Strong nonnegative linearization of orthogonal polynomials

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One of the main problems in the theory of orthogonal polynomials is to determine whether any expansion of the product of two orthogonal polynomials in terms of these polynomials has nonnegative coefficients. We want to decide which orthogonal systems $\{p_n\}_{n=0}^{N}$ have the property

$$p_n(x)p_m(x) = \sum c(n,m,k)p_k(x)$$

with nonnegative coefficients $c(n,m,k)$ for every n, m and k. Numerous classical orthogonal polynomials as well as their q-analogues satisfy nonnegative linearization property (Askey, Gasper, Rahman). There are many criteria for nonnegative linearization given in terms of the coefficients of the recurrence relation the orthogonal polynomials satisfy, that can be applied to general orthogonal polynomials systems (Askey, Sz.). These criteria are based on the connection between the linearization property and a certain discrete boundary value problem of hyperbolic type.

Let orthogonal polynomials $\{p_n\}_{n=0}^N$, where N may be infinite, satisfy

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1}, \quad 0 \le n \le N,$$

where $\gamma_n, \alpha_{n+1} > 0$, β_n are real numbers and $\alpha_0 = \gamma_N = p_{-1} = 0$. Assume that for $0 \le n < N$ we have

(i) $\alpha_n \le \alpha_{n+1}$.

(ii) $\beta_n \le \beta_{n+1}$.

(iii) $\alpha_n + \gamma_n \le \alpha_{n+1} + \gamma_{n+1}$.

(iv) $\alpha_n \le \gamma_n$ for $n < N$.

Then $c(n, m, k) \ge 0$ for any n, m, k (Sz1992).

This criterion yields nonnegative linearization for the associated polynomials of any order.

There is a wide class of orthogonal polynomials which resist any general criteria known so far. These are finite systems of orthogonal polynomials. The simplest family consists of Krawtchouk polynomials, orthogonal with respect to the binomial distribution

$$\mu = \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \delta_n, \quad q = 1 - p,$$

where $0 < p < 1$. Upon normalization

$$K_n(0) = (-1)^n (p/q)^{n/2}$$

they satisfy

$$\begin{aligned} xK_n = & \sqrt{pq}\,(N-n)K_{n+1} + [p(N-n) + qn]K_n \\ & + \sqrt{pq}\,nK_{n-1}, \quad 0 \le n \le N. \end{aligned}$$

Eagleson (1960) has shown that K_n admit non-negative linearization if and only if $0 < p \le \frac{1}{2}$.

When we apply our criterion to this case the assumptions (i)–(iii) are satisfied for $0 < p \leq \frac{1}{2}$. The assumption (iv) is valid only for $n \leq N/2$. Let's modify the recurrence relation by removing the middle term. Then we obtain

$$x\widetilde{K}_n = \sqrt{pq}\,(N-n)\widetilde{K}_{n+1} + \sqrt{pq}\,n\widetilde{K}_{n-1},\ 0 \leq n \leq N.$$

It is possible to show nonnegative linearization for these polynomials by using the criterion given previously and some symmetry property of this new system.

Now the following question arises. Given a system of polynomials satisfying

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1}.$$

What property should this system satisfy so that for any nondecreasing sequence ε_n of real numbers the new system

$$xq_n = \gamma_n q_{n+1} + (\beta_n + \varepsilon_n)q_n + \alpha_n q_{n-1}$$

satisfied nonnegative linearization property ? Clearly the system p_n should satisfy nonnegative linearization property, but it is not sufficient.

This new property will be called strong nonnegative linearization. Unfortunately the polynomials $\widetilde{K}_n$ do not satisfy it. Indeed, let

$$\varepsilon_n = \begin{cases} 0 & 0 \le n \le N-1, \\ \varepsilon & n = N. \end{cases}$$

Let

$$xq_n = (N-n)q_{n+1} + \varepsilon_n q_n + nq_{n-1}.$$

Then the nonnegative linearization fails for $0 < \varepsilon < \sqrt{N-2}$. Therefore the finite system case remains unsolved.

Let $p_n^{(l)}(x)$ denote the associated polynomials of order l. By definition they satisfy the shifted recurrence relation

$$xp_n^{(l)} = \gamma_{n+l} p_{n+1}^{(l)} + \beta_{n+l} p_n^{(l)} + \alpha_{n+l} p_{n-1}^{(l)}.$$

The system p_n satisfy the strong nonnegative linearization property if and only if

$$p_n^{(l)} p_m^{(l)} = \sum_{k=|n-m|}^{n+m} C_l(n,m,k) p_k^{(l)},$$

with nonnegative coefficients $C_l(n,m,k)$ for all n, m, k and $l \geq 0$.

For convenience we will use the polynomials $p_n^{[l]}$ satisfying

$$xp_n^{[l]} = \gamma_n p_{n+1}^{[l]} + \beta_n p_n^{[l]} + \alpha_n p_{n-1}^{[l]}, \quad n \geq l+1,$$
$$p_0^{[l]} = p_1^{[l]} = \ldots = p_l^{[l]} = 0, \; p_{l+1}^{[l]} = \frac{1}{\gamma_l}.$$

Then we have

$$p_n^{[l]} = p_{n-l-1}^{(l+1)}.$$

The advantage is that the polynomials $p_n^{[l]}$ satisfy the same recurrence relation for each l.

We have

$$p_n^{[l]} p_m^{[l]} = \sum_{k=|n-m|+l+1}^{n+m-l-1} c_l(n,m,k) p_k^{[l]}.$$

The nonnegativity of the coefficients $C_l(n,m,k)$, is equivalent to the nonnegativity of the linearization coefficients $c(n,m,k) = C_0(n,m,k)$ and that of $c_l(n,m,k)$ because

$$c_l(n,m,k) = C_{l+1}(n-l-1, m-l-1, k-l-1).$$

We are going to show three equivalent conditions for each of the properties nonnegative linearization and strong nonnegative linearization.

Let $u(n,m)$ be a matrix of complex numbers indexed by $0 \leq n,m < N$. Define two operators L_1 and L_2 acting on such matrices by the rule

$$\begin{aligned}
(L_1 u)(n,m) &= \gamma_n u(n+1,m) \\
&\quad + \beta_n u(n,m) + \alpha_n u(n-1,m), \\
(L_2 u)(n,m) &= \gamma_m u(n,m+1) \\
&\quad + \beta_m u(n,m) + \alpha_m u(n,m-1).
\end{aligned}$$

Let

$$H = L_1 - L_2.$$

Proposition 1. *(i) The polynomials* p_n *admit nonnegative product linearization if and only if every matrix* $u = \{u(n,m)\}$ *such that*

$$\begin{cases} Hu(n,m) &= 0, \text{ for } 0 \le m < n < N \\ u(n,0) &\ge 0, \text{ for } 0 \le n \le N, \end{cases}$$

satisfies $u(n,m) \ge 0$ *for* $0 \le m \le n \le N$.

(ii) The polynomials $\{p_n\}_{n=0}^N$ *admit strong nonnegative product linearization if and only if every matrix* $u = \{u(n,m)\}$ *such that*

$$\begin{cases} Hu(n,m) &\le 0, \text{ for } 0 \le m < n < N \\ u(n,0) &\ge 0, \text{ for } 0 \le n \le N, \end{cases}$$

satisfies $u(n,m) \ge 0$ *for* $0 \le m \le n \le N$.

The proof of both propositions follows from the lemma.

Lemma 1. *Given a matrix* $v = \{v(n,m)\}_{n>m\geq 0}$ *and a sequence* $f = \{f(n)\}_{n\geq 0}$. *Let a matrix* $u = \{u(n,m)\}_{n\geq m\geq 0}$ *satisfy*

$$\begin{aligned} Hu(n,m) &= v(n,m), \quad && \text{for } 0 \leq m < n < N, \\ u(n,0) &= f(n), && \text{for } 0 \leq n \leq N. \end{aligned}$$

Then

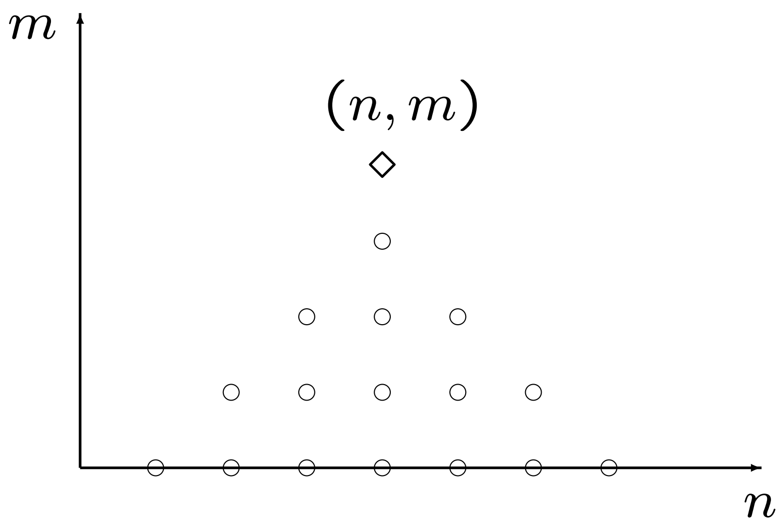
$$u(n,m) = -\sum_{k>l\geq 0} v(k,l)c_l(n,m,k) + \sum_{k\geq 0} f(k)c(n,m,k).$$

The summations are finite because $c_l(n,m,k) \neq 0$ implies $|n-m| \leq k+l+1 \leq n+m$.

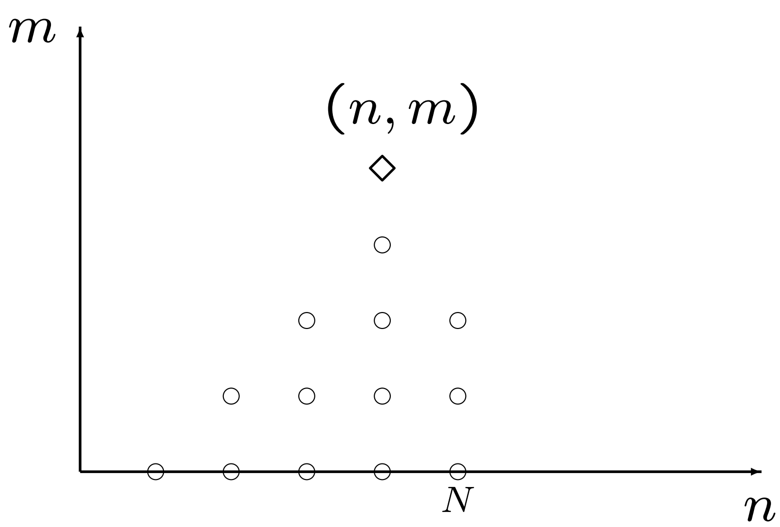
For each point (n, m) with $1 \leq m \leq n \leq N$, let $\Delta_{n,m}$ denote the set of lattice points in the plane defined by

$$\Delta_{n,m} = \{(i,j) \,|\, 0 \leq j \leq i \leq N, \; |n-i| < m-j\}.$$

The set $\Delta_{n,m}$ is depicted in below for $n+m \leq N$. (the points in $\Delta_{n,m}$ are marked with empty circles).



In case N is finite and $n + m > N$ the corresponding picture is



Let H^* denote the adjoint operator to H with respect to the inner product of matrices

$$\langle u, v\rangle = \sum_{n,m=0}^{N-1} u(n,m)\overline{v(n,m)}.$$

This operator acts according to

$$(H^*v)(n,m) =$$
$$\alpha_{n+1}v(n+1,m) + \beta_n v(n,m) + \gamma_{n-1}v(n-1,m)$$
$$-\alpha_{m+1}v(n,m+1) - \beta_m v(n,m) - \gamma_{m-1}v(n,m-1).$$

Theorem 1. *(a) The orthogonal polynomials* $\{p_n\}_{n=0}^{N}$ *admit nonnegative linearization if for every* (n, m), *with* $1 \leq m \leq n \leq N$, *there exists a matrix* $v(i, j)$ *such that*

(i) $\operatorname{supp} v \subset \Delta_{n,m}$.

(ii) $(H^*v)(n, m) < 0$.

(iii) $(H^*v)(i, j) \geq 0$ *for* $(i, j) \neq (n, m)$.

(b) The orthogonal polynomials $\{p_n\}_{n=0}^{N}$ *admit strong nonnegative linearization if for every* (n, m), *with* $1 \leq m \leq n \leq N$, *there exists a matrix* $v(i, j)$ *such that*

(i) $\operatorname{supp} v \subset \Delta_{n,m}$.

(ii) $(H^*v)(n, m) < 0$.

(iii) $(H^*v)(i, j) \geq 0$ *for* $(i, j) \neq (n, m)$.

(iv) $v(i, j) \geq 0$ *for* $(i, j) \in \Delta_{n,m}$.

Let $v_{n,m}$ denote a matrix such that

$$\begin{aligned} \operatorname{supp} v_{n,m} &\subset \Delta_{n,m}, \\ (H^* v_{n,m})(n,m) &= -1, \\ (H^* v_{n,m})(i,j) &= 0, \text{ for } 0 < j < m. \end{aligned}$$

This is a special choice of the matrix which satisfies conditions (i), (ii) and partially (iii), of Theorem 1. The matrix $v_{n,m}$ is uniquely determined. Moreover we have the following.

Theorem 2. *For any $n \geq m \geq 0$ and $k > l \geq 0$ we have*

$$v_{n,m}(k,l) = c_l(n,m,k).$$

Moreover

$$H^* v_{n,m} = -\delta_{(n,m)} + \sum_{k=n-m}^{n+m} c(n,m,k)\delta_{(k,0)}.$$

In this way the conditions given in Theorem 1 become equivalent to nonnegative (or strong nonnegative) linearization.

By Theorem 2 the following becomes evident.

Theorem 3. *(a) The orthogonal polynomials* $\{p_n\}_{n=0}^N$ *admit nonnegative linearization if and only if for every* (n, m), *with* $1 \leq m \leq n \leq N$, *the matrix* $v_{n,m}$ *satisfies* $(H^* v_{n,m})(j, 0) \geq 0$.

(b) The orthogonal polynomials $\{p_n\}_{n=0}^N$ *admit strong nonnegative linearization if and only if for every* (n, m), *with* $1 \leq m \leq n \leq N$, *the matrix* $v_{n,m}$ *satisfies* $(H^* v_{n,m})(j, 0) \geq 0$ *and* $v_{n,m}(i, j) \geq 0$ *for* $(i, j) \in \Delta_{n,m}$.

Proof of Theorem 1(b)
Let $u = \{u(n,m)\}_{n \geq m \geq 0}$ satisfy

$$(Hu)(n,m) \leq 0, \text{ for } n > m \geq 0,$$
$$u(n,0) \geq 0.$$

We will show that $u(n,m) \geq 0$, by induction on m. Assume that $u(i,j) \geq 0$ for $j < m$. Let v be a matrix satisfying the assumptions of Theorem 1(b). Then

$$0 \geq \langle Hu, v \rangle = \langle u, H^*v \rangle$$
$$= u(n,m)(H^*v)(n,m) + \sum_{\substack{i \geq j \geq 0 \\ j < m}} u(i,j)(H^*v)(i,j)$$

Therefore

$$-u(n,m)(H^*v)(n,m) \geq \sum_{\substack{i \geq j \geq 0 \\ j < m}} u(i,j)(H^*v)(i,j),$$

and the conclusion follows. $\diamondsuit$

Theorem 4. *Assume that*

(i) $\beta_m \leq \beta_n$ *for* $m \leq n$.

(ii) $\alpha_m \leq \alpha_n$ *for* $m < n$.

(iii) $\alpha_m + \gamma_m \leq \alpha_n + \gamma_n$ *for* $m < n$.

(iv) $\alpha_m \leq \gamma_n$ *for* $m \leq n$.

Then the system $\{p_n\}_{n=0}^{\infty}$ *satisfies the strong nonnegative linearization property.*

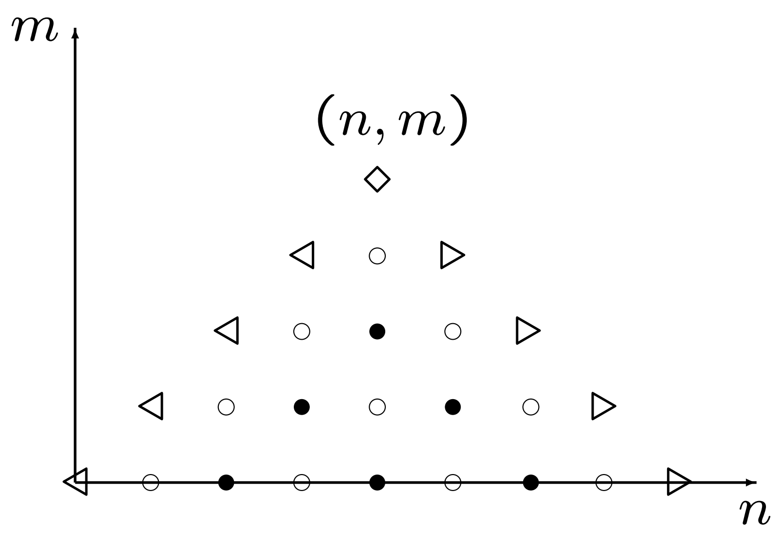
It suffices to construct for every (n, m) with $n \geq m$, a matrix v satisfying the assumptions of Theorem 1. Fix (n, m). Define the matrix v according to the following.

$$v(i,j) = \begin{cases} c_i c_j & (i,j) \in \Delta_{n,m},\ n+m-i-j \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

where

$$c_0 = 1, \qquad c_i = \frac{\gamma_0 \gamma_1 \cdots \gamma_{i-1}}{\alpha_1 \alpha_2 \ldots \alpha_i}.$$

The points in the support of v are marked by empty circles in the picture below.



Then supp H^*v consists of the points marked by $\circ, \bullet, \triangleleft, \triangleright$ and $\diamond$. A straightforward computation gives the following.

$$\frac{(H^*v)(i,j)}{c_i c_j} = \begin{cases} -\alpha_m & (i,j) = (n,m) \\ \beta_i - \beta_j & (i,j) \ - \ \circ \\ \alpha_i + \gamma_i - \alpha_j - \gamma_j & (i,j) \ - \ \bullet \\ \alpha_i - \alpha_j & (i,j) \ - \ \triangleright \\ \gamma_i - \alpha_j & (i,j) \ - \ \triangleleft \end{cases}$$

Theorem 5. *Assume orthogonal polynomial system* $\{p_n\}_{n=0}^N$ *satisfies strong nonnegative linearization property. Let* ε_n *be a nondecreasing sequence. Let* q_n *be a sequence of polynomials satisfying the perturbed recurrence relation*

$$xq_n = \gamma_n q_{n+1} + (\beta_n + \varepsilon_n)q_n + \alpha_n q_{n-1},$$

for $n \geq 0$. *Then the system* $\{q_n\}_{n=0}^N$ *satisfies strong nonnegative linearization property.*

Proof. We will make use of Theorem 1(b). Let H and H_ε denote the hyperbolic operators corresponding to the unperturbed and perturbed system, respectively. For any matrix $v(i,j)$ we have

$$(H_\varepsilon^* v)(i,j) = (H^* v)(i,j) + (\varepsilon_i - \varepsilon_j)v(i,j).$$

By assumptions for any $n \geq m \geq 0$, there exists a matrix v satisfying the assumptions of Theorem 1(b), with respect to H.

The same matrix v satisfies these assumptions with respect to H_ε. Indeed, the assumptions (i) and (iv) do not depend on the perturbation. Since $v(n,m) = 0$ the assumption (ii) is not affected, as well. Concerning (iii), since $v \geq 0$ and ε_n is nondecreasing we have

$$(H_\varepsilon^* v)(i,j) \geq (H^* v)(i,j) \geq 0,$$

for $i \geq j \geq 0$ and $j < m$. Hence the perturbed system of polynomials satisfies the strong non-negative linearization property $\diamondsuit$

This theorem is not valid for standard nonnegative linearization property e.g. the Krawtchouk polynomials case.

Since

$$\gamma_0 p_1 p_n = \gamma_n p_{n+1} + (\beta_n - \beta_0) p_n + \alpha_n p_{n-1},$$

nonnegative linearization requires $\beta_n \geq \beta_0$. There are many examples of orthogonal polynomial systems satisfying nonnegative linearization for which $\beta_1 \geq \beta_2 \geq ... \geq \beta_0$. For example the Jacobi polynomials with $\alpha > \beta > -1$ and $-1 \leq \alpha + \beta < 0$, are such. Therefore the associated polynomials do not satisfy nonnegative linearization. Moreover the following holds.

Proposition 2. *Assume orthogonal polynomial system* $\{p_n\}_{n=0}^N$ *satisfies strong nonnegative linearization property and*

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1}.$$

Then β_n *is nondecreasing.*

Proposition 3. *The Jacobi polynomials satisfy strong nonnegative linearization property if and only if either* $\alpha > \beta > -1$ *and* $\alpha + \beta \geq 0$ *or* $\alpha = \beta \geq -\frac{1}{2}$.

By Gasper's result the Jacobi polynomials satisfy nonnegative linearization property if $\alpha \geq \beta > -1$ and $c(2,2,2) \geq 0$. In particular the conditions $\alpha \geq \beta > -1$ and $\alpha + \beta \geq -1$ are sufficient.

<table>
<tr><th>Nonnegative linearization</th><th>Strong nonnegative linearization</th></tr>
<tr><td>$c(n,m,k) \geq 0$</td><td>$c(n,m,k) \geq 0$
$c_l(n,m,k) \geq 0$</td></tr>
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We always assume that $n \geq m \geq 0$ i $i \geq j \geq 0$.