Exercises - Algebraic Topology 1. List 8

Good pairs, homology of spheres, degrees of maps between spheres, cellular homology

1. Recall that, given a space X, the *cone* over X, denoted CX, is the quotient $CX := (X \times [0,1])/(X \times \{1\})$, and the *suspension* over X is the quotient $SX := (X \times [0,1])/(X \times \{0,1\})$. View naturally X as a subspace in CX, by identifying it with the subspace $X \times \{0\}$.
   (1) Show that $H_{k+1}(CX, X) = \widetilde{H}_k X$ for each $k \geq 0$.
   (2) Show that (CX, X) is a good pair, and that $CX/X \cong SX$. Deduce that $H_{k+1}SX \cong \widetilde{H}_k X$ for any $k \geq 0$.
2. Given a subspace $A \subset X$, define $X \cup CA$ as the quotient $X \sqcup CA/\sim$, where the relation $\sim$ corresponds to the identical identification of the subset $A \subset X$ with the subset $A \subset CA$. Show that if (X, A) is a good pair then $\widetilde{H}_n(X \cup CA) \cong H_n(X, A)$ for all $n \geq 0$.
3. Given a continuous map $f : X \to Y$, consider the *cone* of this map as $Cf : CX \to CY$ given by $Cf([(x,t)]) = [(f(x), t)]$.
   (A) Use naturality of exact sequences to show that, under isomorphisms

   $$H_{n+1}(CX, X) = \widetilde{H}_n X \quad \text{and} \quad H_{n+1}(CY, Y) = \widetilde{H}_n Y$$

   provided by the long exact sequence of pairs, the induced homomorphisms $f_* : \widetilde{H}_n X \to \widetilde{H}_n Y$ and $(Cf)_* : H_{n+1}(CX, X) \to H_{n+1}(CY, Y)$ coincide.
   (B) Formulate and prove a similar property for suspensions.
   (C) Assume that we know that the map $f : S^1 \to S^1$ given by $f(e^{it}) = e^{int}$ has degree n. Use (B) to construct, for any $k > 1$ and any integer n, a map $S^k \to S^k$ of degree n.
4. (A) Prove that any map $f : D^n \to D^n$, for any $n > 2$, has a fixed point, by imitating the proof for $n = 2$, which was using the fundamental groups, but now use homology.
   (B) Prove the same by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n (both naturally and consistently identified with D^n) to the southern hemisphere via f.
   [This was Brouwer's original proof.]
5. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continous vector field defined on the unit ball D^n in R^n such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.
6. Compute the homology groups of the following 2-complexes:
   (1) the quotient of S^2 obtained by identifying north and south poles to a point;
   (2) $S^1 \times (S^1 \vee S^1)$;
   (3) the space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles;

(4) the quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Apply cellular homology to conveniently and efficiently chosen cell structures for the corresponding spaces. To derive incidence numbers for relevant pairs (2-cell, 1-cell) use the fact that degrees of maps between 1-circles can be expressed equivalently in terms of the induced homomorphisms of their fundamental groups.

7. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.
8. The *real projective space* RP^n is the quotient of S^n via the antipodal identifications (for all pairs of antipodal points in S^n).
   (1) Compute homology groups of RP^n.
   (2) Show that when n is odd then the quotient map $q : S^n \to RP^n$ induces a homomorphism $q_* : H_n S^n \to H_n RP^n$ that sends a generator to twice a generator.
   (3) Describe the homomorphism q_* in the case when n is even.