



Inhomogeneous Jacobi Matrices on Trees

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Received: 9 May 2016 / Revised: 31 January 2018 / Accepted: 12 February 2018 /
Published online: 7 May 2018
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Abstract We study Jacobi matrices on trees with one end at infinity. We show that the defect indices cannot be greater than 1 and give criteria for essential self-adjointness.

Keywords Jacobi matrix · Tree · Essential self-adjointness

Mathematics Subject Classification Primary 47B36

1 Introduction

The aim of the paper is to study a special class of symmetric unbounded operators and their spectral properties. These are Jacobi operators defined on trees. They are immediate generalizations of classical Jacobi matrices that act on sequences $\{u_n\}_{n=0}^{\infty}$ by the rule

$$(Ju)_n = \lambda_n u_{n+1} + \beta_n u_n + \lambda_{n-1} u_{n-1}, \quad n \geq 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive and real numbers, respectively, with the convention $u_{-1} = \lambda_{-1} = 0$. These matrices are closely related to the set of polynomials defined recursively by

Communicated by Erik Koelink.

The author acknowledges support by Polish Grant NCN 2013/11/B/ST1/02308.

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$$xp_n(x) = \lambda_n p_{n+1}(x) + \beta_n p_n(x) + \lambda_{n-1} p_{n-1}(x), \quad n \geq 0, \quad (1)$$

with $p_{-1} = 0$, $p_0 = 1$.

In the case when the coefficients of the matrix are bounded, the matrix J represents a self-adjoint operator on $\ell^2(\mathbb{N}_0)$. If $E(x)$ denotes the resolution of identity associated with J , then the polynomials $p_n(x)$ are orthonormal with respect to the measure $d\mu(x) = d(E(x)e_0, e_0)$, where e_0 is the sequence taking value 1 at $n = 0$ and vanishing elsewhere and (u, v) denotes the standard inner product in $\ell^2(\mathbb{N}_0)$. The measure μ has bounded support.

When the coefficients are unbounded, the operator J is well defined on the domain $D(J)$ consisting of sequences with finitely many nonzero terms. In that case, if this operator is essentially self-adjoint, then again the polynomials p_n are orthonormal with respect to the measure $d\mu(x) = d(E(x)e_0, e_0)$, except that this measure has unbounded support. Moreover, there is a unique orthogonality measure for polynomials p_n . By a classical theorem, if the operator J is not essentially self-adjoint, there are many measures μ on the real line so that the polynomials belong to $L^2(\mu)$; i.e.,

$$\int_{-\infty}^{\infty} x^{2n} d\mu(x) < \infty, \quad n \in \mathbb{N}_0,$$

and the polynomials p_n are orthogonal with respect to the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} d\mu(x).$$

Therefore essential self-adjointness is a crucial property that distinguishes between the so-called determinate and indeterminate cases. Intuitively the unbounded matrix J is essentially self-adjoint when the coefficients have moderate growth. But the converse is not true in general. For the classical theory of Jacobi matrices, orthogonal polynomials, and moment problems, we direct the reader to [1, 2, 6], and to [5] for a modern treatment.

In a recent paper [3], homogeneous Jacobi matrices on homogeneous trees were studied. Two types of homogeneous trees were considered. One of them was the tree with infinitely many origin points called leaves, like in Fig. 1.

The tree Γ consists of vertices with heights from zero to infinity. Every vertex x with height $n \geq 1$ is connected to a unique vertex $\eta(x)$, the parent, with height $n - 1$, and d vertices x_1, \dots, x_d with height n , the children, like in Fig. 2.

Every vertex x with height zero determines the infinite sequence $\eta^k(x)$ of vertices with height k . Moreover, for every two vertices x and y with height zero, the sequences $\eta^k(x)$ and $\eta^k(y)$ coincide for k large enough. Therefore, we say that the tree Γ has one end at infinity:

The Jacobi matrices were defined on $\ell^2(\Gamma)$, where Γ denotes the set of all vertices of the tree. The formula is as follows:

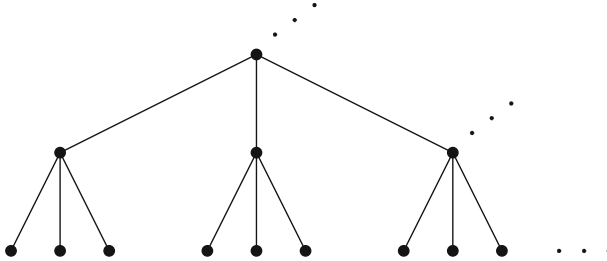
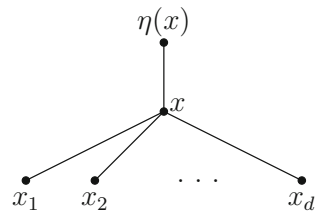


Fig. 1 A tree with infinitely many origin points

Fig. 2 The vertex x and its neighborhood



$$(Jv)(x) = \lambda_n v(\eta(x)) + \beta_n v(x) + \lambda_{n-1} [v(x_1) + v(x_2) + \dots + v(x_d)],$$

where n denotes the height of the vertex x .

An interesting phenomenon occurred. It turned out that the operator J defined on functions $\{v(x)\}_{x \in \Gamma}$, with finitely many nonzero terms, is always essentially self-adjoint, regardless of the growth of the coefficients λ_n and β_n . For example, the operator J with coefficients $\lambda_n = (n + 1)^2$ and $\beta_n = 0$ is not essentially self-adjoint when considered as the classical Jacobi matrix on $\ell^2(\mathbb{N}_0)$. But it is essentially self-adjoint when it acts on $\ell^2(\Gamma)$.

Moreover, its spectrum is discrete and consists of the zeros of all the polynomials p_n associated with the classical Jacobi matrix with coefficients $\sqrt{d} \lambda_n$ and β_n , i.e., satisfying

$$x p_n(x) = \sqrt{d} \lambda_n p_{n+1}(x) + \beta_n p_n(x) + \sqrt{d} \lambda_{n-1} p_{n-1}(x), \quad n \geq 0.$$

Every eigenvalue is of infinite multiplicity.

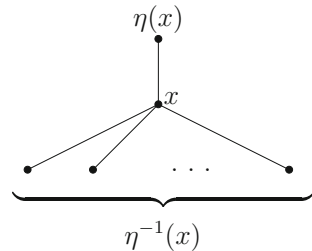
Our aim is to study the *inhomogeneous* Jacobi matrix on that tree. This means we do not require that the coefficients of the matrix depend only on the height of the vertex. With every vertex x , we associate a positive number λ_x and a real number β_x . We are going to study operators of the form

$$Jv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \dots + \lambda_{x_d} v(x_d).$$

One of the main differences between the classical case and the case of the tree Γ is that the eigenvalue equation

$$z v(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \dots + \lambda_{x_d} v(x_d) \quad (2)$$

Fig. 3 The vertex x and its direct relatives



cannot be solved recursively, unlike the equation

$$zv(n) = \lambda_n v(n + 1) + \beta_n v(n) + \lambda_{n-1} v(n - 1).$$

Nonetheless, we will show that the equation has a nonzero solution for every nonreal number z (Corollary 5).

Actually, when we give up homogeneity of the matrix J , we can as well give up homogeneity of the tree. This means the number of descendants of vertices of Γ is not constant; i.e., the quantities $\#\eta^{-1}(x)$ may vary.

The operator J is symmetric on $\ell^2(\Gamma)$ with respect to the natural inner product

$$(u, v) = \sum_{x \in \Gamma} u(x) \overline{v(x)}.$$

We are interested in studying the essential self-adjointness of the matrix J . It turns out that, unlike in the homogeneous case, the matrix J may not be essentially self-adjoint. However, the defect indices cannot be greater than 1 (Proposition 6). We derive certain criteria assuring essential self-adjointness. For example, the analog of the Carleman condition holds (see Theorem 16). Moreover, we relate essential self-adjointness of J with essential self-adjointness of the classical Jacobi matrix J_0 obtained from J by restriction to an infinite path of the tree (see Theorem 9).

2 Definitions and Basic Properties

We will consider a tree Γ with one end at infinity. Its vertices are located on heights from zero to infinity. Let $l(x)$ denote the height of the vertex x . Every vertex x with height $l(x) \geq 0$ is directly connected to a unique vertex $\eta(x)$ with height $l(x) + 1$, the parent. When $l(x) \geq 1$, the vertex x is directly connected to a finite nonempty set of vertices y on height $l(x) - 1$, called its children. The set of children of x will be denoted by $\eta^{-1}(x)$ (see Fig. 3). The number of vertices in $\eta^{-1}(x)$ may vary with x . Vertices x with $l(x) = 0$ have no children; i.e., $\eta^{-1}(x) = \emptyset$.

For a given vertex x , let Γ_x denote the finite subtree containing the vertex x together with all its descendants, i.e., vertices y such that $\eta^k(y) = x$ for some k . Thus $l(y) = l(x) - k$.

Define $\mathcal{F}(\Gamma)$ to be the set of all complex valued functions on Γ , while $\mathcal{F}_0(\Gamma)$ denotes the subspace consisting of functions with finite support. By $\ell^2(\Gamma)$, we denote the space of square summable functions on Γ with standard inner product

$$(f, g) = \sum_{x \in \Gamma} f(x) \overline{g(x)}.$$

Let

$$\delta_x(y) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases}$$

Consider the operator J acting on $\mathcal{F}_0(\Gamma)$ according to the rule

$$J\delta_x = \lambda_x \delta_{\eta(x)} + \beta_x \delta_x + \sum_{y \in \eta^{-1}(x)} \lambda_y \delta_y, \quad l(x) \geq 1, \tag{3}$$

$$J\delta_x = \lambda_x \delta_{\eta(x)} + \beta_x \delta_x, \quad l(x) = 0, \tag{4}$$

where λ_x are positive constants while β_x are real ones.

Remark In the case when $\#(\eta^{-1}(x)) = 1$, for all $x \in \Gamma$ with $l(x) \geq 1$, the tree Γ consists of one vertical line that can be identified with \mathbb{N}_0 . The matrix J becomes a classical Jacobi matrix.

Let S and S^* be the operators acting on $\mathcal{F}_0(\Gamma)$ as follows:

$$S\delta_x = \lambda_x \delta_{\eta(x)}, \quad S^*\delta_x = \begin{cases} \sum_{y \in \eta^{-1}(x)} \lambda_y \delta_y, & l(x) > 0, \\ 0, & l(x) = 0. \end{cases}$$

Then

$$(Sf, g) = (f, S^*g), \quad f, g \in \mathcal{F}_0(\Gamma).$$

The operators S and S^* are straightforward generalizations of weighted shift and backward weighted shift operators usually acting on $\ell^2(\mathbb{N}_0)$. Let M be a multiplication operator on $\mathcal{F}_0(\Gamma)$ defined by

$$M\delta_x = \beta_x \delta_x.$$

Then

$$J = S + S^* + M.$$

In particular, J is a symmetric linear operator on $\mathcal{F}_0(\Gamma)$.

Let $v \in \mathcal{F}_0(\Gamma)$ and $x \in \Gamma$. Then

$$Jv(x) = \begin{cases} \lambda_x v(\eta(x)) + \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y) & l(x) \geq 1, \\ \lambda_x v(\eta(x)) + \beta_x v(x) & l(x) = 0. \end{cases}$$

The formula makes sense for all functions v in $\mathcal{F}(\Gamma)$. Hence we may extend the definition of J to the whole space $\mathcal{F}(\Gamma)$, by setting

$$Jv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y), \quad v \in \mathcal{F}(\Gamma).$$

On the way, we have simplified the notation as $\eta^{-1}(x) = \emptyset$ for $l(x) = 0$.

Observe that the adjoint of $(J, \mathcal{F}_0(\Gamma))$ in $\ell^2(\Gamma)$ is the restriction of this extension to the domain of the adjoint operator, i.e., to the space of all functions $v \in \ell^2(\Gamma)$, so that Jv belongs to $\ell^2(\Gamma)$.

We will study eigenfunctions of the operator J , i.e., functions $v \in \mathcal{F}(\Gamma)$ satisfying

$$Jv = zv$$

or equivalently,

$$zv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y). \tag{5}$$

Unlike in the classical case, this equation cannot be solved recursively; i.e., setting $v(x_0) = 1$ at a vertex x_0 , with $l(x_0) = 0$, doesn't allow recursive computation of all other values $v(x)$, for $x \in \Gamma$. Therefore the existence of nonzero solutions of (5) is not obvious. Our aim is to show that such solutions exist for every nonreal number z .

For $x \in \Gamma$, let J_x denote the truncation of the Jacobi matrix J to the subtree Γ_x , i.e., the matrix with the parameters λ_y^x, β_y^x so that

$$\lambda_y^x = \begin{cases} \lambda_y & \text{for } y \in \Gamma_x \setminus \{x\}, \\ 0 & \text{for } y = x, \end{cases} \quad \beta_y^x = \beta_y \text{ for } y \in \Gamma_x.$$

Lemma 1 *Fix a vertex $x \in \Gamma$. Assume there exists a nonzero function $v \in \mathcal{F}(\Gamma_x \cup \{\eta(x)\})$ and $z \notin \mathbb{R}$ such that $Jv(y) = zv(y)$ for $y \in \Gamma_x$. Then $v(\eta(x)) \neq 0$.*

Proof Assume for a contradiction that $v(\eta(x)) = 0$. Let w denote the truncation of v to Γ_x . Thus $J_x w = zw$. Moreover $w \neq 0$. Therefore z must be a real number, as J_x is a finite dimensional symmetric linear operator. □

Lemma 2 *Fix a vertex $x \in \Gamma$. Assume there exists $0 \neq v \in \mathcal{F}(\Gamma_x)$ and $z \notin \mathbb{R}$ such that $(Jv)(y) = zv(y)$ for $y \in \Gamma_x \setminus \{x\}$. Then*

$$zv(x) \neq \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y).$$

Proof Assume for a contradiction that

$$zv(x) = \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y).$$

Define the function $u \in \mathcal{F}(\Gamma_x \cup \{\eta(x)\})$ by setting $u(y) = v(y)$ for $y \in \Gamma_x$ and $u(\eta(x)) = 0$. Then $(Ju)(y) = zu(y)$ for $y \in \Gamma_x$. In view of Lemma 1, we get a contradiction. \square

Corollary 3 *Assume there exists a function $v \neq 0$ on Γ and $z \notin \mathbb{R}$ such that $(Jv)(x) = zv(x)$ for $x \in \Gamma$. Then v does not vanish on Γ .*

Proof Assume for a contradiction that $v(y) = 0$ for a vertex y . First we will show that v vanishes on Γ_y . If $\ell(y) = 0$, then $\Gamma_y = \{y\}$, and the conclusion follows. Assume $\ell(y) \geq 1$. Consider any vertex $x \in \eta^{-1}(y)$; i.e., $y = \eta(x)$. Then by Lemma 1, we get that the function v vanishes identically on Γ_x . But

$$\Gamma_y = \{y\} \cup \bigcup_{x \in \eta^{-1}(y)} \Gamma_x;$$

hence v vanishes on Γ_y . From the recurrence relation

$$zv(y) = \lambda_y v(\eta(y)) + \beta_y v(y) + \sum_{u \in \eta^{-1}(y)} \lambda_u v(u),$$

we get $v(\eta(y)) = 0$. Therefore, by the first part of the proof, v vanishes identically on $\Gamma_{\eta(y)}$. Applying the same procedure infinitely many times we achieve that v vanishes at every vertex of Γ , as

$$\Gamma = \bigcup_{k=0}^{\infty} \Gamma_{\eta^k(y)}.$$

\square

Lemma 4 *For any nonreal number z and any $x_0 \in \Gamma$ with $l(x_0) \geq 1$, there exists a nonzero function v defined on Γ_{x_0} satisfying*

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_0} \setminus \{x_0\}. \tag{6}$$

Moreover, the function v cannot vanish and is unique up to a constant multiple.

Proof We will use induction on the height $l(x_0)$. Assume $l(x_0) = 1$. Set $v(x_0) = 1$. Let $x \in \eta^{-1}(x_0)$. Then $l(x) = 0$. We want to have

$$zv(x) = \lambda_x v(x_0) + \beta_x v(x).$$

Thus we may set

$$v(x) = \frac{\lambda_x v(x_0)}{z - \beta_x}.$$

In this way (6) is fulfilled.

Assume the conclusion is true for all vertices on height n . Let $l(x_0) = n + 1$. Consider vertices $x_1, x_2, \dots, x_k \in \eta^{-1}(x_0)$. Then $l(x_j) = n$ for $j = 1, 2, \dots, k$. By the induction hypothesis, for every vertex x_j , there exists a nonzero function v_j defined on Γ_{x_j} satisfying

$$(Jv_j)(x) = zv_j(x), \quad x \in \Gamma_{x_j} \setminus \{x_j\}.$$

We have

$$\Gamma_{x_0} = \bigcup_{j=1}^k \Gamma_{x_j} \cup \{x_0\}.$$

We are going to define the function v on Γ_{x_0} in the following way. Set $v(x_0) = 1$ and

$$v(x) = c_j v_j(x), \quad \text{for } x \in \Gamma_{x_j},$$

for some constants c_j , which will be specified later. In this way, we get

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_j} \setminus \{x_j\}, \quad j = 1, 2, \dots, k.$$

In order to conclude the proof, we must show that

$$(Jv)(x_j) = zv(x_j), \quad j = 1, 2, \dots, k.$$

Thus we want to have

$$zc_j v_j(x_j) = \lambda_{x_j} v(x_0) + \beta_{x_j} c_j v_j(x_j) + \sum_{y \in \eta^{-1}(x_j)} \lambda_y c_j v_j(y);$$

i.e.,

$$\lambda_{x_j} v(x_0) = c_j \left(zv_j(x_j) - \beta_{x_j} v_j(x_j) - \sum_{y \in \eta^{-1}(x_j)} \lambda_y v_j(y) \right). \tag{7}$$

The expression in the brackets on the right-hand side is nonzero for every $j = 1, 2, \dots, k$, by Lemma 4. Therefore (7) is satisfied for an appropriate choice of nonzero constants $c_1, c_2, c_3, \dots, c_k$.

By Lemma 1, the function v cannot vanish at any vertex. Moreover, if there was another function \tilde{v} satisfying the conclusion of Lemma 4, then $v - c\tilde{v}$ would also satisfy the conclusion and would vanish for an appropriate choice of the constant c . Thus $v = c\tilde{v}$. □

Corollary 5 *For any nonreal number z , there exists a nonzero function v so that*

$$(Jv)(x) = zv(x) \quad x \in \Gamma.$$

The function v cannot vanish and is unique up to a constant multiple.

Proof Fix a vertex y , $l(y) = 0$. By Lemma 4, for any subtree $\Gamma_{\eta^k(y)}$, $k \geq 1$, there exists a unique function v_k defined on $\Gamma_{\eta^k(y)}$ so that

$$v_k(y) = 1, \quad (Jv_k)(x) = zv_k(x) \quad \text{for } x \in \Gamma_{\eta^k(y)} \setminus \{\eta^k(y)\}.$$

As

$$\Gamma_{\eta^k(y)} \setminus \{\eta^k(y)\} \subset \Gamma_{\eta^{k+1}(u)} \setminus \{\eta^{k+1}(y)\},$$

the function v_{k+1} is a multiple of v_k on $\Gamma_{\eta^k(y)}$, by Lemma 4 applied to $x_0 = \eta^k(y)$. As $v_{k+1}(y) = v_k(y) = 1$, we get

$$v_{k+1}(x) = v_k(x) \quad \text{for } x \in \Gamma_{\eta^k(y)}.$$

Define

$$v(x) = v_k(x) \quad \text{for } x \in \Gamma_{\eta^k(y)}.$$

Since

$$\Gamma = \bigcup_{k=1}^{\infty} \Gamma_{\eta^k(y)},$$

the function v is defined at every vertex of Γ , and the conclusion follows. □

Remark The conclusion may not be true for some real values of z . Indeed, consider a tree Γ with infinitely many vertices at height 0. Assume $\#(\eta^{-1}(x)) = 2$ for any vertex $x \in \Gamma$ with $l(x) \geq 1$. Let $\lambda_y = 1$ and $\beta_y = 0$ for all vertices y such that $l(y) = 0$. Let $l(y_1) = l(y_2) = 0$ and $\eta(y_1) = \eta(y_2)$. Then the function $v = \delta_{y_1} - \delta_{y_2}$ satisfies $Jv = 0$. In this way, we obtain infinitely many solutions, which vanish on x with $l(x) \geq 1$.

3 Essential Self-Adjointness and Defect Indices

Let $z \notin \mathbb{C}$. The function $v \in \ell^2(\Gamma)$ belongs to the defect space N_z if v is orthogonal to $\text{Im}(zI - J) = (zI - J)(\mathcal{F}_0(\Gamma))$. In particular, v is orthogonal to $(zI - J)\delta_x$ for any $x \in \Gamma$. This implies $Jv = \bar{z}v$. The dimension of the defect space N_z is called the defect index. It is known that the defect index is constant for z in the upper half-plane and for z in the lower half-plane. In our case, the defect index is constant on $\mathbb{C} \setminus \mathbb{R}$ as $Jv = \bar{z}v$ is equivalent to $J\bar{v} = z\bar{v}$. We refer to [4, 6] for the theory of symmetric operators in Hilbert space and their self-adjoint extensions.

Proposition 6 *The defect indices of the operator J cannot be greater than 1.*

Proof Fix a nonreal number z . Assume J is not essentially self-adjoint. Then there exists $0 \neq v \in \ell^2(\Gamma)$ satisfying $Jv = zv$. By Corollary 5, the function v is unique up to a constant multiple. Hence the defect space is one-dimensional. □

Proposition 6 implies the following.

Corollary 7 *Let J be a Jacobi matrix on Γ . Fix a nonreal number z , and let v denote the unique, up to a constant multiple, nonzero solution of the equation $Jv = zv$. Then J is essentially self-adjoint if and only if $v \notin \ell^2(\Gamma)$.*

Theorem 8 *There exist Jacobi matrices on Γ that are not essentially self-adjoint.*

Proof We set $\beta_x \equiv 0$. Fix a nonreal number z . Choose an infinite path $\{x_n\}$ in Γ so that $l(x_n) = n$. We will construct a matrix J by induction on n . Assume that we have constructed a matrix J on $\Gamma_{x_{n-1}} \setminus \{x_{n-1}\}$ and a nonvanishing function v on $\Gamma_{x_{n-1}}$ so that

$$\|v|_{\Gamma_{x_{n-1}}}\|_2^2 \leq 1 - 2^{-(n-1)}$$

and

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_{n-1}} \setminus \{x_{n-1}\}.$$

We want to extend the definition of J and v so that the conclusion remains valid when $n - 1$ is replaced by n .

Our first task is to define $\lambda_{x_{n-1}}$ and $v(x_n)$ so that

$$zv(x_{n-1}) = \lambda_{x_{n-1}}v(x_n) + \sum_{y \in \eta^{-1}(x_{n-1})} \lambda_y v(y);$$

i.e.,

$$\lambda_{x_{n-1}}v(x_n) = zv(x_{n-1}) - \sum_{y \in \eta^{-1}(x_{n-1})} \lambda_y v(y). \tag{8}$$

The right-hand side of (8) cannot vanish by Lemma 2. We will define $\lambda_{x_{n-1}}$ and $v(x_n)$ so as to satisfy (8). By specifying $\lambda_{x_{n-1}}$ large enough, we may assume that

$$|v(x_n)|^2 \leq 2^{-n-1}.$$

For any $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$, consider the subtree $\Gamma_y \setminus \{y\}$. Set $\lambda_x = 1$ for any $x \in \Gamma_y \setminus \{y\}$. By Lemma 4, there is a nonzero solution v_y defined on Γ_y satisfying

$$(Jv_y)(x) = zv_y(x), \quad x \in \Gamma_y \setminus \{y\}.$$

By rescaling, we may assume that

$$\sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \|v_y|_{\Gamma_y}\|_2^2 \leq 2^{-n-1}.$$

We want to define the numbers λ_y for $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$ so that

$$z v_y(y) = (J v_y)(y) = \lambda_y v_y(x_n) + \sum_{x \in \eta^{-1}(y)} \lambda_x v_y(x).$$

Hence we want to get

$$\lambda_y = \frac{z v_y(y) - \sum_{x \in \eta^{-1}(y)} \lambda_x v_y(x)}{v(x_n)}. \tag{9}$$

By Lemma 2, the numerator (9) cannot vanish. We may multiply v_y by a constant of absolute value 1 so that the expression on the right-hand side of (9) becomes positive. In this way, the values λ_y for $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$ are defined. We extend the definition of v to Γ_{x_n} by setting

$$v(x) = v_y(x), \quad x \in \Gamma_y, \quad y \neq x_{n-1}.$$

On the way, we have also extended the definition of J so that

$$(J v)(x) = z v(x), \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Moreover, by construction, we have

$$\begin{aligned} \|v|_{\Gamma_{x_n}}\|_2^2 &= \|v|_{\Gamma_{x_{n-1}}}\|_2^2 + \sum_{y \in \eta^{-1}(x_n), y \neq x_{n-1}} \|v|_{\Gamma_y}\|_2^2 + |v(x_n)|^2 \\ &\leq 1 - 2^{-(n-1)} + 2^{-n-1} + 2^{-n-1} = 1 - 2^{-n}. \end{aligned}$$

□

Remark The Jacobi matrix J constructed in the proof satisfies $\beta_x \equiv 0$ and $\lambda_x = 1$ for vertices x whose distance from the path $\{x_n\}$ is greater than 2.

Remark Another way of proving Theorem 8 is as follows. Fix any Jacobi matrix J_0 so that the operator J_0 is bounded on $\ell^2(\Gamma)$. For example, we may set $\beta_x \equiv 0$ and $\lambda_x = (\#\eta^{-1}(y))^{-1/2}$, whenever $x \in \eta^{-1}(y)$. Let S denote the operator acting according to the rule

$$Sv(x) = \lambda_x v(\eta(x)).$$

By Γ_0 we denote the set of leaves, i.e., vertices with height 0. Then

$$\begin{aligned} \|Sv\|_2^2 &= \sum_{x \in \Gamma} |Sv(x)|^2 = \sum_{x \in \Gamma} \lambda_x^2 |v(\eta(x))|^2 \\ &= \sum_{y \in \Gamma \setminus \Gamma_0} |v(y)|^2 \sum_{x \in \eta^{-1}(y)} \lambda_x^2 = \sum_{y \in \Gamma \setminus \Gamma_0} |v(y)|^2 \leq \|v\|_2^2. \end{aligned}$$

The operator S is thus bounded on $\ell^2(\Gamma)$ with $\|S\| \leq 1$. The adjoint operator S^* acts by the rule

$$S^*v(x) = \sum_{y \in \eta^{-1}(x)} \lambda_y v(y), \quad x \notin \Gamma_0,$$

$$S^*v(x) = 0, \quad x \in \Gamma_0.$$

Then for $J_0 = S + S^*$, we get $\|J_0\| \leq 2$. Fix an infinite path $\{x_n\}$ and a sequence of positive numbers $\{\lambda_n\}$. Let J_1 be the degenerate Jacobi matrix defined by $\beta_x \equiv 0$ and $\lambda_{x_n} = \lambda_n, \lambda_x = 0$, for $x \notin \{x_n\}$. Choose the coefficients λ_n so that the classical Jacobi matrix associated with the coefficients λ_n and $\beta_n \equiv 0$ is not essentially self-adjoint. For example, set $\lambda_n = 2^n$. Let $J = J_0 + J_1$. The matrix J is nondegenerate. Moreover, J is not essentially self-adjoint as a bounded perturbation of a not essentially self-adjoint operator ([6], cf. Prop. 8.6 [4]).

The next theorem provides a relation between Jacobi matrices on the tree Γ and classical Jacobi matrices associated with an infinite path of Γ .

Theorem 9 *Assume a Jacobi matrix J on Γ is not essentially self-adjoint and $\beta_x \equiv 0$. Choose an infinite path $\{x_n\}$ with $l(x_0) = 0$ and $x_n = \eta^n(x_0)$. Then the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$ is not essentially self-adjoint.*

Before proving Theorem 9 we will need the following lemma.

Lemma 10 *Let J be a Jacobi matrix on Γ with $\beta_x \equiv 0$. Let $Jv = iv$ and $v(x_0) = 1$ for a vertex x_0 on height 0. Then the function $\tilde{v}(x) = i^{-l(x)}v(x)$ is positive.*

Proof By assumption, we have

$$iv(x) = \lambda_x v(\eta(x)) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y).$$

Thus

$$\tilde{v}(x) = \lambda_x \tilde{v}(\eta(x)) - \sum_{y \in \eta^{-1}(x)} \lambda_y \tilde{v}(y). \tag{10}$$

We know that \tilde{v} cannot vanish and is unique up to a constant multiple. The function $\operatorname{Re} \tilde{v}$ also satisfies (10) and takes the value 1 at x_0 . Thus $\tilde{v} = \operatorname{Re} \tilde{v}$; i.e., \tilde{v} is real valued. Multiplying (10) by $\tilde{v}(x) = \tilde{v}(\eta(y))$ results in

$$\lambda_x \tilde{v}(x)\tilde{v}(\eta(x)) = \tilde{v}(x)^2 + \sum_{y \in \eta^{-1}(x)} \lambda_y \tilde{v}(y)\tilde{v}(\eta(y)). \tag{11}$$

By (11), if the quantity $\tilde{v}(x)\tilde{v}(\eta(x))$ is positive for vertices $x \in \Gamma$, with $l(x) = n$, then it is positive for $l(x) = n + 1$. For $l(x) = 0$, the sum in (11) is empty; hence $\tilde{v}(x)\tilde{v}(\eta(x)) > 0$. Summarizing $\tilde{v}(x)\tilde{v}(\eta(x)) > 0$ for any $x \in \Gamma$. In other words, the values $\tilde{v}(x)$ and $\tilde{v}(\eta(x))$ have the same sign for any $x \in \Gamma$. Since Γ is connected, the function $x \mapsto \tilde{v}(x)$ must have constant sign on Γ . \square

Proof of Theorem 9 By Corollary 5, there exists a nonvanishing function v on Γ so that $Jv = iv$ and $v(x_0) = 1$. In view of Corollary 7, the function v is square summable on Γ . By (10) evaluated at $x = x_n$, we obtain

$$\tilde{v}(x_n) = \lambda_{x_n} \tilde{v}(x_{n+1}) - \sum_{y \in \eta^{-1}(x_n)} \lambda_y \tilde{v}(y).$$

Hence

$$\lambda_{x_n} \tilde{v}(x_{n+1}) - \lambda_{x_{n-1}} \tilde{v}(x_{n-1}) = \tilde{v}(x_n) + \sum_{\substack{y \in \eta^{-1}(x_n) \\ y \neq x_{n-1}}} \lambda_y \tilde{v}(y) > 0.$$

The last inequality follows from Lemma 10. Therefore

$$\begin{aligned} \tilde{v}(x_{2n}) &\geq \frac{\lambda_{x_0} \lambda_{x_2} \dots \lambda_{x_{2n-2}}}{\lambda_{x_1} \lambda_{x_3} \dots \lambda_{x_{2n-1}}} \tilde{v}(x_0), \\ \tilde{v}(x_{2n+1}) &\geq \frac{\lambda_{x_1} \lambda_{x_3} \dots \lambda_{x_{2n-1}}}{\lambda_{x_2} \lambda_{x_4} \dots \lambda_{x_{2n}}} \tilde{v}(x_1). \end{aligned}$$

Since the sequence $\tilde{v}(x_n)$ is square summable, we get

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_{x_0} \lambda_{x_2} \dots \lambda_{x_{2n-2}}}{\lambda_{x_1} \lambda_{x_3} \dots \lambda_{x_{2n-1}}} \right)^2 + \left(\frac{\lambda_{x_1} \lambda_{x_3} \dots \lambda_{x_{2n-1}}}{\lambda_{x_2} \lambda_{x_4} \dots \lambda_{x_{2n}}} \right)^2 < \infty. \tag{12}$$

The last inequality is equivalent to not essential self-adjointness of the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$. Indeed, let p_n and q_n denote the polynomials of the first and the second kind associated with J_0 ; i.e.,

$$\begin{aligned} xp_n(x) &= \lambda_n p_{n+1}(x) + \lambda_{n-1} p_{n-1}(x), \quad n \geq 0, \\ xq_n(x) &= \lambda_n q_{n+1}(x) + \lambda_{n-1} q_{n-1}(x), \quad n \geq 1, \end{aligned}$$

where $p_{-1} = 0$, $p_0 = 1$ and $q_0 = 0$, $q_1 = 1/\lambda_0$. Then (12) is equivalent to

$$\sum_{n=1}^{\infty} [p_n^2(0) + q_n^2(0)] < \infty,$$

as

$$p_{2n}(0) = (-1)^n \frac{\lambda_0 \lambda_2 \dots \lambda_{2n-2}}{\lambda_1 \lambda_3 \dots \lambda_{2n-1}}, \quad q_{2n+1}(0) = (-1)^n \frac{\lambda_1 \lambda_3 \dots \lambda_{2n-1}}{\lambda_2 \lambda_4 \dots \lambda_{2n}}.$$

In view of [1, Problem 10, p. 84] or [5, Thm. 3 (i), (iv)], the matrix J_0 is not essentially selfadjoint. □

Fix a vertex x_0 , $l(x_0) = 0$, and let $x_1 = \eta(x_0)$. Assume also that

$$\#\eta^{-1}(x_1) > 1,$$

i.e., that x_1 has at least two children, i.e., x_0 has a sibling.

By Corollary 5 applied to Γ , there exists a nonvanishing function v_z on Γ satisfying

$$(Jv_z)(x) = zv_z(x), \quad \text{for } x \in \Gamma, \quad v_z(x_0) = 1. \tag{13}$$

Similarly, Corollary 5 applied to $\Gamma \setminus \{x_0\}$ implies existence of a nonvanishing function u_z on $\Gamma \setminus \{x_0\}$ so that

$$(Ju_z)(x) = zu_z(x), \quad \text{for } x \in \Gamma \setminus \{x_0\}, \quad u_z(x_1) = \frac{1}{\lambda_{x_0}}. \tag{14}$$

We extend the definition of the function u_z to Γ by setting $u_z(x_0) = 0$.

The functions v_z and u_z satisfying (13) and (14) will be called *the solution* and *the associated solution* of the equation

$$(Jf)(x) = zf(x), \quad x \in \Gamma \setminus \{x_0\}. \tag{15}$$

By Corollary 5, every solution of (15) is a linear combination of v_z and u_z . The following lemma is straightforward but useful.

Lemma 11 *Consider a densely defined symmetric operator A on a Hilbert space \mathcal{H} . Let \mathcal{H}_0 be a finite dimensional subspace of $D(A) \subset \mathcal{H}$, and let $P_{\mathcal{H}_0}$ denote the orthogonal projection onto \mathcal{H}_0 . Define the operator $\tilde{A} : D(A) \cap \mathcal{H}_0^\perp \rightarrow \mathcal{H}_0^\perp$ by*

$$\tilde{A} = (I - P_{\mathcal{H}_0})A(I - P_{\mathcal{H}_0}).$$

The operator \tilde{A} is essentially self-adjoint if and only if A is essentially self-adjoint.

Theorem 12 *Assume J is not essentially self-adjoint. Fix a vertex x_0 with $l(x_0) = 0$, so that x_0 has a sibling, and a nonreal number z . Then the associated solution u_z is square summable on Γ .*

Proof Let $\mathcal{H}_0 = \mathbb{C}\delta_{x_0}$. The operator \tilde{J} acts on $\ell^2(\Gamma \setminus \{x_0\})$ and is not essentially self-adjoint by Lemma 11. Moreover, if \tilde{u}_z denotes the truncation of u_z to $\tilde{\Gamma} = \Gamma \setminus \{x_0\}$, we have

$$(\tilde{J}\tilde{u}_z)(x) = z\tilde{u}_z(x), \quad x \in \tilde{\Gamma}.$$

By Corollary 3 applied to $\tilde{\Gamma}$, we know that \tilde{u}_z cannot vanish. Since \tilde{J} is not essentially self-adjoint, there exists a function $0 \neq \tilde{v} \in \ell^2(\tilde{\Gamma})$ such that

$$(\tilde{J}\tilde{v})(x) = z\tilde{v}(x), \quad x \in \tilde{\Gamma}.$$

By Lemma 4, applied to $\tilde{\Gamma}$, we get that $\tilde{u}_z(x) = c\tilde{v}(x)$ for $x \in \tilde{\Gamma}$. Hence \tilde{u}_z is square summable, which implies the conclusion. \square

Corollary 7 and Theorem 12 imply:

Corollary 13 *Assume a Jacobi matrix J on Γ is not essentially self-adjoint. Fix a vertex x_0 with $l(x_0) = 0$, so that x_0 has a sibling. Then for any nonreal number z , every solution of the equation*

$$zv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y), \quad x \neq x_0$$

is square summable.

Fix a vertex x_0 , $l(x_0) = 0$, and remove from Γ all the edges from the infinite path $\{x_n\}_{n=0}^\infty$, where $x_n = \eta^n(x_0)$. Note that we do not remove vertices x_n , $n \geq 0$. In this way the tree Γ splits into an infinite number of finite subtrees of the form $\Gamma_n := \Gamma_{x_n} \setminus \Gamma_{x_{n-1}}$. In other words, Γ_n consists of x_n and all its descendants with the exception of x_{n-1} and its descendants.

Lemma 14 *Let $x \in \Gamma_n$ for some $n \geq 1$. Then $v_z(x_n)u_z(x) = u_z(x_n)v_z(x)$.*

Proof By Lemma 1, we know that v_z and u_z cannot vanish. Both functions satisfy $(Ju_z)(x) = zu_z(x)$, $(Jv_z)(x) = zv_z(x)$ for $x \in \Gamma_n \setminus \{x_n\}$. By Lemma 4, we get $v_z(x) = cu_z(x)$ for $x \in \Gamma_n$. Plugging in $x = x_n$ implies $c = v_z(x_n)/u_z(x_n)$, and the conclusion follows. \square

Proposition 15 *For the solution v_z and the associated solution u_z , we have*

$$\left| \begin{matrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{matrix} \right| = \frac{1}{\lambda_{x_n}}.$$

Proof By (14), we get for $n \geq 1$,

$$\begin{aligned} \lambda_{x_n} u_z(x_{n+1}) &= zu_z(x_n) - \beta_{x_n} u_z(x_n) - \sum_{y \in \eta^{-1}(x_n)} \lambda_y u_z(y) \\ &= zu_z(x_n) - \beta_{x_n} u_z(x_n) - \lambda_{x_{n-1}} u_z(x_{n-1}) - \sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \lambda_y u_z(y), \\ \lambda_{x_n} v_z(x_{n+1}) &= zv_z(x_n) - \beta_{x_n} v_z(x_n) - \sum_{y \in \eta^{-1}(x_n)} \lambda_y v_z(y) \\ &= zv_z(x_n) - \beta_{x_n} v_z(x_n) - \lambda_{x_{n-1}} v_z(x_{n-1}) - \sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \lambda_y v_z(y). \end{aligned}$$

Observe that $\eta^{-1}(x_n) \setminus \{x_{n-1}\} \subset \Gamma_n$. Hence Lemma 14 implies

$$v_z(x_n)u_z(y) = u_z(x_n)v_z(y), \quad y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}.$$

Now, multiplying the equations by $v_z(x_n)$ and $u_z(x_n)$, respectively, and subtracting sidewise leads to

$$\lambda_{x_n} \begin{vmatrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{vmatrix} = \lambda_{x_{n-1}} \begin{vmatrix} v_z(x_{n-1}) & u_z(x_{n-1}) \\ v_z(x_n) & u_z(x_n) \end{vmatrix}.$$

The conclusion follows as

$$\lambda_{x_0} \begin{vmatrix} v_z(x_0) & u_z(x_0) \\ v_z(x_1) & u_z(x_1) \end{vmatrix} = 1.$$

□

The following theorem provides a natural analog of the Carleman criterion for essential self-adjointness.

Theorem 16 *Let J be a Jacobi matrix associated with the coefficients λ_x and β_x . Let x_n denote any infinite path such that $l(x_0) = 0$ and $x_n = \eta^n(x_0)$. Assume*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_{x_n}} = \infty.$$

Then the operator J is essentially self-adjoint.

Proof Assume first that x_0 has a sibling. Then the result follows by a standard argument from Corollary 13 and Proposition 15. Indeed, if J was not essentially self-adjoint, then the functions v_z and u_z would be square summable for any nonreal z . Thus the series $\sum \lambda_{x_n}^{-1}$ would be summable, as well.

Assume now that x_0 has no siblings. Set $x_1 = \eta(x_0)$, and consider the tree $\Gamma' = \Gamma \cup \{x'_0\}$ augmented by one vertex x'_0 , so that $\eta(x'_0) = x_1$. Let J' denote the Jacobi matrix on Γ' with coefficients λ'_x, β'_x defined by

$$\lambda'_x = \begin{cases} \lambda_x & x \in \Gamma, \\ 1 & x = x'_0, \end{cases} \quad \beta'_x = \begin{cases} \beta_x & x \in \Gamma, \\ 0 & x = x'_0. \end{cases}$$

We have $\lambda'_{x_n} = \lambda_{x_n}$. By the first part of the proof, the operator J' is essentially self-adjoint. However J' is a one-dimensional extension of the operator J . Hence, in view of Lemma 11, the operator J is essentially selfadjoint. □

Remark The assumption does not depend on the choice of the infinite path, as any two such paths meet at a certain vertex.

Acknowledgements I would like to thank all the referees. Their remarks improved the exposition of the paper immensely.

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