where

$$|R| \leqslant \frac{1}{720}|f'''(x)|.$$

In particular, for $f(x) = x^{-1}$, we have

$$f^{(n)}(x) = (-1)^n n! x^{-n-1},$$

so

$$\psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^4}\right)$$

as $x \to \infty$. From the Taylor expansion

$$\log(a+t) = \log a + \frac{t}{a} - \frac{t^2}{2a^2} + O\left(\frac{t^3}{a^3}\right) \qquad \left(\frac{t}{a} \to 0\right),$$

we deduce

$$\log \psi(x) = \log\log x - \frac{1}{2x \log x} - \frac{1}{12x^2 \log x} - \frac{1}{8x^2(\log x)^2} + O\left(\frac{1}{x^3(\log x)^2}\right).$$

For $f(x) = (x \log x)^{-1}$ we have

$$f^{(n)}(x) = \frac{(-1)^n}{x^{n+1}\log x} Q_n\left(\frac{1}{\log x}\right),$$

where Q_n is a polynomial of degree n determined by the recurrence relation

$$Q_{n+1}(t) = (n+1+t)Q_n(t) + t^2 Q_n'(t)$$

and the initial condition $Q_0 = 1$. Clearly, Q_n has only positive coefficients, so the monotonicity condition is satisfied for all n. We have, then, again by use of the Euler-Maclaurin formula,

$$P(x) = \log\log x - \frac{1}{2x\log x} - \frac{1}{12x^2 \log x}\left(1 + \frac{1}{\log x}\right) + O\left(\frac{1}{x^4 \log x}\right),$$

and therefore

$$P(x) - \log\psi(x) = \frac{1}{24x^2(\log x)^2} + O\left(\frac{1}{x^3(\log x)^2}\right),$$

which is clearly positive for sufficiently large x.

Also solved by I. E. Leonard, O. P. Lossers (The Netherlands), and the proposer.

Spectral Radii as Possibly Unattained Infima

6496 [1985, 362]. *Proposed by Ryszard Szwarc, University of Wrocław, Poland.*

Let T be a bounded operator with spectral radius 1 on a given Hilbert space H, and let $c > 1$. Prove that there is an invertible operator A on H such that $\|ATA^{-1}\| \leqslant c$.

Combined solution. Since the spectral radius of T is

$$\lim_{n\to\infty} \|T^n\|^{1/n} = 1,$$

we may define a positive Hermitian operator A by

$$A^2 = \sum_{n=0}^{\infty} c^{-2n}(T^*)^n T^n = I + c^{-2}T^*T + \cdots .$$

This A is invertible since $I \leqslant A$. Define a norm equivalent to $\|\cdot\|$ by

$$\|x\|_c = \|Ax\|.$$

By the definition of A,

$$\|Tx\|_c^2 = c^2\|x\|_c^2 - c^2\|x\|^2 \leqslant c^2\|x\|_c^2,$$

so for any y in H we have

$$\|ATA^{-1}y\|^2 = \|TA^{-1}y\|_c^2 \leqslant c^2\|A^{-1}y\|_c^2 = \|y\|^2$$

and the result follows.

If we replace T by rT where $r > 0$, the result may be stated in a formally more general way: if r is the spectral radius of T and $c > r$, then

$$\|ATA^{-1}\| \leqslant c$$

for some invertible A. In other words,

$$\inf\{\|ATA^{-1}\| : A \text{ is invertible}\} = \text{spectral radius}(T).$$

In this form it was proved by G. -C. Rota, On models for linear operators, Comm. Pure Appl. Math., 13(1960), 469–472. Related questions and generalizations are considered by B. Sz.-Nagy, Completely continuous operators with uniformly bounded iterates, Publ. Math. Inst. Hung. Acad. Sci., 4 (1959), 89–92; F. Gilfeather, Norm conditions on resolvents..., Proc. Amer. Math. Soc., 68(1978), 44–48; and P. Halmos, *A Hilbert Space Problem Book*, Cor. 4 to Problem 153.

The infimum in the last formula is attained for spaces of dimension at most one, but need not be attained in any higher dimension. In fact, the matrix operator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is already a counterexample in dimension 2. Here $(T - I)^2 = 0$, so T has spectral radius 1. Also,

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

so if the 2-vector v satisfies

$$A^{-1}v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then

$$AT^nA^{-1}v = A\begin{pmatrix} n \\ 1 \end{pmatrix},$$

and

$$|(ATA^{-1})v| = \left|A\binom{n}{1}\right| \to \infty.$$

But

$$\|(ATA^{-1})^n\| \leqslant \|ATA^{-1}\|^n,$$

so the infimum of $\|ATA^{-1}\|$ is not attained. (The editor thanks H. Lotz, M. J. Pelling, and especially H. Porta for helpful comments and analysis.)

Solutions were received from K. N. Boyadzhiev (Bulgaria), F. Gilfeather, B. Sz.-Nagy (Hungary), Pei Yuan Wu (Taiwan), and the proposer.

---

q-Analogues of a Gamma Function Identity

6497 [1985, 362]. *Proposed by Richard Askey, University of Wisconsin.*

Let $0 < q < 1$, $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$. Show that

$$\int_0^\infty \frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\frac{d_qt}{t} = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} \tag{1}$$

and

$$\int_0^\infty \frac{(-tq^b;q)_\infty(-q^{a+1}/t;q)_\infty}{(-t;q)_\infty(-q/t;q)_\infty}\frac{dt}{t} = \frac{-\log q}{1-q}\,\frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \tag{2}$$

where

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1-xq^n),$$

$$\Gamma_q(x) := (q;q)_\infty(1-q)^{1-x}/(q^x;q)_\infty,$$

and

$$\int_0^\infty f(t)\,d_qt := (1-q)\sum_{-\infty}^{\infty} f(q^n)q^n.$$

These extend the gamma function identity

$$\int_0^\infty \frac{dt}{t(1+t)^b(1+t^{-1})^a} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

to q-gamma functions (for properties see R. Askey, Ramanujan's extensions of the gamma and beta functions, this MONTHLY, 87 (1980) 346–359).