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A SHORT PROOF OF THE GRIGORCHUK-COHEN COGROWTH THEOREM

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ABSTRACT. Let G be a group generated by $g_1, \dots, g_r$. There are exactly $2r(2r-1)^{n-1}$ reduced words in $g_1, \dots, g_r$ of length n. Part of them, say γ_n represents identity element of G. Let $\gamma = \limsup \gamma_n^{1/n}$. We give a short proof of the theorem of Grigorchuk and Cohen which states that G is amenable if and only if $\gamma = 2r - 12$. Moreover we derive some new properties of the generating function $\sum \gamma_n z^n$.

Let G be a finitely generated discrete group. Consider G as an epimorphic image $\pi\colon \mathbf{F}_r \to G$ of the free group $\mathbf{F}_r$ on r generators. Thus G is isomorphic to the quotient group $\mathbf{F}_r/N$ where $N = \ker \pi$ is a normal subgroup of $\mathbf{F}_r$. Once we fix a set of free generators in $\mathbf{F}_r$ we introduce $|x|$ the length of the word x in $\mathbf{F}_r$ with respect to the generators and their inverses. Let $N_n = \{x \in N \colon |x| = n\}$, $\gamma_n = \operatorname{card} N_n$ and $\gamma = \limsup \gamma_n^{1/n}$ which is called the growth exponent of N is $\mathbf{F}_r$ with respect to the fixed set of free generators in $\mathbf{F}_r$. Because there are exactly $2r(2r-1)^{n-1}$ elements of length n in $\mathbf{F}_r$, $\gamma \le 2r - 1$. Grigorchuk [2] and Cohen [1] have shown that a group G is amenable if and only if γ attains maximal possible value, i.e. $\gamma = 2r - 1$. We propose a new rather simple proof without any estimates which allows us to draw out new information on the behaviour of the generating function $N(z) = \sum \gamma_n z^n$.

As in [1] and [2] we will base our proof on a characterization of discrete amenable groups given by Kesten [3]. Any absolutely summable function f on a group G defines a translation invariant operator of $l^2(G)$ as $g \mapsto f * g$. The norm of corresponding map we denote as usual by $\|f\|_{C^*_{\mathrm{red}}(G)}$.

Theorem 1 (Kesten [3]). *Let f be any selfadjoint summable positive function on G whose support generates G. Then G is amenable if and only if* $\|f\|_{C^*_{\mathrm{red}}(G)} = \sum_{x \in G} f(x)$.

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As in [1] let χ_n, $n = 0, 1, 2, \ldots$ denote the characteristic function of the set of works of length n in $\mathbf{F}_r$. Observe that the support of the function $\pi(\chi_1)$ generates G by assumption. We are going to compute the norm $\|\pi(\chi_1)\|_{C^*_{\text{red}}(G)}$.

Theorem 2 (Grigorchuk [2], Cohen [1]). *Let* $q = 2r - 1$. *then* $\|\pi(\chi_1)\|_{C^*_{\text{red}}(G)} = \gamma + q/\gamma$ *only if* $\gamma > 1$ (*or equivalently* $\ker \pi$ *is nontrivial*).

Together with Kesten's theorem it gives immediately the following

Theorem 3 (Grigorchuk [2], Cohen [1]). *G is amenable if and only if* $\gamma = 2r - 1$.

Proof of Theorem 2. The linear functional $C^*_{\text{red}}(G) \ni f \mapsto f(e)$ is the faithful trace $\mathbf{Tr}$ on $C^*_{\text{red}}(G)$ thus (see [6])

$$\|f\|_{C^*_{\text{red}}(G)} = \lim_n (\mathbf{Tr} f^{*2n})^{1/2n} = \lim_n \left(\frac{\mathbf{Tr} f^{*(2n+2)}}{\mathbf{Tr} f^{*2n}} \right)^{1/2}.$$

Let t be a positive number such that $0 < t < q^{-1/2}$. If we let $\alpha(t) = qt + t^{-1}$ then by [4, proof of Theorem 3.1, p. 128] or by [5, Theorem 1] we have

$$(\alpha(t)\delta_e - \chi_1)^{-1}(x) = \frac{t}{1 - t^2} t^{|x|} \qquad \text{in } C^*_{\text{red}}(\mathbf{F}_r).$$

Thus using $(\alpha - x)^{-1} = \sum a^{-(n+1)} x^n$ we obtain

$$\sum_{n=0}^{\infty} \alpha(t)^{-(n+1)} \chi_1^{*n} = \frac{t}{1 - t^2} \sum_{n=0}^{\infty} t^n \chi_n.$$

Applying successively π and then $\mathbf{Tr}$ to both sides and observing that $\mathbf{Tr}\pi(\chi_n) = \gamma_n$ gives

$$(*) \qquad \sum_{n=0}^{\infty} \mathbf{Tr}\pi(\chi_1)^{*n} \alpha(t)^{-(n+1)} = \frac{t}{1 - t^2} \gamma_n t^n.$$

(all this makes sense at least for values of t small enough).

Now the point is that $(*)$ relates the radii of convergence: r_1 of the series $\sum \gamma_n t^n$ and r_2 of $\sum \mathbf{Tr}\pi(\chi_1)^{*n} y^{n+1}$. Clearly we have $r_1 = \gamma^{-1}$ and $r_2 = \|\pi(\chi_1)\|^{-1}_{C^*_{\text{red}}(G)}$. Furthermore $\|\pi(\chi_1)\|_{C^*_{\text{red}}(G)} \geq \|\chi_1\|_{C^*_{\text{red}}(\mathbf{F}_r)} = 2q^{1/2}$ implies $r_2 \leq \frac{1}{2} q^{-1/2}$ (see [5, Corollary 2], [1, Theorem 4]). Next using the fact that the function $y(t) = \alpha(t)^{-1}$: $[0, q^{-1/2}] \to [0, \frac{1}{2} q^{-1/2}]$ is increasing and the coefficients of the both series are nonnegative yields $y(r_1) = r_2$. But this gives the desired $\|\pi(\chi_1)\|_{C^*_{\text{red}}(G)} = \gamma + q/\gamma$.

Remarks. Let $\|\pi(\chi_1)\|_{C^*_{\text{red}}(G)} = \gamma + q/\gamma$. Then the function $z \mapsto \mathbf{Tr}[\alpha(z)\delta_e - \pi(\chi_1)]^{-1}$ is analytic in the domain $\{z \in \mathbf{C}\colon \alpha(z) \notin [-(\gamma + q\gamma^{-1}), \gamma + q\gamma^{-1}]\}$. In particular it is analytic in $D = \{z \in \mathbf{C}\colon |z| < q^{-1/2}\} \setminus ([-q^{-1/2}, \frac{1}{\gamma}] \cup [\frac{1}{\gamma}, q^{-1/2}])$. Hence by $(*)$ the function $z/(1 - z^2) \sum \gamma_n z^n$ as well as $\sum \gamma_n z^n$ can be continued analytically to D. It means that $-\gamma^{-1}$ and γ^{-1} are the only possible singular points on the circle of convergence of the power series $N(z) = \sum \gamma_n z^n$.

Added in proof. After submission of the manuscript I was informed of the paper by W. Woess, *Cogrowth of groups and simple random walks*, Archiv der mathematic **41** (1983), 363–370, where the same arguments are used. In particular a formula similar to $(*)$ is proved there.

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