomorphisms induced by r and i, respectively.
   (1) Show that each i_k is injective.
   (2) Show that for each $k \geq 0$ we have $H_kX \cong H_kA \oplus \ker(r_k)$.

Relative singular homology and exact sequence of a pair

5. Show that for $B \subset A \subset X$ and for any fixed n the sequence of homomorphisms

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

induced by inclusions is exact.

6. Justify that the quotient homomorhisms $j : C_nX \to C_n(X, A) = C_nX/C_nA$ form a chain map between the corresponding singular chain complexes (verify that they commute with the corresponding boundary homomorphisms).
7. Check, both directly from the definition and by applying the exact sequence for pairs, what is the relationship between the homology groups H_nX and $H_n(X, x)$, where $x \in X$ is any point.
8. Check directly, by referring to geometric interpretations of the corresponding homomorphisms (as recalled below), that the exact sequence of a pair (X, A)

$$\to H_nA \to H_nX \to H_n(X, A) \to H_{n-1}A \to H_{n-1}X \to$$

   is indeed exact.
   HINT: use the following facts. For the inclusion $i : A \to X$ and for a cycle $c \in C_nA$, we have $i_\star([c]) = [c]$. For the quotient map $j : C_nX \to C_n(X, A)$ and for a cycle $c \in C_nX$, we have $j_\star([c]) = [c + C_nA]$. For the adjoint boundary homomorphism $\partial : H_n(X, A) \to H_{n-1}A$, and for a cycle $c + C_nA \in C(X, A)$ we have $\partial([c + C_nA]) = [\partial c]$.
9. Show that for a pair of spaces (X, A), the inclusion $A \to X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.
10. Show that $H_0(X, A) = 0$ iff A intersects each path-component of X.
11. Show that $H_1(X, A) = 0$ iff $H_1A \to H_1X$ is surjective and each path-component of X contains at most one path component of A.

Simplicial homology

12. Let X be an n-dimensional simplicial complex.
    (a) Show that each nontrivial (i.e. $\neq 0$) simplicial n-cycle in X represents a nontrivial homology class in H_nX.
    (b) Show that for $k > n$ we have $H_kX = 0$.
13. Consider a closed connected n-dimensional manifold M with a fixed triangulation. Suppose that this manifold is orientable, i.e. the n-simplices of its triangulation can be oriented *consistently*, which means that for any $(n-1)$-simplex τ of this triangulation the orientations induced from the orientations of the two n-simplices containing τ are opposite.
    (a) Using simplicial homology, show that $H_nM = Z$.
    (b) Suppose M is closed, connected, n-dimensional, triangulated and non-orientable. Show that then $H_nM = 0$.
14. Let $K(3, 3, 3)$ be a 2-dimensional simplicial complex described as follows. Consider sets A, B, C consisting of 3 elements. Identify the vertex set of $K(3, 3, 3)$ with the disjoint union $A \sqcup B \sqcup C$, and the set od 2-simplices with the family of all such subsets $T \subset A \sqcup B \sqcup C$ which have precisely one element in each of A, B and C. Compute the simplicial homology of $K(3, 3, 3)$.
15. Let X be a simplicial complex, and let CX be its simplicial cone. Check directly that for $k > 0$ any simplicial k-cycle in CX is null-homologous, so that $H_k(CX) = 0$.