

Exercises - Algebraic Topology 1. List 8 (extended version)

Good pairs, homology of spheres, degrees of maps between spheres,

1. Recall that, given a space X , the *cone* over X , denoted CX , is the quotient $CX := (X \times [0, 1]) / (X \times \{1\})$, and the *suspension* over X is the quotient $SX := (X \times [0, 1]) / (X \times \{0, 1\})$. View naturally X as a subspace in CX , by identifying it with the subspace $X \times \{0\}$.
 - (1) Show that $H_{k+1}(CX, X) = \tilde{H}_k X$ for each $k \geq 0$.
 - (2) Show that (CX, X) is a good pair, and that $CX/X \cong SX$. Deduce that $H_{k+1}SX \cong \tilde{H}_k X$ for any $k \geq 0$.
2. Given a subspace $A \subset X$, define $X \cup CA$ as the quotient $X \sqcup CA / \sim$, where the relation \sim corresponds to the identical identification of the subset $A \subset X$ with the subset $A \subset CA$. Show that if (X, A) is a good pair then $\tilde{H}_n(X \cup CA) \cong H_n(X, A)$ for all $n \geq 0$.
3. Given a continuous map $f : X \rightarrow Y$, consider the *cone* of this map as $Cf : CX \rightarrow CY$ given by $Cf([(x, t)]) = [(f(x), t)]$.
 - (A) Use naturality of exact sequences to show that, under isomorphisms

$$H_{n+1}(CX, X) = \tilde{H}_n X \quad \text{and} \quad H_{n+1}(CY, Y) = \tilde{H}_n Y$$

provided by the long exact sequence of pairs, the induced homomorphisms $f_* : \tilde{H}_n X \rightarrow \tilde{H}_n Y$ and $(Cf)_* : H_{n+1}(CX, X) \rightarrow H_{n+1}(CY, Y)$ coincide.

- (B) Formulate and prove a similar property for suspensions.
- (C) Assume that we know that the map $f : S^1 \rightarrow S^1$ given by $f(e^{it}) = e^{int}$ has degree n . Use (B) to construct, for any $k > 1$ and any integer n , a map $S^k \rightarrow S^k$ of degree n .
4. (A) Prove that any map $f : D^n \rightarrow D^n$, for any $n > 2$, has a fixed point, by imitating the proof for $n = 2$, which was using the fundamental groups, but now use homology.
 - (B) Prove the same by applying degree theory to the map $S^n \rightarrow S^n$ that sends both the northern and southern hemispheres of S^n (both naturally and consistently identified with D^n) to the southern hemisphere via f .
[This was Brouwer's original proof.]
5. Let $f : S^n \rightarrow S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in R^n such that $F(x) \neq 0$ for all x , then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.
6. Prove, using homology, that if $m \neq n$, then R^m and R^n are not homeomorphic.

CW-complexes and cellular homology

7. Verify that for any CW-complex X the pair (X^n, X^{n-1}) is a good pair of spaces.
8. Provide details of the (inductive) argument for showing that any compact subset of a CW-complex X is contained in some finite subcomplex of X .
9. For a given finite CW-complex X the cellular boundary map $\partial_{n+1}^{CW} : C_{n+1}^{CW} X \rightarrow C_n^{CW} X$ is the “ \mathbb{Z} -linear” map given by the matrix with integer coefficients $i_{\alpha, \beta}$ (incidence coefficients for pairs of $(n+1)$ - and n -cells). Check how is this matrix modified if one changes
 - (a) the orientation of one of the $(n+1)$ -cells,
 - (b) the orientation of one of the n -cells.

10. Let $f : S^n \rightarrow S^n$ be a map of degree m , and let $X = S^n \cup_f D^{n+1}$ be the space obtained from S^n by attaching an $(n+1)$ -cell via f . Describe some convenient CW -structure on X and use it to compute the homology of X .
11. Let G be an arbitrary finitely generated abelian group (i.e. a finite direct sum of cyclic groups, infinite or finite). For any $n \geq 1$, construct a connected CW -complex X such that $H_n X \cong G$ and $\tilde{H}_i X = 0$ for all $i \neq n$.
12. Compute the homology groups of the following 2-complexes:
 - (1) the quotient of S^2 obtained by identifying north and south poles to a point;
 - (2) $S^1 \times (S^1 \vee S^1)$;
 - (3) the space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles;
 - (4) the quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Apply cellular homology to conveniently and efficiently chosen cell structures for the corresponding spaces. To derive incidence numbers for relevant pairs (2-cell, 1-cell) use the fact that degrees of maps between 1-circles can be expressed equivalently in terms of the induced homomorphisms of their fundamental groups.

13. Show that the quotient map $S^1 \times S^1 \rightarrow S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2 . On the other hand, show via covering spaces that any map $S^2 \rightarrow S^1 \times S^1$ is nullhomotopic.
14. The *real projective space* RP^n is the quotient of S^n via the antipodal identifications (for all pairs of antipodal points in S^n).
 - (1) Compute homology groups of RP^n .
 - (2) Show that when n is odd then the quotient map $q : S^n \rightarrow RP^n$ induces a homomorphism $q_* : H_n S^n \rightarrow H_n RP^n$ that sends a generator to twice a generator.
 - (3) Describe the homomorphism q_* in the case when n is even.
15. Let X be a CW -complex, and let Y be obtained by attaching an n -cell to $X^{(n-1)}$ using the attaching map $f : S^{n-1} \rightarrow X^{(n-1)}$, which is itself cellular for the standard cellular structure on S^{n-1} consisting of one 0-cell and one $(n-1)$ -cell. Let $i : X^{(n-1)} \rightarrow X$ be the inclusion map, and let $\bar{f} := i \circ f : S^{n-1} \rightarrow X$ (where we view \bar{f} also as a cellular map). Use cellular homology to show that:
 - (1) the induced homomorphism $i_* : \tilde{H}_k X \rightarrow \tilde{H}_k Y$ is an isomorphism when $k \neq n$ and $k \neq n-1$.

Describe the appropriate natural homomorphisms appearing in the sequences below, and show that these sequences are exact:

$$\begin{aligned}
 (2) \quad & 0 \rightarrow \text{image}(\bar{f}_*) \rightarrow \tilde{H}_{n-1} X \rightarrow \tilde{H}_{n-1} Y \rightarrow 0, \\
 (3) \quad & 0 \rightarrow \tilde{H}_n X \rightarrow \tilde{H}_n Y \rightarrow \ker(\bar{f}_*) \rightarrow 0.
 \end{aligned}$$