Closable Hankel operators and moment problems

Christian Berg and Ryszard Szwarc

September 28, 2019

Abstract

In a paper from 2016 D. R. Yafaev considers Hankel operators associated with Hamburger moment sequences q_n and claims that the corresponding Hankel form is closable if and only if the moment sequence tends to 0. The claim is not correct, since we prove closability for any indeterminate moment sequence but also for certain determinate moment sequences corresponding to measures with finite index of determinacy. Yafaev's result holds under some quasi-analyticity assumptions, in particular if Carleman's condition for the moments is satisfied.

Mathematics Subject Classification: Primary 47A05; Secondary 47B25, 47B35

Keywords. Hankel operators, moment problems.

1 Introduction

In [9] Yafaev considers Hankel operators associated with Hamburger moment sequences

$$q_n = \int_{-\infty}^{\infty} x^n \, dM(x), \quad n = 0, 1, \ldots, \tag{1}$$

where M is a positive measure on the real line such that the set of polynomials $\mathbb{C}[x]$ is contained in the Hilbert space $L^2(M)$.

We use the notation of [9] and let $\mathcal{D}$ denote the dense subspace of $\ell^2 = \ell^2(\mathbb{Z}_+)$ of complex sequences with only finitely many non-zero terms. The standard orthonormal basis in ℓ^2 is denoted $e_n, n = 0, 1, \ldots$.

Furthermore, we let $A : \mathcal{D} \to \mathbb{C}[x]$ denote the operator

$$Ag(x) = \sum_{n \geq 0} g_n x^n, \quad g = (g_0, g_1, \dots) \in \mathcal{D}, \tag{2}$$

considered as a densely defined operator from the Hilbert space ℓ^2 to $L^2(M)$.

The Hankel form $q[g, g]$ defined on $\mathcal{D}$ by

$$q[g, g] := \sum_{n,m \geq 0} q_{n+m} g_n \overline{g_m}, \quad g \in \mathcal{D} \tag{3}$$

clearly satisfies

$$q[g, g] = ||Ag||^2_{L^2(M)}, \tag{4}$$

which gives the following result, see [9, Lemma 2.1].

Lemma 1.1. *The form $q[g, g]$ is closable in ℓ^2 if and only if A is closable.*

Because of this result we shall only consider closability of A and leave aside closability of the form q.

The main result [9, Theorem 1.2] can be stated like this.

Theorem 1.2. *Let q_n denote the moments* (1)*. Then the following conditions are equivalent:*

(i) *The operator A in* (2) *is closable.*

(ii) $\lim_{n\to\infty} q_n = 0$.

(iii) *The measure satisfies $M(\mathbb{R} \setminus (-1, 1)) = 0$, in other words* $\operatorname{supp}(M) \subseteq [-1, 1]$ *and $M(\{\pm 1\}) = 0$.*

It is elementary that (ii) and (iii) are equivalent and that these conditions imply that (i) holds. However, (i) does not imply (ii). After having informed the author of [9] about this, he has provided an Erratum [10] which establishes Theorem 1.2 under some quasi-analyticity assumption about the moments.

We give next our main results:

Theorem 1.3. *If the measure M is indeterminate, then A is closable.*

Theorem 1.4. *There exist determinate measures with unbounded support such that A is closable. This holds in particular for all determinate measures with finite index of determinacy.*

These two theorems show that some kind of ”strong” determinacy condition is necessary for $(i) \implies (iii)$ to hold. The next theorem is a special case of a theorem in the Erratum [10].

Theorem 1.5. *[10, Remark 2] Suppose that the moments satisfy Carleman's sufficient condition for determinacy*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{q_{2n}}} = \infty.$$

If A is closable, then condition (iii) holds.

Remark 1.6. In a previous version of this paper, see [7], we gave a proof of Theorem 1.5, where Carleman's condition was replaced by the stronger assumption $\sqrt[2n]{q_{2n}} = o(n)$.

Let us give some background material for Theorems 1.3, 1.4, based on [1].

Associated with the moments (1) we have the orthonormal polynomials (P_n), which are uniquely determined by the conditions

$$\int_{-\infty}^{\infty} P_n(x) P_m(x)\, dM(x) = \delta_{n,m}, \tag{5}$$

when we assume that all P_n have positive leading coefficients.

If the moment problem is indeterminate, there exists an infinite convex set V of measures M satisfying (1). All measures $M \in V$ have unbounded support. Among the solutions are the Nevalinna extremal or in short the N-extremal (in [8] called von Neumann solutions), which are precisely the measures $M \in V$ for which $\mathbb{C}[x]$ is dense in $L^2(M)$ by a theorem of M. Riesz ([1, Chapter 2.2, 2.3]). The N-extremal measures are discrete measures supported by the zero set Λ of certain entire functions of minimal exponential type, i.e., of the form

$$M = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda > 0,$$

cf. [1, p.101].

By a theorem going back to Stieltjes in special cases, the following remarkable fact holds: If one mass is removed from M, then the new measure becomes determinate, i.e.,

$$\widetilde{M} := M - c_{\lambda_0} \delta_{\lambda_0}, \quad \lambda_0 \in \Lambda$$

is determinate. For details see e.g. [2], where this result was exploited. The measure $\widetilde{M}$ is a so-called determinate measure of index of determinacy 0 and if further $n \geq 1$ masses are removed we arrive at a determinate measure M' of index of determinacy n, in symbols $\mathrm{ind}(M') = n$. See [3], which contains an intrinsic definition of such measures M by the study of an index $\mathrm{ind}_z(M)$ associated to a point $z \in \mathbb{C}$. Finally, in [4, Equation (1.5)] we define $\mathrm{ind}(M) := \mathrm{ind}_z(M)$ for $z \in \mathbb{C} \setminus \mathrm{supp}(M)$ because $\mathrm{ind}_z(M)$ is independent of z outside the support of M.

In the indeterminate case the series

$$P(z) := \left(\sum_{n=0}^{\infty} |P_n(z)|^2 \right)^{1/2}, \quad z \in \mathbb{C} \tag{6}$$

converges uniformly on compact subsets of $\mathbb{C}$, cf. [1, Theorem 1.3.2], so P is a positive continuous function.

For any polynomial $p(z) = \sum g_k P_k(z), g = (g_k) \in \mathcal{D}$ and any compact subset $K \subset \mathbb{C}$ we therefore have

$$\sup_{z \in K} |p(z)| \leq P_K ||g||_{\ell^2}, \quad P_K := \sup_{z \in K} P(z), \tag{7}$$

and independent of the measure $M \in V$ we have

$$||p||_{L^2(M)} = ||g||_{\ell^2}. \tag{8}$$

In the indeterminate case the polynomials P_n form an orthonormal basis in $L^2(M)$ for all the N-extremal solutions M, and for the other solutions M they form an orthonormal basis in the closure $\overline{\mathbb{C}[x]}^{L^2(M)}$.

It follows that this closure is isometrically isomorphic as Hilbert space with the space $\mathcal{E}$ of functions of the form

$$u(x) = \sum_{k=0}^{\infty} g_k P_k(x), \quad g \in \ell^2,\ x \in \mathbb{R} \tag{9}$$

under the norm

$$||u||_{L^2(M)} = ||g||_{\ell^2},$$

and the series above converges in $L^2(M)$. However, because of (7) the series in (9) converges locally uniformly in $\mathbb{C}$ to an entire function, which is a representative of u. Furthermore, the inequality (7) holds with p replaced by u.

Note that if $u_n, u \in \mathcal{E}$ satisfies $u_n \to u$ in $L^2(M)$, then $u_n \to u$ uniformly on compact subsets of $\mathbb{C}$. By a classical result of complex analysis it also holds that $D^k u_n \to D^k u$ locally uniformly in $\mathbb{C}$ for any $k \in \mathbb{N}$.

For all $M \in V$ we have the orthogonal decomposition

$$L^2(M) = \mathcal{E} \oplus \mathbb{C}[x]^\perp. \tag{10}$$

2 Proofs

Proof of Theorem 1.3

Assume $g^{(n)} \in \mathcal{D} \to 0$ in ℓ^2 and that $Ag^{(n)} \to f$ in $L^2(M)$. We have to prove that $f = 0$. Clearly $f \in \mathcal{E}$.

From the previous discussion we know that convergence in the Hilbert norm implies locally uniform convergence in the complex plane, not only for the functions but also for derivatives of any order. Therefore

$$g_k^{(n)} = \frac{D^k(Ag^{(n)})(0)}{k!} \to \frac{D^k f(0)}{k!},$$

so the Taylor series of f vanishes because in particular $g_k^{(n)} \to 0$ for $n \to \infty$ for any fixed k. $\square$

Proof of Theorem 1.4

Let us for simplicity first consider an N-extremal measure M with mass $c > 0$ at 0 and consider $\widetilde{M} := M - c\delta_0$, which is a discrete determinate measure with unbounded support. A concrete example is studied in [5, p. 128]. The measure $\widetilde{M}$ does not satisfy condition (iii) of Theorem 1.2. Let A and $\widetilde{A}$ denote the operators (2) with values in $L^2(M)$ and $L^2(\widetilde{M})$ respectively. We know that the operator $A : \mathcal{D} \to L^2(M)$ is closable by Theorem 1.3.

Assume that $g^{(n)} \to 0$ in ℓ^2, where $g^{(n)} \in \mathcal{D}$, and that $\widetilde{A}g^{(n)} \to f$ in $L^2(\widetilde{M})$. We have $\widetilde{A}g^{(n)}(0) = g_0^{(n)} \to 0$, and therefore

$$Ag^{(n)}(x) \to \begin{cases} f(x), & x \in \operatorname{supp}(\widetilde{M}) \\ 0, & x = 0 \end{cases}$$

in $L^2(M)$ because $M = \widetilde{M} + c\delta_0$. Since A is closable, we conclude that $f = 0$.

Let us next modify the proof just given by removing one or finitely many masses one by one at mass-points λ_0 satisfying $|\lambda_0| < 1$ of an N-extremal measure M. In fact, for $n \to \infty$ also

$$\widetilde{A}g^{(n)}(\lambda_0) = \sum_{k\geq 0} g_k^{(n)} \lambda_0^k \to 0,$$

because

$$|\sum_{k\geq 0} g_k^{(n)} \lambda_0^k| \leq ||g^{(n)}||_{\ell^2} \left(1 - |\lambda_0|^2\right)^{-1/2}.$$

We finally claim that if M is an arbitrary determinate measure with $\mathrm{ind}(M) = n \geq 0$, then the corresponding operator A is closable. In fact let $\Lambda \subset (-1, 1)$ denote a set of $n+1$ points disjoint with $\mathrm{supp}(M)$. Such a choice is clearly possible since the support is discrete in $\mathbb{R}$. By [3, Theorem 3.9] the measure

$$M^+ := M + \sum_{\lambda\in\Lambda} \delta_\lambda$$

is N-extremal and the corresponding operator A^+ is closable by Theorem 1.3. By removing the masses δ_λ for $\lambda \in \Lambda$ one by one we obtain that the operator A associated with M is closable. □

Comments on the proof of Theorem 1.5

Yafaev's proof is based on a study of the set $\mathcal{D}_* \subset L^2(M)$ for an arbitrary positive measure M with moments of any order as in (1), namely

$$\mathcal{D}_* := \left\{ u \in L^2(M) : u_n := \int_{-\infty}^{\infty} u(t) t^n \, dM(t) \in \ell^2 \right\}. \tag{11}$$

Lemma 2.2 in [9] states that the adjoint A^* of the operator A from (2) is given by $\mathrm{dom}(A^*) = \mathcal{D}_*$ and

$$(A^* u)_n = \int_{-\infty}^{\infty} u(t) t^n \, dM(t), \; n = 0, 1, \ldots, \quad u \in \mathcal{D}_*. \tag{12}$$

Yafaev uses the following result, Theorem 2.3 in [9], which is not true in general, but true under the quasi-analyticity assumption in [10]:

Claim The following conditions are equivalent:

(iii) of Theorem 1.2,

(iv) $\mathcal{D}_*$ is dense in $L^2(M)$.

While it is correct that (iii) implies (iv), the converse is not true. In Theorem 3.1 we prove that (iv) holds, if M is an indeterminate measure and hence (iii) does not hold.

3 Additional results

We use the following notation for the orthonormal polynomials (5).

$$\begin{aligned} P_n(x) &= b_{n,n}x^n + b_{n-1,n}x^{n-1} + \ldots + b_{1,n}x + b_{0,n}, & (13)\\ x^n &= c_{n,n}P_n(x) + c_{n-1,n}P_{n-1}(x) + \ldots + c_{1,n}P_1(x) + c_{0,n}P_0(x). & (14) \end{aligned}$$

The matrices $\mathcal{B} = \{b_{i,j}\}$ and $\mathcal{C} = \{c_{i,j}\}$ with the assumption

$$b_{i,j} = c_{i,j} = 0 \qquad \text{for } i > j$$

are upper-triangular. Since $\mathcal{B}$ and $\mathcal{C}$ are transition matrices between two sequences of linearly independent systems of functions, we have

$$\mathcal{B}\mathcal{C} = \mathcal{C}\mathcal{B} = \mathcal{I}. \tag{15}$$

Both matrices define operators in ℓ^2 with domain $\mathcal{D}$ by defining the image of $e_n \in \mathcal{D}$ to be the n'th column of the matrix. We use the same symbol for these operators as their matrices.

In the following we assume the moment problem (1) to be indeterminate. In this case $\mathcal{B}$ extends to a bounded operator on ℓ^2 which is Hilbert-Schmidt by [5, Proposition 4.2]. We denote it here $\overline{\mathcal{B}}$, since it is the closure of $\mathcal{B}$. We know that $\overline{\mathcal{B}}$ is one-to-one by [5, Proposition 4.3], and then it is easy to see that $\mathcal{C}$ is closable and

$$\operatorname{dom}(\overline{\mathcal{C}}) = \overline{\mathcal{B}}(\ell^2), \quad \overline{\mathcal{C}} = \overline{\mathcal{B}}^{-1}. \tag{16}$$

Theorem 3.1. *Suppose M is indeterminate. Then the set $\mathcal{D}_*$ is dense in $L^2(M)$.*

Proof. For $u \in \mathbb{C}[x]^\perp$ we have

$$u_n = \int_{-\infty}^{\infty} u(x)x^n\, dM(x) = 0, \quad n = 0, 1, \ldots,$$

and for $u \in \mathcal{E}$ given by (9) we find

$$\begin{aligned} u_n &= \int_{-\infty}^{\infty} u(x)x^n\,dM(x) = \sum_{k=0}^{\infty} g_k \int_{-\infty}^{\infty} P_k(x)x^n\,dM(x) \\ &= \sum_{k=0}^{n} c_{k,n} g_k = (\mathcal{C}^t g)_n, \end{aligned}$$

where we have used (14).

By the orthogonal decomposition (10) we find

$$\mathcal{D}_* = \left\{ u = \sum_{k=0}^{\infty} g_k P_k \mid g \in \ell^2,\ \mathcal{C}^t g \in \ell^2 \right\} \oplus \mathbb{C}[x]^\perp,$$

so $\mathcal{D}_*$ is dense in $L^2(M)$ if and only if

$$X := \{g \in \ell^2 \mid \mathcal{C}^t g \in \ell^2\} \ \text{ is dense in } \ \ell^2.$$

However, $\{\mathcal{B}^t \eta \mid \eta \in \mathcal{D}\} \subset X$ and the subset is already dense in ℓ^2.

In fact, for $\eta \in \mathcal{D}$ we have $\mathcal{B}^t\eta \in \ell^2$ because the matrix $\mathcal{B}$ is Hilbert-Schmidt. Furthermore, $\mathcal{C}^t(\mathcal{B}^t\eta) = \eta \in \ell^2$ because of (15).

Finally, since $\overline{\mathcal{B}}$ is a bounded operator and one-to-one on ℓ^2, the set $\{\mathcal{B}^t\eta \mid \eta \in \mathcal{D}\}$ is dense in ℓ^2. □

By Theorem 1.3 we know that the operator A given by (2) is closable, when M is indeterminate. We shall now describe the closure $\overline{A}$ in this case. For this we need the unitary operator $U : \ell^2 \to \mathcal{E}$ given by $U(e_n) = P_n$, $n = 0, 1, \ldots$.

Theorem 3.2. *Suppose M is indeterminate. Then*

$$\mathrm{dom}(\overline{A}) = \overline{\mathcal{B}}(\ell^2), \quad \overline{A} = U\overline{\mathcal{C}}. \tag{17}$$

For $\xi \in \mathrm{dom}(\overline{A})$ we have $\xi = \overline{\mathcal{B}}y$ for a unique $y \in \ell^2$ and the following series expansions hold

$$\overline{A}\xi(z) = \sum_{k=0}^{\infty} \xi_k z^k = \sum_{n=0}^{\infty} y_n P_n(z), \quad z \in \mathbb{C}, \tag{18}$$

uniformly for z in compact subsets of $\mathbb{C}$.

Proof. We clearly have $A = U\mathcal{C}$, hence $\overline{A} = U\overline{\mathcal{C}}$, and therefore $\mathrm{dom}(\overline{A}) = \mathrm{dom}(\overline{\mathcal{C}}) = \overline{\mathcal{B}}(\ell^2)$.

For $\xi = \overline{\mathcal{B}}y$ for $y \in \ell^2$, we have $\overline{A}\xi = Uy$ and

$$f(z) := Uy(z) = \sum_{n=0}^{\infty} y_n P_n(z),$$

uniformly for z in compact subsets of $\mathbb{C}$. By Cauchy's integral formula we therefore get

$$\begin{aligned} \frac{f^{(k)}(0)}{k!} &= \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z^{k+1}}\,dz = \sum_{n=0}^{\infty} y_n \frac{1}{2\pi i}\int_{|z|=1} \frac{P_n(z)}{z^{k+1}}\,dz \\ &= \sum_{n=0}^{\infty} y_n b_{k,n} = \xi_k. \end{aligned}$$

This shows the first expression in (18). □

We end with an example related to Yafaev's condition (iii).

Example 3.3. *Let M be a positive measure on $[-1, 1]$ with $M(\{1\}) = c > 0$. The operator A is not closable.*

In fact, define

$$g_k^{(n)} = \begin{cases} 1/n, & 0 \le k \le n-1, \\ 0, & k \ge n. \end{cases}$$

Then $g^{(n)} \to 0$ in ℓ^2. We have,

$$Ag^{(n)}(x) = \begin{cases} 1, & x = 1, \\ \frac{1}{n}\frac{1-x^n}{1-x}, & -1 \le x < 1. \end{cases}$$

Hence $Ag^{(n)}(x) \to \chi_1(x)$ pointwise and also in $L^2(M)$, where χ_B denotes the indicator function of a subset B of the real line. Thus A is not closable.

References

[1] N. I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis. English translation, Oliver and Boyd, Edinburgh, 1965.

[2] C. Berg and J. P. R. Christensen, Density questions in the classical theory of moments, Ann. Inst. Fourier **31**, no. 3 (1981), 99–114.

[3] C. Berg and A. J. Durán, The index of determinacy for measures and the ℓ^2-norm of orthonormal polynomials. Trans. Amer. Math. Soc. **347** (1995), 2795–2811.

[4] C. Berg and A. J. Durán, When does a discrete differential perturbation of a sequence of orthonormal polynomials belong to ℓ^2? Journal of Functional Analysis **136** (1996), 127–153.

[5] C. Berg and R. Szwarc, The smallest eigenvalue of Hankel matrices, Constr. Approx. **34** (2011), 107–133.

[6] C. Berg and R. Szwarc, Inverse of infinite Hankel moment matrices, SIGMA **14** (2018), 109, 48 pages

[7] C. Berg and R. Szwarc, Closable Hankel operators and moment problems, ArXiv:1905.03010v1 8 May 2019.

[8] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. **137** (1998), 82–203.

[9] D. R. Yafaev, Unbounded Hankel operators and moment problems, Integr. Equ. Oper. Theory **85** (2016), 289–300. DOI 10.1007/s00020-016-2289-y.

[10] D. R. Yafaev, Unbounded Hankel operators and moment problems: Erratum, Integr. Equ. Oper. Theory, to appear.

Christian Berg
Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen, Denmark
e-mail: `berg@math.ku.dk`

Ryszard Szwarc
Institute of Mathematics, University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: szwarc2@gmail.com