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Chain sequences and compact perturbations of orthogonal polynomials

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0 Introduction

The main objective of this paper is to study relations between two systems of orthogonal polynomials corresponding to two measures which differ by a point mass. The intuition suggests that the difference operator associated with polynomials via recurrence formula shouldn't change too much if we add a point mass to the measure. One expects that the new difference operator is a compact perturbation of the original one. In Sect. 3 we show that under some conditions on the measure it is so. This generalizes partially results of Nevai [3]. However the statement is not true in general. We show an example of two measures equal modulo a point mass such that corresponding difference operators are not compact perturbations of each other.

The basic tools we use are chain sequences and quadratic transformations. These are described in Sect. 1 and 2. The notion of chain sequences is due to Wall [5]. We refer to Chihara's book [2] for results on chain sequences that are frequently used in the present work. In particular a kind of master key is Chihara's theorem on the convergence of chain sequences [2, Theorem 6.4]. As his proof involves several results concerning chain sequences that are not discussed here, we give an alternative straightforward proof in the Appendix. Also, in Sect. 2, we characterize so called maximal parameter sequences for chain sequences.

In Sect. 4 we discuss the growth of orthonormal polynomials on the interval of orthogonality. Nevai et al. [4] showed that if the coefficients in the recurrence formula are convergent then the polynomials have uniform subexponential growth on the interval of orthogonality. 12 years earlier Nevai [3] showed almost uniform subexponential growth in the interior of the interval.

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We show using tricks with quadratic transformations that one can derive uniform subexponential growth from the almost uniform one. The method works for measures whose support consists of an interval and finitely many points off the interval.

1 Chain sequences

Let μ be a probability measure on the real line $\mathbb{R}$ all of whose moments are finite. We will always assume that the support of μ is an infinite so that the monomials $1, x, x^2, \ldots$ are linearly independent. Let $\{P_n\}_{n=0}^{\infty}$ be a system of orthonormal polynomials obtained from the sequence $1, x, x^2, \ldots$ by the Gram–Schmidt procedure. Then P_n obey a three-term recurrence formula of the form

$$xP_n = \lambda_n P_{n+1} + \beta_n P_n + \lambda_{n-1} P_{n-1} \tag{1}$$

where λ_n are positive coefficients while β_n are real ones. With this relation we usually associate the difference operator L acting on sequences as

$$La_n = \lambda_n a_{n+1} + \beta_n a_n + \lambda_{n-1} a_{n-1} \tag{2}$$

L is a symmetric operator on the space $\ell^2(\mathbb{N})$ of square summable sequences. For any complex number z and initial value a_0 there exists a unique eigenvector $\{a_n\}_{n=0}^{\infty}$ corresponding to the eigenvalue z. This is due to the fact that λ_n are nonzero numbers. The correspondence

$$J: \{a_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} a_n P_n$$

is an isometry from $\ell^2(\mathbb{N})$ into the Hilbert space $\mathscr{L}^2(\mathbb{R}, d\mu)$. By (1) and (2) we have

$$L = J^* M_x J$$

where M_x is a linear operator on $\mathscr{L}^2(\mathbb{R}, d\mu)$ whose action is to multiply by the variable x. As such the operator L has a simple spectrum which coincides with that of M_x, the latter being, as is well known, the support of μ. By this reasoning the operator L is positive definite if and only if the support of the measure μ is contained in the half-axis $[0, +\infty)$. The positivity of L can be stated also in terms of the coefficients λ_n and β_n. This is where the chain sequence turn up in natural way. The following result is well-known. We will give a short proof for the sake of completeness.

Proposition 1 [2, Theorem 9.2] *The support of μ is contained in $[0, +\infty)$ if and only if $\beta_n > 0$, $n \in \mathbb{N}$, and there is a sequence of numbers m_n satisfying $0 \leqq m_n \leqq 1$ and*

$$\frac{\lambda_{n-1}^2}{\beta_n \beta_{n-1}} = m_n(1 - m_{n-1}) \quad n = 0, 1, \ldots. \tag{3}$$

Proof. Let Δ_n denote the $n\times n$ minor of the matrix of L, i.e.

$$\Delta_n=\begin{vmatrix} \beta_0 & \lambda_0 & \cdots & 0 & 0 \\ \lambda_0 & \beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{n-1} & \lambda_{n-1} \\ 0 & 0 & \cdots & \lambda_{n-1} & \beta_n \end{vmatrix}.$$

Expanding the determinant relative to the last row two times successively gives a recurrence formula for Δ_n.

$$\Delta_n=\beta_n\Delta_{n-1}-\lambda_{n-1}^2\Delta_{n-2}\quad n\geqq 2\,.$$

Assume that $\operatorname{supp}\mu\subset[0,+\infty)$. Then L is positive definite operator. We then have

$$\beta_n=\int_0^{+\infty} xP_n(x)^2\,d\mu(x)\,.$$

Since the support of μ is an infinite subset of nonnegative reals, then $\beta_n>0$. The positivity of L implies that $\Delta_n\geqq 0$ for every n. Actually we have $\Delta_n>0$. Indeed, if $\Delta_n=0$, then there would exist a sequence a with finite support such that $La=0$. As L has simple spectrum, so a is a multiple of the sequence $\{P_n(0)\}_{n=0}^{\infty}$, which doesn't vanish for infinitely many n. Hence $a=0$. For the same reason $\beta_n>0$, for every n. Set $\Delta_{-1}=1$, $m_0=0$ and

$$m_n=\frac{\lambda_{n-1}^2}{\beta_n}\frac{\Delta_{n-2}}{\Delta_{n-1}}\quad\text{for } n=1,2,\ldots.$$

Then $0\leqq m_n\leqq 1$, and also (3) is satisfied.

Conversely if $\beta_n>0$ and there is a sequence m_n satisfying (3), then the sequence $\tilde{m}_n$ defined by $\tilde{m}_0=0$, and

$$\frac{\lambda_{n-1}^2}{\beta_n\beta_{n-1}}=\tilde{m}_n(1-\tilde{m}_{n-1})\quad n=1,2,\ldots$$

fulfills $0\leqq\tilde{m}_n\leqq m_n$ (cf. [2, Theorem 5.2, p. 93]). But as we have seen in the first part of the proof

$$\tilde{m}_n=\frac{\lambda_{n-1}^2}{\beta_n}\frac{\Delta_{n-2}}{\Delta_{n-1}}.$$

Hence $\Delta_n>0$, for every $n\in\mathbb{N}$ and L is positive definite operator. □

Remark. The first part of the theorem can be also obtained by means of [1, Theorem 1]. It is worthwhile comparing the second part of the theorem with [1, Theorem 2].

Sequences $\{a_n\}_{n=0}^{\infty}$ that admit a representation of the form

$$a_n=g_n(1-g_{n-1})\quad 0\leqq g_n\leqq 1\,,$$

are called **chain sequences**. $\{g_n\}_{n=0}^{\infty}$ is called a parameter sequence for $\{a_n\}_{n=0}^{\infty}$. When $g_0=0$, then g_n is called the **minimal parameter sequence**. By [2, Theorem 5.3, p. 94] there exists a **maximal parameter sequence** $\{M_n\}$ such that $g_n \leqq M_n$ for every parameter sequence $\{g_n\}_{n=0}^{\infty}$. If the maximal and minimal sequences coincide we say that $\{a_n\}_{n=0}^{\infty}$ determines its parameter sequence uniquely. We are going to describe minimal and maximal parameter sequences by means of quadratic transformations. We refer the reader to [2, Chaps. III.5, IV.2] for more detailed treatment of the above notions.

2 Quadratic transformations

Let $d\mu_0(y)$ and $d\mu_1(y)$ be probability measures with infinite support contained in $[0, +\infty)$, having all moments finite, and related by

$$d\mu_1(y)=c^{-1}y\,d\mu_0(y), \quad c=\int_0^{+\infty} y\,d\mu_0(y)\,.$$

The corresponding orthonormal polynomials $Q_{0,n}$ and $Q_{1,n}$ satisfy the recurrence formula

$$yQ_{i,n}=\lambda_{i,n}Q_{i,n+1}+\beta_{i,n}Q_{i,n}+\lambda_{i,n-1}Q_{i,n-1}\,, \quad i=0,1\,. \tag{4}$$

Let $dv_i(x)$ be a symmetric measure such that

$$dv_i(x)=\frac{1}{2}d\mu_i(x^2) \quad x\geqq 0$$

and $P_{i,n}$ be the polynomials orthonormal with respect to $dv_i(x)$. Then the polynomials $P_{i,2n}$ are even functions while $P_{i,2n+1}$ are odd ones, hence they satisfy a recurrence formula of the form

$$xP_{i,n}=\alpha_{i,n}P_{i,n+1}+\alpha_{i,n-1}P_{i,n-1}\,, \quad i=0,1\,. \tag{5}$$

There are certain relations between those polynomials as well as between coefficients of the corresponding recurrence formulas. They are summarized in the following proposition which should be compared with [2, Theorem 9.1].

Proposition 2 *We have*

$$P_{i,2n}(y^{1/2})=Q_{i,n}(y) \quad i=0,1 \tag{6}$$

$$y^{-1/2}P_{0,2n+1}(y^{1/2})=c^{-1/2}Q_{1,n}(y) \tag{7}$$

$$\alpha_{i,2n}\alpha_{i,2n+1}=\lambda_{i,n} \quad \alpha_{i,2n-1}^2+\alpha_{i,2n}^2=\beta_{i,n}\,, \quad i=0,1 \tag{8}$$

$$\alpha_{0,2n+1}\alpha_{0,2n+2}=\lambda_{1,n} \quad \alpha_{0,2n}^2+\alpha_{0,2n+1}^2=\beta_{1,n}\,. \tag{9}$$

Proof. The first formula follows from the fact that the polynomials $P_{i,2n}(y^{1/2})$, are orthonormal relative to $d\mu_i(y)$. Next observe that the polynomials $c^{1/2}y^{-1/2}P_{0,2n+1}(y^{1/2})$ are orthonormal relative to $c^{-1}y\,d\mu_0(y)=d\mu_1(y)$. This gives (7).

By (5) we obtain

$$x^2 P_{i,2n} = \alpha_{i,2n}\alpha_{i,2n+1} P_{i,2n+2} + (\alpha_{i,2n-1}^2 + \alpha_{i,2n}^2) P_{i,2n} + \alpha_{i,2n-2}\alpha_{i,2n-1} P_{i,2n-2} \tag{10}$$

$$x^2 P_{i,2n+1} = \alpha_{i,2n+1}\alpha_{i,2n+2} P_{i,2n+3} + (\alpha_{i,2n}^2 + \alpha_{i,2n+1}^2) P_{i,2n+1} + \alpha_{i,2n-1}\alpha_{i,2n} P_{i,2n-1} . \tag{11}$$

Now combining (4) through (7), we get the conclusion. □

The formulas (8) and (9) imply that $\beta_{i,n}^{-1}\alpha_{i,2n-1}^2$ are parameter sequences for $(\beta_{i,n}\beta_{i,n+1})^{-1}\lambda_{i,n}^2$. These are the minimal parameter sequences as they take value zero for $n=0$. Also $\beta_{1,n}^{-1}\alpha_{0,2n}^2$ is a parameter sequence for $(\beta_{1,n}\beta_{1,n+1})^{-1}\lambda_{1,n}^2$. We are going to show that this one is a maximal parameter sequence.

Theorem 1 *Let $d\mu(y)$ be a probability measure with all moments finite, whose support is infinite and contained in $[0, +\infty)$. Let the corresponding orthonormal polynomials Q_n satisfy*

$$yQ_n = \lambda_n Q_{n+1} + \beta_n Q_n + \lambda_{n-1} Q_{n-1} .$$

If $\int_0^{+\infty} y^{-1}\,d\mu(y) = +\infty$, then $(\beta_n\beta_{n+1})^{-1}\lambda_n^2$ is a chain sequence that determines its parameter sequence uniquely. If the integral $c = \int_0^{+\infty} y^{-1}\,d\mu(y)$ is finite then letting $d\mu_0(y) = c^{-1}y^{-1}\,d\mu(y)$, and adopting the notation preceding the theorem, implies that the sequence $\beta_n^{-1}\alpha_{0,2n}^2$ is the maximal parameter sequence for $(\beta_n\beta_{n+1})^{-1}\lambda_n^2$.

Proof. Suppose that g_n is a parameter sequence for $(\beta_n\beta_{n+1})^{-1}\lambda_n^2$, and $g_0>0$. First we are going to show that $d\mu(y)$ has finite moment of order -1. Then we will show that $g_n \leqq \beta_n^{-1}\alpha_{0,2n}^2$.

We may assume that

$$g_n = \frac{\tilde{\alpha}_{2n}^2}{\beta_n} ,$$

for a sequence of numbers $\tilde{\alpha}_{2n}$. Set

$$\tilde{\alpha}_{2n+1}^2 = \beta_n - \tilde{\alpha}_{2n}^2 .$$

Then

$$\begin{cases} \lambda_{n-1} = \tilde{\alpha}_{2n-1}\tilde{\alpha}_{2n} \\ \beta_n \;\;\;= \tilde{\alpha}_{2n}^2 + \tilde{\alpha}_{2n+1}^2 . \end{cases} \tag{12}$$

Define the polynomials $\tilde{P}_n$ by $\tilde{P}_0 = 1$ and

$$x\tilde{P}_n = \tilde{\alpha}_n \tilde{P}_{n+1} + \tilde{\alpha}_{n-1}\tilde{P}_{n-1} . \tag{13}$$

Hence they are orthonormal with respect to a symmetric probability measure $d\tilde{\nu}(x)$. By (12) and (13) the polynomials $\tilde{P}_{2n+1}$ satisfy the formula

$$x^2\tilde{P}_{2n+1} = \lambda_n\tilde{P}_{2n+3} + \beta_n\tilde{P}_{2n+1} + \lambda_{n-1}\tilde{P}_{2n-1} . \tag{14}$$

As the polynomials $xQ_n(x^2)$ satisfy the same formula we get

$$\tilde{P}_{2n+1}(x)=\tilde{c}^{1/2}xQ_n(x^2), \quad \tilde{c}>0 .$$

The polynomials $y^{-1/2}\tilde{P}_{2n+1}(y^{1/2})$, are orthogonal with respect to $y\,d\tilde{v}(y^{1/2})$, hence

$$2\tilde{c}y\,d\tilde{v}(y^{1/2})=d\mu(y) . \tag{15}$$

This implies that $d\mu(y)$ has finite moment of order -1. Set

$$d\mu_0(y)=c^{-1}y^{-1}\,d\mu(y) .$$

Then $d\mu_1(y)=d\mu(y)$. From now on we follow the notation introduced in the beginning of this section, except for the subscript 1, which we tend to drop when denoting objects associated to $d\mu(y)$. We thus have

$$y^{-1}d\mu(y)=y^{-1}d\mu_1(y)=2c\,dv_0(y^{1/2}) . \tag{16}$$

Therefore by (15) and (16) the measures $dv_0(x)$ and $d\tilde{v}(x)$ can differ by a point mass at 0. However by (16) $dv_0(x)$ cannot carry an atom at the origin. In conclusion we get

$$d\tilde{v}=av_0+(1-a)\delta_0 , \tag{17}$$

for a number a, $0\leqq a<1$.

Now observe that by (5) and (13) we obtain

$$x=xP_{0,0}=\alpha_{0,0}P_{0,1}$$
$$x=x\tilde{P}_0=\tilde{\alpha}_0\tilde{P}_1 .$$

Since $\tilde{P}_n$ and $P_{0,n}$ are orthonormal relative to the measures $\tilde{v}$ and v_0 respectively, we have

$$\begin{aligned}\tilde{\alpha}_0^2&=\tilde{\alpha}_0^2\int_{-\infty}^{+\infty}\tilde{P}_1(x)^2\,d\tilde{v}(x)\\ &=\alpha_{0,0}^2\int_{-\infty}^{+\infty}P_{0,1}(x)^2\,d\{av_0+(1-a)\delta_0\}(x)\\ &=a\alpha_{0,0}^2\int_{-\infty}^{+\infty}P_{0,1}(x)^2\,dv_0(x)=a\alpha_{0,0}^2 .\end{aligned}$$

This implies

$$g_0=\frac{\tilde{\alpha}_0^2}{\beta_0}\leqq\frac{\alpha_{0,0}^2}{\beta_0} .$$

Both g_n and $\beta_n^{-1}\alpha_{0,n}^2$ are parameter sequences for $(\beta_n\beta_{n+1})^{-1}\lambda_n^2$, the first one by assumption, the other one by the considerations preceding the theorem (attention: we have set $\beta_{1,n}=\beta_n$). In view of [2, Theorems 5.2, 5.3, pp. 93–94] this yields that $\beta_n^{-1}\alpha_{0,n}^2$ is the maximal parameter sequence. □

Remark. The first part of the Theorem can be also obtained by means of [1, Theorem 1]. Is is also worthwhile comparing the second part of the theorem with [1, Theorem 2].

3 Perturbations of orthogonal polynomials

Let $d\mu(y)$ be a probability measure with infinite support contained in $[0, +\infty)$. The corresponding orthonormal polynomials Q_n satisfy the recurrence formula

$$yQ_n = \lambda_n Q_{n+1} + \beta_n Q_n + \lambda_{n-1} Q_{n-1} .$$

By Proposition 1 we know that $\beta_n > 0$, and $(\beta_n \beta_{n+1})\lambda_n^2$ is a chain sequence. For the purpose of this paper we introduce the following notion.

Definition 1 *Let $A \geqq 0$. We say that $d\mu(y)$ belongs to the class $\mathscr{C}(A)$, if* $\operatorname{supp}\mu \subset [0, +\infty)$ *and*

$$\frac{\lambda_n}{\lambda_{n+1}} \xrightarrow{n} 1 , \quad \frac{\lambda_n}{\beta_n} \xrightarrow{n} \sqrt{A} .$$

Observe that

$$d\mu(y) \in \mathscr{C}(A) \Rightarrow \left(\frac{\beta_n}{\beta_{n+1}} \xrightarrow{n} 1, \frac{\lambda_n^2}{\beta_n \beta_{n+1}} \xrightarrow{n} A \right) . \tag{18}$$

Thus by [2, Theorem 6.4] (see also Appendix) the number A can take values between 0 and $\frac{1}{4}$.

The class $\mathscr{C}(A)$ is invariant for two types of perturbations. For $0 < a < 1$, let $\mu_a = a\mu + (1-a)\delta_0$.

Theorem 2 (i) $d\mu(y) \in \mathscr{C}(A)$ iff $y\,d\mu(y) \in \mathscr{C}(A)$.
(ii) $d\mu(y) \in \mathscr{C}(A)$ iff $d\mu_a(y) \in \mathscr{C}(A)$.

Proof. (i) Adopting the notation of Sect. 2, we have to show that $d\mu_0(y) \in \mathscr{C}(A)$ iff $d\mu_1(y) \in \mathscr{C}(A)$.

Assume that $d\mu_1(y) \in \mathscr{C}(A)$. By Theorem 1 the sequence $\beta_{1,n}^{-1}\alpha_{0,2n}^2$ is the maximal parameter sequence for $(\beta_{1,n}\beta_{1,n+1})^{-1}\lambda_{1,n}^2$. As by (18) the latter converges to A, so by [2, Theorems 6.3, 6.4, p. 102] (see Appendix) we have

$$g_n = \frac{\alpha_{0,2n}^2}{\beta_{1,n}} \xrightarrow{n} \frac{1+\sqrt{1-4A}}{2} . \tag{19}$$

By Proposition 2 (9) and (19) we have

$$\begin{aligned} \frac{\alpha_{0,2n+1}^2}{\alpha_{0,2n}^2} &= \frac{\beta_{1,n} - \alpha_{0,2n}^2}{\alpha_{0,2n}^2} \\ &= \frac{1-g_n}{g_n} \xrightarrow{n} \frac{1-\sqrt{1-4A}}{1+\sqrt{1-4A}} = B . \end{aligned} \tag{20}$$

By (18) and (19) we also have

$$\frac{\alpha_{0,2n+2}^2}{\alpha_{0,2n}^2}=\frac{\beta_{1,n+1}}{\beta_{1,n}}\frac{g_{n+1}}{g_n}\xrightarrow{n}1\,. \tag{21}$$

Now (20) and (21) imply

$$\frac{\alpha_{0,2n}^2}{\alpha_{0,2n-1}^2}\xrightarrow{n}\frac{1+\sqrt{1-4A}}{1-\sqrt{1-4A}}\,. \tag{22}$$

Combining (20), (21) and (22) gives

$$\frac{\alpha_{0,n+2}}{\alpha_{0,n}}\xrightarrow{n}1\,. \tag{23}$$

Therefore by (8)

$$\frac{\lambda_{0,n+1}}{\lambda_{0,n}}=\frac{\alpha_{0,2n+2}\alpha_{0,2n+3}}{\alpha_{0,2n}\alpha_{0,2n+1}}\xrightarrow{n}1\,.$$

Next, (8), (20), (22) and (23) give

$$\begin{aligned}\frac{\beta_{0,n}}{\lambda_{0,n}}&=\frac{\alpha_{0,2n-1}^2+\alpha_{0,2n}^2}{\alpha_{0,2n}\alpha_{0,2n+1}}\\&=\frac{\alpha_{0,2n-1}}{\alpha_{0,2n+1}}\frac{\alpha_{0,2n-1}}{\alpha_{0,2n}}+\frac{\alpha_{0,2n}}{\alpha_{0,2n+1}}\xrightarrow{n}\sqrt{B}+\frac{1}{\sqrt{B}}=\frac{1}{\sqrt{A}}\,.\end{aligned}$$

Hence $d\mu_0\in\mathscr{C}(A)$. The proof of the opposite implication of (i) is similar and is left to the reader.

(ii) Observe that if we set $\mu_0=\mu_a$, the measure μ_1 does not depend on the choice of a. Hence by the first part of the theorem, $\mu\in\mathscr{C}(A)$ if and only if $\mu_1\in\mathscr{C}(A)$, and this holds if and only if $\mu_a\in\mathscr{C}(A)$. □

Theorem 3 *Let Q_n and $Q_n^{(a)}$ be the polynomials orthonormal relative to the measures μ and $\mu_a=a\mu+(1-a)\delta_0$, respectively. Assume that the support* supp μ *is bounded. Let*

$$\begin{aligned}yQ_n&=\lambda_nQ_{n+1}+\beta_nQ_n+\lambda_{n-1}Q_{n-1}\quad.\\ yQ_n^{(a)}&=\lambda_n^{(a)}Q_{n+1}^{(a)}+\beta_n^{(a)}Q_n^{(a)}+\lambda_{n-1}^{(a)}Q_{n-1}^{(a)}\,.\end{aligned}$$

If $d\mu(y)\in\mathscr{C}(A)$, then

$$\lambda_n^{(a)}-\lambda_n\xrightarrow{n}0 \quad \textit{and} \quad \beta_n^{(a)}-\beta_n\xrightarrow{n}0\,.$$

Proof. Consider pairs of measures of the form (μ_0,μ_1), where $\mu_0=\mu$, or $\mu_0=\mu^{(a)}$. As we have already seen, in both cases the measures μ_1 coincide. However the coefficients $\alpha_{0,n}$ and $\beta_{0,n}$ are not equal, and so we must denote them differently according to the case. Let $\alpha_{0,n}$, $\beta_{0,n}$ and $\alpha_{0,n}^{(a)},\beta_{0,n}^{(a)}$ denote the sequences corresponding to $\mu_0=\mu$ and $\mu_0=\mu^{(a)}$, respectively. The corresponding sequences with subscripts 1 are identical.

By Proposition 2 and remarks following it, the sequences

$$h_n = \frac{\alpha_{0,2n}^2}{\beta_{1,n}} \quad h_n^{(a)} = \frac{(\alpha_{0,2n}^{(a)})^2}{\beta_{1,n}} \tag{24}$$

are the parameter sequences for $(\beta_{1,n}\beta_{1,n+1})^{-1}\lambda_{1,n}^2$. By Theorem 2 we have $d\mu_1 \in \mathscr{C}(A)$. Hence by already quoted Chihara's result [2, p. 102] the sequences h_n and $h_n^{(a)}$ are convergent and

$$h_n \xrightarrow{n} \frac{1 \pm \sqrt{1-4A}}{2} \quad h_n^{(a)} \xrightarrow{n} \frac{1 \pm \sqrt{1-4A}}{2} . \tag{25}$$

Actually, since by Theorem 1 $h_n^{(a)}$ is not a maximal parameter sequence, then by [2, Theorems 6.3, 6.4] it tends to $\frac{1}{2}(1-\sqrt{1-4A})$. Moreover we can assume without loss of generality, that $\mu(\{0\})=0$. Hence by Theorem 1, h_n is the maximal parameter sequence and as such must tend to $\frac{1}{2}(1+\sqrt{1-4A})$. In any case (25) is sufficient for our purposes.

By Proposition 2 and (24) we have

$$\begin{aligned}
(\lambda_n)^2 - (\lambda_n^{(a)})^2 &= (\lambda_{0,n})^2 - (\lambda_{0,n}^{(a)})^2 \\
&= \beta_{1,n}\beta_{1,n+1}\{h_n(1-h_n) - h_n^{(a)}(1-h_n^{(a)})\} \\
\beta_n - \beta_n^{(a)} &= \beta_{0,n} - \beta_{0,n}^{(a)} \\
&= \alpha_{0,2n}^2 - \alpha_{0,2n-2}^2 - (\alpha_{0,2n}^{(a)})^2 + (\alpha_{0,2n-2}^{(a)})^2 \\
&= \beta_{1,n}h_n - \beta_{1,n-1}h_{n-1} - \beta_{1,n}h_n^{(a)} + \beta_{1,n-1}h_{n-1}^{(a)} .
\end{aligned}$$

Now the conclusion follows as sequences h_n and $h_n^{(a)}$ are convergent, $\beta_{1,n}$ is bounded and $\beta_{1,n}/\beta_{1,n+1}$ is convergent to 1 (in view of $\mu_1 \in \mathscr{C}(A)$), and finally because

$$h_n(1-h_n) \xrightarrow{n} A \quad h_n^{(a)}(1-h_n^{(a)}) \xrightarrow{n} A .$$

□

Remark. It is worth observing that the polynomials Q_n and $Q_n^{(a)}$ are cases of the family denoted by P_n^h in [1, Theorem 2].

Let Q_n denote the polynomials orthonormal relative to the measure μ, and

$$yQ_n = \lambda_n Q_{n+1} + \beta_n Q_n + \lambda_{n-1} Q_{n-1} .$$

Following [3] we say that a measure μ belongs to $M(a, b)$, if

$$\lambda_n \xrightarrow{n} 2a, \quad \beta_n \xrightarrow{n} b .$$

Corollary 1 (Nevai) *Let $\mu \in M(a, b)$, and $(y-A)$ be positive on* $\operatorname{supp}\mu$. *Then $(y-A)\,d\mu(y) \in M(a, b)$.*

Proof. Translating the measure if necessary, we may assume that $A=0$. Then $\operatorname{supp}\mu \subset [0, +\infty)$. In that case we have $b>0$. Hence the measure μ_0 belongs to $\mathscr{C}(a^2/4b^2)$. By Theorem 2 also the measure $d\mu_1(y) = cyd\mu(y)$ is in $\mathscr{C}(a^2/4b^2)$. To complete the proof we need to show that either $\lambda_{1,n}$ or $\beta_{1,n}$ is convergent.

This can be read off from the proof of Theorem 2. Indeed, by (8) and (9) we have

$$\lambda_{1,n}=\alpha_{0,2n+1}\alpha_{0,2n+2}=\lambda_{0,n}\frac{\alpha_{0,2n+2}}{\alpha_{0,2n}}.$$

Thus by (23)

$$\lim_{n\to\infty}\lambda_{1,n}=\lim_{n\to\infty}\lambda_{0,n}.$$

□

Let L and $L^{(a)}$ denote difference operators associated with μ and $\mu^{(a)}$, i.e.

$$La_n=\lambda_n a_{n+1}+\beta_n a_n+\lambda_{n-1}a_{n-1}$$

$$L^{(a)}a_n=\lambda_n^{(a)}a_{n+1}+\beta_n^{(a)}a_n+\lambda_{n-1}^{(a)}a_{n-1}.$$

Theorem 3 states that if $\mu\in\mathscr{C}(A)$ then $L^{(a)}$ is a compact perturbation of the operator L. It is not so in general, as the following example shows.

Example 1 Roughly the construction consists in finding a chain sequence with two parameter sequences that are not approximately equal at infinity. Let

$$\alpha_n=\begin{cases}2^{-1/2} & \text{for } n=4k,\ 4k+1\\ 3^{-1/2} & \text{for } n=4k+2\\ 2^{1/2}3^{-1/2} & \text{for } n=4k+3\end{cases}$$

$$\tilde{\alpha}_n=\begin{cases}2^{1/2}3^{-1/2} & \text{for } n=4k\\ 3^{-1/2} & \text{for } n=4k+1\\ 2^{-1/2} & \text{for } n=4k+2,\ 4k+3.\end{cases}$$

Observe that

$$\alpha_{2n+1}\alpha_{2n+2}=\tilde{\alpha}_{2n+1}\tilde{\alpha}_{2n+2},\tag{26}$$

$$\alpha_{2n}^2+\alpha_{2n+1}^2=\tilde{\alpha}_{2n}^2+\tilde{\alpha}_{2n+1}^2.\tag{27}$$

On the other hand

$$\alpha_{2n}\alpha_{2n+1}-\tilde{\alpha}_{2n}\tilde{\alpha}_{2n+1}=\pm\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\right).\tag{28}$$

Consider orthogonal polynomials defined by the recurrence relations

$$xP_n=\alpha_n P_{n+1}+\alpha_{n-1}P_{n-1},\quad P_0=1,$$

$$x\tilde{P}_n=\tilde{\alpha}_n\tilde{P}_{n+1}+\tilde{\alpha}_{n-1}\tilde{P}_{n-1},\quad \tilde{P}_0=1.$$

The polynomials P_n and $\tilde{P}_n$ are orthonormal relative to symmetric measures $d\nu(x)$ and $d\tilde{\nu}(x)$, respectively. Let

$$d\mu(y)=2d\nu(y^{1/2})\quad d\tilde{\mu}(y)=2d\tilde{\nu}(y^{1/2}).$$

The polynomials $y^{-1/2}P_{2n+1}(y^{1/2})$ and $y^{-1/2}\tilde{P}_{2n+1}(y^{1/2})$ are orthogonal relative to $y\,d\mu(y)$ and $y\,d\tilde{\mu}(y)$, respectively. By (26) and (27) they satisfy the same recurrence formula. Thus

$$y\,d\mu(y)=y\,d\tilde{\mu}(y)\,.$$

Hence the measures $d\mu(y)$ and $d\tilde{\mu}(y)$ can differ by a point mass at 0. The corresponding orthogonal polynomials are $Q_n(y)=P_{2n}(y^{1/2})$ and $\tilde{Q}_n(y)=\tilde{P}_{2n}(y^{1/2})$ which in view of (26), (27) satisfy the recurrence relations

$$yQ_n=\lambda_n Q_{n+1}+\beta_n Q_n+\lambda_{n-1}Q_{n-1}\,,$$

$$y\tilde{Q}_n=\tilde{\lambda}_n Q_{n+1}+\tilde{\beta}_n \tilde{Q}_n+\tilde{\lambda}_{n-1}\tilde{Q}_{n-1}\,,$$

where $\lambda_n=\alpha_{2n}\alpha_{2n+1}$, and $\tilde{\lambda}_n=\tilde{\alpha}_{2n}\tilde{\alpha}_{2n+1}$. By (28) the difference $\tilde{\lambda}_n-\lambda_n$ stays away from 0, despite the fact that the corresponding measures are equal modulo point mass at 0.

4 Growth of orthogonal polynomials

Let Q_n denote the polynomials orthonormal relative to the measure μ, and

$$yQ_n=\lambda_n Q_{n+1}+\beta_n Q_n+\lambda_{n-1}Q_{n-1}\,.$$

Assume $\mu\in M(a,b)$. It is well-known that since the formula is a compact transformation of the constant-coefficients recurrence formula with $\lambda_n=a/2$, and $\beta_n=b$, the support of the measure μ consists of the interval $[b-a,b+a]$ and the denumerable set, whose accumulation points belong to this interval. In [3, Theorem 3.9, p. 11] it was shown that if a measure μ is in $M(a,b)$, then

$$\lim_{n\to\infty}\,[b^2-(y-a)^2]\frac{Q_n(y)^2}{Q_0(y)^2+Q_1(y)^2+\cdots+Q_n(y)^2}=0\,, \tag{29}$$

uniformly for $y\in[b-a,b+a]$. In particular the polynomials $Q_n(y)$ have almost uniform subexponential growth in $(a-b,a+b)$, i.e.

$$\limsup\sqrt[n]{|Q_n(y)|}\leqq 1\,,$$

uniformly for $y\in[c,d]\subset(b-a,b+a)$. It took twelve years of efforts to drop the word "almost" from the estimate. Namely in [4, Theorem 2] it is shown that (29) holds without the factor $b^2-(y-a)^2$, and also the exponent 2 can be replaced by any positive p. In particular the polynomials have uniform subexponential growth in the entire closed interval $[b-a,b+a]$.

In this section we are going to show that there are cases when subexponential growth at the end points of the interval can be derived from Nevai's result (29) itself. By using affine transformations we can reduce considerations only to the class $M(\frac{1}{2},\frac{1}{2})$, We start with the simplest case.

Proposition 3 (Nevai, Totik, Zhang) *Assume that* $\mu \in M(\frac{1}{2}, \frac{1}{2})$, *and* $\operatorname{supp}\mu = [0, 1]$. *Then*

$$\limsup \sqrt[n]{|Q_n(y)|} \leqq 1 ,$$

uniformly in $[0, 1]$.

Proof. Let $dv(x)$ be the symmetric measure defined by $dv(x) = \frac{1}{2} d\mu(x^2)$, for $x > 0$. As we have seen in Sect. 2 the polynomials $P_n(x)$ that are orthonormal relative to $dv(x)$, satisfy

$$xP_n = \alpha_n P_{n+1} + \alpha_{n-1} P_{n-1} ,$$

and $P_{2n}(x) = Q_n(x^2)$. Also by (8) the sequence $g_n = \alpha_{2n-1}^2/\beta_n$ is a parameter sequence for $a_n = \lambda_{n-1}^2/(\beta_{n-1}\beta_n)$. By assumption we have $a_n \to 1/4$. Hence by [2, Theorem 6.4] (see also Appendix) we have

$$\alpha_{2n-1}^2 = g_n \beta_n \to \frac{1}{4} .$$

By (8) we also have

$$\alpha_{2n} = \frac{\lambda_n}{\alpha_{2n+1}} \to \frac{1}{2} .$$

This implies that $v \in M(0, 1)$. By Nevai's theorem (see (29)) we conclude that

$$\limsup \sqrt[n]{|P_n(x)|} \leqq 1 ,$$

uniformly for $x \in [-1+\varepsilon, 1-\varepsilon]$, for any $\varepsilon > 0$. In particular

$$\limsup \sqrt[n]{|Q_n(y)|} = \limsup \sqrt[n]{|P_{2n}(y^{1/2})|} \leqq 1 ,$$

uniformly for any $y \in [0, 1-\varepsilon]$. By considering the reflected measure $d\check{\mu}(y) = d\mu(1-y)$, we can deduce that also

$$\limsup \sqrt[n]{|Q_n(y)|} \leqq 1 ,$$

uniformly for any $y \in [\varepsilon, 1]$. This completes the proof. □

This method works also for measures that admit finitely many points off the interval $[0, 1]$.

Proposition 4 (Nevai, Totik, Zhang) *Assume that* $\mu \in M(\frac{1}{2}, \frac{1}{2})$, *and* $\operatorname{supp}\mu = [0, 1] \cup F$, *where* F *is a finite set of points disjoint from* $[0, 1]$. *Then*

$$\limsup \sqrt[n]{|Q_n(y)|} \leqq 1 ,$$

uniformly in $[0, 1]$.

Proof. We will show the conclusion for $y \in [0, 1-\varepsilon]$, and then extend it to the entire interval by considering, as at the end of the preceding proof, the measure reflected about the point $\frac{1}{2}$.

Let y_0 be the least negative point in F. Let $d\mu_0(y)=d\mu(y-y_0)$. Then $\mu_0 \in M(a, b-y_0)$. By Corollary 1 we have $\mu_1 \in M(a, b-y_0)$, where $d\mu_1(y)=cy\,d\mu_0(y)$. By (5) and Proposition 2 (6), (7) the polynomials $Q_{0,n}$ and $Q_{1,n}$ are related by

$$Q_{0,n}=\alpha_{0,2n}Q_{1,n}+\alpha_{0,2n-1}Q_{1,n-1} \ .$$

The sequence $\alpha_{0,n}$ is bounded as the corresponding measure $d\nu_0$ has bounded support. If polynomials $Q_{1,n}$ satisfy

$$\limsup \sqrt[n]{|Q_{1,n}(y)|} \leqq 1 \ ,$$

uniformly on $[0, 1-\varepsilon]$, then also

$$\limsup \sqrt[n]{|Q_{0,n}(y)|} \leqq 1 \ .$$

Thus it suffices to consider the measure $d\mu_1(y+y_0)$, whose support contains fewer points off the interval $[0, 1]$. Repeating this argument we can reduce the problem to the case of measure in $M(\frac{1}{2}, \frac{1}{2})$, when we don't have any mass point to the left from 0. Then we can apply Proposition 3. □

It would be interesting to know if there is a similar way of handling the case of infinitely many points off the interval.

Appendix

For the sake of selfcontainment we will give an alternative proof of the fact that if a chain sequence is convergent then any parameter sequence is convergent as well. The result is due to Chihara [1], and can be found in [2], too. The proof is based on a theory of chain sequences developed by Wall [5]. We present a straightforward proof for the sake of completeness.

Proposition 5 *Let a sequence a_n be of the form $a_n=(1-g_n)g_{n+1}$, where $0 \leqq g_n \leqq 1$. If a_n is convergent then also g_n is convergent. Moreover, if $\lim_n a_n = a$, then $0 \leqq a \leqq 1/4$, and*

$$\lim_n a_n = \frac{1 \pm \sqrt{1-4a}}{2} \ .$$

Proof. First we show that $a \leqq 1/4$. For a contradiction assume $a > 1/4$. Then $g_{n+1}(1-g_n) \geqq 1/4$ for n large. Thus

$$g_{n+1} \geqq \frac{1}{4(1-g_n)} \geqq g_n \ .$$

This means g_n is increasing, hence it converges to g, for $0 \leqq g \leqq 1$. Consequently

$$a=(1-g)g \leqq 1/4 \ .$$

Let

$$s_n=\frac{1}{2}(1-\sqrt{(1-4a_n)_+}) \quad S_n=\frac{1}{2}(1+\sqrt{(1-4a_n)_+}) \ .$$

Observe that

$$g_n \geqq g_{n+1} \Leftrightarrow a_n \leqq 1/4 \text{ and } s_n \leqq g_n \leqq S_n . \tag{30}$$

We have $s_n \to s$, and $S_n \to S$, where

$$s = \frac{1}{2}(1 - \sqrt{1-4a}), \quad S = \frac{1}{2}(1 + \sqrt{1-4a}) .$$

Let $\delta > 0$. Take N large so that $|S_N - S| \leqq \delta$, for $n > N$. Assume that $g_N > S + \delta$. Thus $g_N \geqq S_N$. In view of (30) this implies

$$g_{N+1} \geqq g_N > S + \delta .$$

Now by induction one can show that g_n is increasing beginning from N. Thus g_n converges to a number strictly greater than S. This gives a contradiction. Therefore we can conclude that

$$\limsup g_n \leqq S . \tag{31}$$

The case $a = \frac{1}{4}$ requires a special approach. If $a = \frac{1}{4}$, then $s = S = \frac{1}{2}$. Let $\varepsilon_n = \frac{1}{2}\sqrt{(1-4a_n)_+}$. Then by (30) we obtain

$$g_{n+1} \geqq g_n \Leftrightarrow \left| g_n - \frac{1}{2} \right| \leqq \varepsilon_n .$$

As $\varepsilon_n \to 0$, we get either $\liminf g_n \geqq \frac{1}{2}$ or g_n is decreasing beginning from some N. In view of (31) this completes the proof in this case.

Let's turn to the case $a < \frac{1}{4}$. Then $s < \frac{1}{2} < S$.

(1) Assume that $d = \liminf g_n \in (s, S)$. Let $0 < \delta < S - d$. Then there is N, such that

$$g_n \geqq s_n, \quad |S_n - S| \leqq \delta \quad \text{for } n \geqq N .$$

Since $d < S - \delta$, there exists $n \geqq N$ with

$$s_n \leqq g_n \leqq S - \delta \leqq S_n .$$

By virtue of (30) we obtain

$$s_{n+1} \leqq g_{n+1} \leqq g_n \leqq S - \delta \leqq S_{n+1} .$$

Now by induction we can show that $\limsup g_n \leqq S - \delta$. Letting δ tend to $S - d$ we end up with

$$\limsup g_n \leqq d = \liminf g_n .$$

(2) Assume that $d = \liminf g_n \leqq s$. Fix $\delta > 0$ such that $d + \delta \leqq \frac{1}{2}$. There exists N, such that

$$a_n \leqq (d+\delta)(1-d-\delta) \quad \text{for } n \geqq N . \tag{32}$$

By assumption $g_n \leqq d + \delta$, for some $n \geqq N$. Then by (32) we have

$$g_{n+1} = \frac{a_n}{1-g_n} \leqq \frac{a_n}{1-d-\delta} \leqq d + \delta .$$

By induction we conclude that $g_m \leqq d+\delta$, for any $m>n$. Consequently $\limsup g_n \leqq d+\delta$. Since δ is arbitrary we get $\liminf g_n = \limsup g_n$.

(3) $\liminf g_n \geqq S$. By (31) this implies

$$\liminf g_n = \limsup g_n = S\ . \quad \square$$

References

1. Chihara, T.: Chain sequences and orthogonal polynomials. Trans. Amer. Math. Soc. **104**, 1–16 (1962)
2. Chihara, T.: "An introduction to orthogonal polynomials," Vol. 13, Mathematics and Its Applications. Gordon and Breach, New York, London, Paris, 1978
3. Nevai, P.: Orthogonal polynomials, Mem. Amer. Math. Soc. **213** (1979)
4. Nevai, P., Totik, V., Zhang, J.: Orthogonal polynomials: their growth relative to their sums. J. Approx. Theory **67**, 215–234 (1991)
5. Wall, H.S.: "Analytic theory of continued fractions." D. van Nostrand Co., New York, 1948