Positivity of Turán determinants for orthogonal polynomials II

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Abstract

The polynomials p_n orthogonal on the interval $[-1,1]$, normalized by $p_n(1)=1$, satisfy Turán's inequality if $p_n^2(x)-p_{n-1}(x)p_{n+1}(x)\geq 0$ for $n\geq 1$ and for all x in the interval of orthogonality. We give a general criterion for orthogonal polynomials to satisfy Turán's inequality. This extends essentially the results of [18]. In particular the results can be applied to many classes of orthogonal polynomials, by inspecting their recurrence relation.

Keywords— orthogonal polynomials, Turán determinants, recurrence formula

1 Introduction

Consider a symmetric probability measure μ such that $\operatorname{supp}\mu=[-1,1]$. By the Gram-Schmidt orthogonalization procedure applied to the system of monomials x^n, $n\geq 0$, we obtain a sequence of orthogonal polynomials $p_n(x)$, $n\geq 0$. Every polynomial p_n is of exact degree n. We may assume that its leading coefficient is positive. It is well known that the polynomials p_n satisfy the three term recurrence relation of the form

$$xp_n=\gamma_n p_{n+1}+\alpha_n p_{n-1},\quad n\geq 0, \tag{1}$$

with convention $\alpha_0=p_{-1}=0$. Due to orthogonality the polynomial p_n has n roots in the open interval $(-1,1)$. Therefore $p_n(1)>0$. Let

$$P_n(x)=\frac{p_n(x)}{p_n(1)},\quad n\geq 0.$$

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The coefficients $\gamma_n,\ \alpha_{n+1}$ are positive for $n \geq 0$. In case the polynomials p_n are orthonormal then the sequences of the coefficients are related by $\gamma_n = \alpha_{n+1}$ and the recurrence relation simplifies to

$$xp_n = \alpha_{n+1}p_{n+1} + \alpha_n p_{n-1}, \qquad n \geq 0.$$

We refer to [5, 14] for the basic theory concerning orthogonal polynomials.

We are interested in determining when

$$\Delta_n(x) := P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad n \geq 1. \tag{2}$$

The expression $\Delta_n(x)$ is called the Turán's determinant. The problem has been studied for many classes of specific orthogonal polynomials (see [1, 2, 3, 4, 6, 7, 8, 10, 12, 13, 15, 16, 20, 21]. We refer to the introduction in [18] for a short account of known results.

Turán determinants can be used to determine the orthogonality measure μ in terms of orthonormal polynomials p_n. Paul Nevai [11] observed if $\alpha_n \overset{n}{\to} 1/2$ then the sequence of measures (perhaps signed)

$$[p_n^2(x) - p_{n-1}(x)p_{n+1}(x)]\, d\mu(x)$$

is weakly convergent to the measure

$$\frac{2}{\pi}\sqrt{1-x^2}\, dx, \quad |x| < 1.$$

Máté and Nevai [9] showed that if additionally sequence α_n has bounded variation then the limit of Turán determinants exists. Moreover the orthogonality measure is absolutely continuous on the interval $(-1, 1)$ its density is given by

$$\frac{2\sqrt{1-x^2}}{\pi f(x)}, \quad |x| < 1,$$

where

$$f(x) := \lim_n [p_n^2(x) - p_{n-1}(x)p_{n+1}(x)] > 0, \quad |x| < 1.$$

It turns out that the way we normalize the polynomials is essential for the Turán inequality to hold. Indeed, assume p_n satisfy (1) and $p_n(1) = 1$, i.e.

$$\alpha_n + \gamma_n = 1. \tag{3}$$

Assume

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0, \quad |x| \leq 1,\ n \geq 1.$$

Define new polynomials by $p_n^{(\sigma)}(x) = \sigma_n p_n(x)$, where σ_n is a sequence of positive coefficients. Then the condition

$$\{p_n^{(\sigma)}(x)\}^2 - p_{n-1}^{(\sigma)}(x)p_{n+1}^{(\sigma)}(x) \geq 0, \quad |x| \leq 1, \ n \geq 1$$

is equivalent to (see Proposition [18])

$$\sigma_n^2 - \sigma_{n-1}\sigma_{n+1} \geq 0, \quad n \geq 1.$$

This means if the Turán determinants are nonnegative, when the polynomials are normalized at $x = 1$, then they stay nonnegative for any other normalization provided that they are nonnegative at $x = 1$, as $\sigma_n = p_n^{(\sigma)}(1)$.

By Theorem 1 [18] if the polynomials are normalized at $x = 1$, i.e. $p_n(1) = 1$, α_n is increasing and $\alpha_n \leq \frac{1}{2}$, the Turán determinants are positive in the interval $(-1, 1)$. This result can be applied to many classes of orthogonal polynomials, including for example the ultraspherical polynomials for which positivity has been obtained in [12, 13]

The result mentioned above can be applied provided that we are given the coefficients α_n explicitly. For many classes of orthogonal polynomials in the interval $[-1, 1]$ we are given recurrence relations, but the values $p_n(1)$ cannot be evaluated in the explicit form. Therefore we are unable to provide a recurrence relation for the polynomials $P_n(x) = p_n(x)/p_n(1)$, in the form for which we can inspect easily the assumptions of Theorem 1 [18]. This occurs when we study the associated polynomials. Indeed assume p_n satisfy (1) and (3). For a fixed natural number the associated polynomials $p_n^{(k)}$ of order k are defined by

$$xp_n^{(k)} = \begin{cases} \gamma_k p_1^{(k)} & n = 0, \\ \gamma_{n+k} p_{n+1}^{(k)} + \alpha_{n+k} p_{n-1}^{(k)} & n \geq 1. \end{cases} \tag{4}$$

These polynomials do not satisfy $p_n^{(k)}(1) = 1$ as

$$p_1^{(k)}(1) = \gamma_k^{-1} = (1 - \alpha_k)^{-1} > 1.$$

The obstacle described above has been partially overcome in Corollary 1 of [18], but it required additional assumptions, in particular $\gamma_0 \geq 1$. Unfortunately many examples including the associated polynomials violate that condition. The aim of this note is to provide a counterpart to Corollory 1 [18] by allowing $\gamma_0 < 1$. This is done in Theorem 1. As the assumptions in this theorem are complicated Corollary 1 provides a wide class of relatively simple recurrence relations for which Theorem 1 applies. General examples are provided at the end of the paper.

2 Results

Theorem 1. *Assume the polynomials p_n satisfy*

$$xp_n = \gamma_n p_{n+1} + \alpha_n p_{n-1}, \quad n \geq 0, \tag{5}$$

where $\alpha_0 = p_{-1} = 0,\ p_0 = 1$. Assume

(a) the sequence α_n is strictly increasing and $\alpha_n \leq 1/2$,

(b) the sequence γ_n is positive and strictly decreasing ,

(c) $\alpha_n + \gamma_n \leq 1$.

Assume also that there holds

$$\frac{\alpha_n - \alpha_{n-1}}{\alpha_n\gamma_{n-1} - \alpha_{n-1}\gamma_n} \leq \frac{\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1}}{\gamma_n - \gamma_{n+1}}, \quad n \geq 1, \tag{6}$$

$$\gamma_0 - \gamma_1 \leq \alpha_1\gamma_0^2. \tag{7}$$

Then for

$$P_n(x) = \frac{p_n(x)}{p_n(1)}$$

we have

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad -1 \leq x \leq 1.$$

Proof. Let

$$g_n = \frac{p_{n+1}(1)}{p_n(1)}.$$

By (5) we get

$$g_n = \frac{1}{\gamma_n}\left(1 - \frac{\alpha_n}{g_{n-1}}\right), \quad n \geq 1. \tag{8}$$

Lemma 1. *Under assumptions of Theorem 1 there holds*

$$1 \leq g_n \leq \frac{\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1}}{\gamma_n - \gamma_{n+1}}, \quad n \geq 0. \tag{9}$$

Proof. (5) gives $g_0 = 1/\gamma_0 \geq 1$. Assume $g_{n-1} \geq 1$ for $n \geq 1$. By (8) and (c) we get

$$g_n \geq \frac{1}{\gamma_n}(1 - \alpha_n) \geq 1.$$

This shows the left hand side inequality.

By (7) we get

$$g_0 = \frac{1}{\gamma_0} \le \frac{\alpha_1\gamma_0}{\gamma_0 - \gamma_1},$$

which shows the right hand side inequality in (9) for $n = 0$. Assume (9) holds for some $n \ge 0$. Then, in view of (8) and (6), we get

$$g_{n+1} = \frac{1}{\gamma_{n+1}}\left(1 - \frac{\alpha_{n+1}}{g_n}\right) \le \frac{1}{\gamma_{n+1}}\left(1 - \frac{\alpha_{n+1}(\gamma_n - \gamma_{n+1})}{\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1}}\right)$$
$$= \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1}} \le \frac{\alpha_{n+2}\gamma_{n+1} - \alpha_{n+1}\gamma_{n+2}}{\gamma_{n+1} - \gamma_{n+2}}.$$

□

Lemma 2. *Under the assumptions of Theorem 1 the sequence $g_n = p_{n+1}(1)/p_n(1)$ is nonincreasing.*

Proof. Let

$$f_k(x) = \frac{1}{\gamma_k}\left(1 - \frac{\alpha_k}{x}\right), \quad x \ge 1.$$

The functions f_k are nondecreasing. Moreover by a straightforward computation we get

$$f_{k+1}(x) \le f_k(x), \qquad 1 \le x \le \frac{\alpha_{k+1}\gamma_k - \alpha_k\gamma_{k+1}}{\gamma_k - \gamma_{k+1}}. \tag{10}$$

We have

$$g_0 = \frac{1}{\gamma_0}, \quad g_1 = \frac{1}{\gamma_1}(1 - \alpha_1\gamma_0).$$

By (7) we get $g_0 \ge g_1$. Assume $g_{n-1} \ge g_n$. Then in view of (8) and Lemma 1 we obtain

$$g_{n+1} = f_{n+1}(g_n) \le f_n(g_n) \le f_n(g_{n-1}) = g_n.$$

□

The polynomials P_n satisfy

$$xP_n = \widetilde{\gamma}_n P_{n+1} + \widetilde{\alpha}_n P_{n-1}, \quad n \ge 0,$$

where

$$\widetilde{\alpha}_n = \alpha_n \frac{p_{n-1}(1)}{p_n(1)}, \qquad \widetilde{\gamma}_n = \gamma_n \frac{p_{n+1}(1)}{p_n(1)}.$$

Since $P_n(1) = 1$ we get

$$\widetilde{\alpha}_n + \widetilde{\gamma}_n = 1.$$

Moreover by Lemma 1, Lemma 2 and (a) the sequence $\widetilde{\alpha}_n$ is nondecreasing and $\widetilde{\alpha}_n \le 1/2$. Thus the conclusion follows from Theorem 1(i) of [18]. □

Remark 1. As a side effect of Theorem 1 we get that the polynomials p_n admit nonnegative linearization as the polynomials P_n satisfy the assumptions of Theorem 1 in [17]. We refer to [19] where this problem is discussed in detail.

The assumption (6) in Theorem 1 can be troublesome for verification in examples. However there is a wide class of examples for which (6) simplifies substantially.

Corollary 1. *Let the polynomials p_n satify (5) with*

$$\alpha_n = \frac{1}{2} - \alpha\delta_n, \qquad \gamma_n = \frac{1}{2} + \gamma\delta_n, \quad n \geq 0.$$

where $\alpha \geq \gamma > 0$ and $\delta_n \searrow 0$. Then the conclusion of Theorem 1 holds.

Proof. We have

$$\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1} = \frac{1}{2}(\alpha+\gamma)(\delta_n - \delta_{n+1}), \quad n \geq 0.$$

Thus (6) takes the form

$$\frac{2\alpha}{\alpha+\gamma} \leq \frac{\alpha+\gamma}{2\gamma},$$

which is true for any numbers $\alpha, \gamma > 0$.

Next, since

$$0 = \alpha_0 = \frac{1}{2} - \alpha\delta_0$$

we get $\alpha\delta_0 = 1/2$. Thus

$$\alpha_1\gamma_0^2 = \left(\frac{1}{2} - \frac{1}{2}\frac{\delta_1}{\delta_0}\right)\left(\frac{1}{2} + \gamma\delta_0\right)^2 \geq \frac{\delta_0 - \delta_1}{2\delta_0}\, 2\gamma\delta_0 = \gamma_0 - \gamma_1.$$

Therefore all the assumptions of Theorem 1 are satisfied. □

Example 1. Consider the symmetric Pollaczek polynomials $P_n^\lambda(x;a)$. They are orthogonal in the interval $[-1,1]$ and satisfy the recurrence relation

$$xP_n^\lambda(x;a) = \frac{n+1}{2(n+\lambda+a)}P_{n+1}^\lambda(x;a) + \frac{n+2\lambda-1}{2(n+\lambda+a)}P_{n-1}^\lambda(x;a),$$

where the parameters satisfy $a > 0,\ \lambda > 0$. Set

$$p_n(x) = \frac{n!}{(2\lambda)_n}P_n^\lambda(x;a),$$

where $(\mu)_n = \mu(\mu+1)\dots(\mu+n-1)$. Then the polynomials p_n satisfy the recurrence relation

$$xp_n = \frac{n+2\lambda}{2(n+\lambda+a)}p_{n+1} + \frac{n}{2(n+\lambda+a)}p_{n-1}.$$

Observe that the assumptions of Corollary 1(i) of [18] are satisfied for $a \geq \lambda$.

Remark 2. There is a misprint in the formulation of Corollary 1 in [18]. The assumptions there required that

$$\lim_n \alpha_n = \frac{1}{2}\widetilde{a}, \qquad \lim_n \gamma_n = \frac{1}{2}\widetilde{a}^{-1}$$

with $0 < \widetilde{a} < 1$. But the conclusion holds also for $\widetilde{a} = 1$ with the same proof as in [18]. For symmetric Pollaczek polynomials we actually have $\widetilde{a} = 1$.

However for $\lambda > a$ the assumptions of Corollary 1(ii) [18] are not satisfied as was wrongly stated in [18], because $\gamma_0 < 1$. Instead we can apply Corollary 1, on the previous page, with

$$\alpha = \lambda + a, \quad \gamma = \lambda - a, \quad \delta_n = \frac{1}{2(n+\lambda+a)}.$$

Remark 3. Corollary 1 requires $\alpha\delta_0 = \frac{1}{2}$, i.e. the quantity δ_0 is determined by α, which limits the range of examples. We will get rid of that assumption in the next corollary, allowing some flexibility for the quantity δ_0.

Corollary 2. *Let the polynomials p_n satify (5) with*

$$\alpha_0 = 0, \qquad \gamma_0 = \frac{1}{2} + \gamma\delta_0,$$

$$\alpha_n = \frac{1}{2} - \alpha\delta_n, \qquad \gamma_n = \frac{1}{2} + \gamma\delta_n, \quad n \geq 1,$$

where $\alpha \geq \gamma > 0$ and $\delta_n \searrow 0$. Assume also that

$$\frac{3\gamma - \alpha}{2\gamma(\alpha+\gamma)} \leq \delta_0 \leq \frac{1}{2\alpha}. \tag{11}$$

Then the conclusion of Corollary 1 holds.

Remark 4. The condition $\delta_0 \leq 1/(2\alpha)$ is not artificial. Instead of setting $\alpha_0 = 0$ we could define

$$\alpha_0 = \frac{1}{2} - \alpha\delta_0.$$

The aformentioned assumption amounts to the condition $\alpha_0 \geq 0$.

Observe also that the possible range for the quantity δ_0 described in (11) is nonempty as we always have

$$\frac{3\gamma - \alpha}{2\gamma(\alpha+\gamma)} \leq \frac{1}{2\alpha}.$$

Proof. We are forced to modify the proof of the preceding corollary at places where δ_0 shows up, as $\alpha_0 = 0$ is no longer equal $\frac{1}{2} - \alpha\delta_0$. Thus we have to make calculations concerning (6), for $n = 1$, and (7), by hand. Since $\alpha\delta_0 \le \frac{1}{2}$ we get

$$\alpha_1\gamma_0^2 \ge \left(\frac{1}{2} - \frac{1}{2}\frac{\delta_1}{\delta_0}\right)\left(\frac{1}{2} + \gamma\delta_0\right)^2 \ge \frac{\delta_0 - \delta_1}{2\delta_0}\,2\gamma\delta_0 = \gamma_0 - \gamma_1.$$

This gives (7). Next we verify (6) for $n = 1$, as the value δ_0 is involved there on the left hand side. The inequality (6) in this case reduces to

$$\frac{1}{\gamma_0} = \frac{2}{1 + 2\gamma\delta_0} \le \frac{\alpha + \gamma}{2\gamma}.$$

This inequality is equivalent to the left hand side of (11). □

Remark 5. Corollary 1 requires that the sequence

$$\frac{\gamma_n - \frac{1}{2}}{\frac{1}{2} - \alpha_n}, \quad n \ge 1 \tag{12}$$

is constant. It is possible to extend Corollary 1 to the case when the sequence in (12) is nondecreasing. Indeed

$$\begin{aligned}\alpha_{n+1}\gamma_n - \alpha_n\gamma_{n+1} = \big[\left(\gamma_{n+1} - \tfrac{1}{2}\right)\left(\tfrac{1}{2} - \alpha_n\right) - \left(\gamma_n - \tfrac{1}{2}\right)\left(\tfrac{1}{2} - \alpha_{n+1}\right)\big] \\ + \tfrac{1}{2}(\alpha_{n+1} - \alpha_n + \gamma_n - \gamma_{n+1}) \ge \tfrac{1}{2}(\alpha_{n+1} - \alpha_n + \gamma_n - \gamma_{n+1}).\end{aligned} \tag{13}$$

Denote

$$u_n = \alpha_{n+1} - \alpha_n, \quad v_n = \gamma_n - \gamma_{n+1}. \tag{14}$$

By (13) the assumption (6) will be satisfied if

$$(u_{n-1} + v_{n-1})(u_n + v_n) \ge 4u_{n-1}v_n. \tag{15}$$

Let

$$v_k = \lambda_k u_k, \quad 0 < \lambda_k \le 1. \tag{16}$$

Then (15) takes the form

$$(1 + \lambda_{n-1})(1 + \lambda_n) \ge 4\lambda_n$$

i.e.

$$\lambda_n \le \frac{1 + \lambda_{n-1}}{3 - \lambda_{n-1}}. \tag{17}$$

Let

$$f(x) = \frac{1 + x}{3 - x}, \quad 0 \le x \le 1.$$

The condition (17) amounts to

$$\lambda_n \le f(\lambda_{n-1}). \tag{18}$$

Thus (18) implies (6), provided that the sequence in (12) is nondecreasing. As $f(x) \ge \frac{1}{3}$, the inequality (18), and consequently (6), is satisfied whenever $\lambda_n \le 1/3$. Observe that for $y \ge 1$ we have

$$f\left(\frac{y-1}{y+1}\right) = \frac{y}{y+2}. \tag{19}$$

Remark 5 gives rise to new examples.

Example 2. For $\varepsilon_n \searrow 0$, $\delta_n \searrow \delta \ge 0$, let

$$\alpha_n = \frac{1}{2} - 3\varepsilon_n(1+\delta_n), \qquad \gamma_n = \frac{1}{2} + \varepsilon_n, \qquad n \ge 0.$$

Then

$$\frac{\gamma_n - \frac{1}{2}}{\frac{1}{2} - \alpha_n} = \frac{1}{3(1+\delta_n)} \nearrow \frac{1}{3(1+\delta)}$$

and (see (14) and (16))

$$\lambda_n = \frac{\varepsilon_n - \varepsilon_{n+1}}{3(\varepsilon_n - \varepsilon_{n+1}) + 3(\varepsilon_n\delta_n - \varepsilon_{n+1}\delta_{n+1})} \le \frac{1}{3}.$$

Next

$$1 + \delta_1 \le 1 + \delta_0 = \frac{1}{6\varepsilon_0}$$

(the last equality follows from $\alpha_0 = 0$). Then

$$\begin{aligned} \alpha_1\gamma_0^2 &= \left[\frac{1}{2} - 3\varepsilon_1(1+\delta_1)\right]\left(\frac{1}{2} + \varepsilon_0\right)^2 \\ &\ge \frac{1}{2}\left(1 - \frac{\varepsilon_1}{\varepsilon_0}\right)\left(\frac{1}{2} + \varepsilon_0\right)^2 \ge \frac{1}{2}\left(1 - \frac{\varepsilon_1}{\varepsilon_0}\right) 2\varepsilon_0 = \gamma_0 - \gamma_1. \end{aligned}$$

This gives (7).

Example 3. For $a > 0$ let

$$\alpha_n = \frac{1}{2} - \frac{a}{2(n+a)}, \qquad \gamma_n = \frac{1}{2} + \frac{a}{2(n+a+1)}.$$

Then the sequence in (12) is increasing. Furthermore (cf. (14) and (16))

$$u_n = \frac{a}{2(n+a)(n+a+1)}, \quad v_n = \frac{a}{2(n+a+1)(n+a+2)}, \quad \lambda_n = \frac{n+a}{n+a+2}.$$

By (19) we have $f(\lambda_{n-1}) = \lambda_n$. Thus (6) is satisfied. Next

$$\gamma_0 - \gamma_1 = v_0 = \frac{a}{2(a+1)(a+2)} \le \frac{a}{2(a+1)^2},$$
$$\alpha_1\gamma_0^2 = \frac{(2a+1)^2}{8(a+1)^3}.$$

As

$$(2a+1)^2 \ge 4(a+1)a,$$

we get

$$\alpha_1\gamma_0^2 \ge \frac{a}{2(a+1)^2} \ge \gamma_0 - \gamma_1,$$

so the condition (7) is also satisfied.

Remark 6. Let

$$\lambda_n = \frac{y_n - 1}{y_n + 1}.$$

Then

$$y_n = \frac{1+\lambda_n}{1-\lambda_n}. \tag{20}$$

Moreover condition (17) is equivalent to

$$y_n \le y_{n-1} + 1. \tag{21}$$

Using Remark 6 we can still generalize Example 3.

Example 4. For $a > 0,\ b \ge 0$ let

$$\alpha_n = \frac{1}{2} - \frac{a}{2(n+a)}, \qquad \gamma_n = \frac{1}{2} + \frac{a}{2(n+a+b+1)}.$$

The sequence in (12) is increasing. Next

$$u_n = \frac{a}{2(n+a)(n+a+1)},$$
$$v_n = \frac{a}{2(n+a+b+1)(n+a+b+2)},$$
$$\lambda_n = \frac{(n+a)(n+a+1)}{(n+a+b+1)(n+a+b+2)}.$$

By (20) we get

$$y_n = \frac{n}{b+1} + \frac{2a+b+2}{2(b+1)} + \frac{b^2+2b}{2(b+1)(2n+2a+b+2)}.$$

Since $b \geq 0$, the inequality (21) holds. Next

$$\gamma_0 - \gamma_1 = v_0 = \frac{a}{2(a+b+1)(a+b+2)} \leq \frac{a}{2(a+b+1)^2},$$
$$\alpha_1\gamma_0^2 = \frac{(2a+b+1)^2}{8(a+1)(a+b+1)^2} \geq \frac{(2a+1)^2}{8(a+1)(a+b+1)^2} \geq \frac{a}{2(a+b+1)^2}.$$

Thus (7) is fulfilled.

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