CONVERGENCE OF WEIGHTED AVERAGES OF RELAXED PROJECTIONS

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Abstract. The convergence of the algorithm for solving convex feasibility problem is studied by the method of sequential averaged and relaxed projections. Some results of H. H. Bauschke and J. M. Borwein are generalized by introducing new methods. Examples illustrating these generalizations are given.

1. Introduction and preliminaries

Many problems in applied mathematics deal with finding a point in the intersection of a family of convex sets in Euclidean or Hilbert space. The solution can be achieved in algorithmic way as a limit of composition of projection onto these convex sets. Due to its importance the problem has been studied heavily for many years. We refer to [1] where the reader can find an extensive account of theorems and literature related to the problem, as well as a general approach which captures the earlier methods and results. Other related results can be found in [2], [3], [4], [5], [6] and [7]. In the present work we will generalize, in case of finite dimension, one of the main theorems of [1], concerning the convergence of the algorithm. Firstly we will improve the estimate for the average of relaxed projections, secondly we will admit repetitive control, and lastly we will make use of perturbation theorem, which allows to ignore projections with small weight coefficients. The first generalization is valid also for infinite dimensional inner product spaces. All these were possible by applying new techniques and new proofs as well.

For a closed convex set $C \subset \mathbb{R}^d$ let P_C denote the projection onto C. For $x \in \mathbb{R}^d$ the symbol $d(x, C)$ will denote the distance from x to C. Assume we are dealing with a fixed finite family of closed convex sets $C_1,\ C_2, \ldots, C_N$. For a sequence of relaxation parameters $\alpha_1, \ldots, \alpha_N$ such that $0 \le \alpha_i \le 2$ and numbers $\lambda_1, \ldots, \lambda_N$ such that $0 \le \lambda_i \le 1$,

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$\sum_{i=1}^N \lambda_i = 1$ we will consider the weighted averages

$$\sum_{j=1}^N \lambda_j\{(1-\alpha_j)I + \alpha_j P_{C_j}\} = I - \sum_{j=1}^N \lambda_j\alpha_j(I - P_{C_j}).$$

It is well known that every operator P_{C_j} is firmly nonexpansive (see [1, Facts 1.5]), thus these weighted averages are nonexpansive as well. Since the expressions depend only on the products $\lambda_i\alpha_i$ we will introduce the set

$$\mathcal{B} = \left\{ \beta = (\beta_1, \ldots, \beta_N) \;\middle|\; \beta_i \geq 0,\ \sum_{i=1}^N \beta_i \leq 2 \right\}$$

and define for $\beta \in \mathcal{B}$

$$Q_\beta = I - \sum_{j=1}^N \beta_j(I - P_{C_j}). \tag{1}$$

Remark. Observe that for any $\beta \in \mathcal{B}$ there exist relaxation parameters $\alpha_1, \ldots, \alpha_N$ and average parameters $\lambda_1, \ldots, \lambda_N$ such that $\beta_i = \lambda_i\alpha_i$. Indeed, if $\sum_{k=1}^N \beta_k > 0$ we may set $\alpha_i = \sum_{k=1}^N \beta_k$ and $\lambda_i = \beta_i\left(\sum_{k=1}^N \beta_k\right)^{-1}$. On the other hand if $\beta_i = 0$ for any i we take $\alpha_i = 0$ and $\lambda_i = 1/N$ for any i. Therefore every operator Q_β is nonexpansive.
Remark. All the results in this work remain valid if we replace P_{C_i} with a firmly nonexpansive mappings T_i such that $T_i(c) = c$ for $c \in C_i$ (see [1, p. 370]).

2. Auxiliary results

Proposition 1. *For any $x \in \mathbb{R}^d$ and any $c \in C_1 \cap C_2 \cap \ldots \cap C_N$ we have*

$$\|Q_\beta(x) - c\|^2 \leq \|x - c\|^2 - \sum_{j=1}^N (2 - \frac{\beta_j}{\kappa_j})\beta_j\|x - P_{C_j}(x)\|^2,$$

where $\kappa_1, \kappa_2, \ldots, \kappa_N$ are any nonnegative numbers such that $\sum_{j=1}^N \kappa_j = 1$.

Remark. We set $\beta_j/\kappa_j = 0$ whenever $\beta_j = 0$. If $\beta_j > 0$ and $\kappa_j = 0$ we set $\beta_j/\kappa_j = +\infty$.

Proof. With no loss of generality we may assume that $c = 0$. Then by the convexity of each set C_j and the fact that P_{C_j} is firmly nonexpansive we get

$$\langle P_{C_j}(x), P_{C_j}(x)\rangle \leq \langle x, P_{C_j}(x)\rangle,$$

which implies

$$\langle P_{C_j}(x), x - P_{C_j}(x)\rangle \geq 0.$$

Hence

$$\begin{aligned}
(2) \quad \|Q_\beta x\|^2 &= \left\| x - \sum_{j=1}^{N} \beta_j (x - P_{C_j}(x)) \right\|^2 \\
&= \|x\|^2 + \left\| \sum_{j=1}^{N} \beta_j (x - P_{C_j}(x)) \right\|^2 - 2\sum_{j=1}^{N} \beta_j \langle x, x - P_{C_j}(x)\rangle \\
&= \|x\|^2 + \left\| \sum_{j=1}^{N} \beta_j (x - P_{C_j}(x)) \right\|^2 - 2\sum_{j=1}^{N} \beta_j \, \|x - P_{C_j}(x)\|^2 \\
&\qquad - 2\sum_{j=1}^{N} \beta_j \, \langle P_{C_j}(x), x - P_{C_j}(x)\rangle \\
&\leq \|x\|^2 + \left\| \sum_{j=1}^{N} \beta_j (x - P_{C_j}(x)) \right\|^2 - 2\sum_{j=1}^{N} \beta_j \, \|x - P_{C_j}(x)\|^2.
\end{aligned}$$

Let $\kappa_1, \kappa_2, \ldots, \kappa_N$ satisfy the assumptions. Then by the convexity of the function $x \mapsto \|x\|^2$ we obtain

$$\begin{aligned}
(3) \quad \left\| \sum_{j=1}^{N} \beta_j (x - P_{C_j}(x)) \right\|^2 &= \left\| \sum_{j=1}^{N} \kappa_j \kappa_j^{-1} \beta_j (x - P_{C_j}(x)) \right\|^2 \\
&\leq \sum_{j=1}^{N} \frac{\beta_j^2}{\kappa_j} \|x - P_{C_j}(x)\|^2.
\end{aligned}$$

Combining (1) and (2) concludes the proof. □

Remark. Setting

$$\kappa_j = \frac{\beta_j}{\sum_{k=1}^{N} \beta_j}$$

implies

$$(4) \quad \|Q_\beta(x) - c\|^2 \leq \|x - c\|^2 - \left(2 - \textstyle\sum_{k=1}^{N} \beta_k\right) \sum_{j=1}^{N} \beta_j \|x - P_{C_j}(x)\|^2$$

the inequality obtained in [1, Lemma 3.2(ii)].

Theorem 1. *Given a family of convex sets $C_1, \ldots, C_N$ with nonempty intersection C. Let $\beta \in \mathcal{B}$ and $I \subset \{1, 2, \ldots, N\}$. Then for any $x \in \mathbb{R}^d$ and any $c \in C$ we have*

$$\|Q_\beta(x) - c\|^2 \le \|x - c\|^2 - \min_{i \in I} \nu_i \, \max_{i \in I} d^2(x, C_i),$$

where

$$\nu_i = \frac{2\beta_i \left(2 - \sum_{k=1}^N \beta_k\right)}{\beta_i + 2 - \sum_{k=1}^N \beta_k}.$$

In particular the inequality holds when I is the set of active indices, i.e.

$$I = \{i \mid 1 \le i \le N, \ \beta_i > 0\} \tag{5}$$

Proof. Fix $i \in I$ and set

$$\kappa_j = \begin{cases} \frac{1}{2}\beta_j & j \ne i, \\ 1 - \frac{1}{2}\sum_{j \ne i} \beta_j & j = i. \end{cases}$$

On substituting theses values into the inequality of Proposition 1 we obtain

$$\|Q_\beta(x) - c\|^2 \le \|x - c\|^2 - \nu_i \, d^2(x, C_i) \le \|x - c\|^2 - \min_{k \in I} \nu_k \, d^2(x, C_i).$$

Now maximizing with respect to $i \in I$ gives the conclusion. □

3. Main result

Theorem 2. *Fix a family of convex sets $C_1, \ldots, C_N$ with nonempty intersection C. Given a sequence $\beta^{(n)} \in \mathcal{B}$. Let $I^{(n)}$ denote the set of active indices for $\beta^{(n)}$. Assume that every index $i \in \{1, 2, \ldots, N\}$ occurs in $I^{(n)}$ for infinitely many n. Let n_k be positive integers such that $n_{k-1} < n_k$ and*

$$\{1, 2, \ldots, N\} \subset I^{(n_{k-1})} \cup I^{(n_{k-1}+1)} \cup \ldots \cup I^{(n_k - 1)},$$

i.e. every index occurs at least once for n such that $n_{k-1} \le n < n_k$. For

$$\nu_i^{(n)} = \frac{2\beta_i^{(n)} \left(2 - \sum_{k=1}^N \beta_k^{(n)}\right)}{\beta_i^{(n)} + 2 - \sum_{k=1}^N \beta_k^{(n)}}.$$

let

$$\nu^{(k)} = \min\{\nu_i^{(n)} \mid n_{k-1} < n \le n_k, i \in I^{(n)}\}.$$

Assume that

$$\sum_{k=1}^{\infty} \nu^{(k)} = +\infty. \tag{6}$$

Then for any $x^{(0)} \in \mathbb{R}^d$ *the sequence* $x^{(n)}$ *defined as*

$$x^{(n)} = Q_{\beta^{(n)}}(x^{(n-1)}), \qquad n \geq 1$$

is convergent to a point in C.

Remark. This result generalizes [1, Thm 3.20(ii)] (see also [1, Cor. 3.25]) in two essential aspects. First of all it allows repetitve control while [1, Thm 3.20] could afford only intermittent control, i.e. when the sequence n_k is of the form $n_k = kp$. Secondly the coefficients $\nu_i^{(n)}$ are smaller than $\mu_i^{(n)}$, introduces in [1] as

$$\mu_i^{(n)} = 2\beta_i^{(n)} \left(2 - \sum_{k=1}^{N} \beta_k^{(n)} \right).$$

For example let $N = 2$ and

$$\beta_1^{(n)} = \frac{1}{n}, \; \beta_2^{(n)} = 2 - \frac{2}{n}.$$

The algorithm is then 1-intermittent, hence we can take $n_k = k$ and $I_k = \{1, 2\}$. Thus

$$\sum_{k=1}^{\infty} \min\{\mu_1^{(k)}, \mu_2^{(k)}\} < +\infty, \quad \sum_{k=1}^{\infty} \min\{\nu_1^{(k)}, \nu_2^{(k)}\} = +\infty.$$

Therefore Theorem 3.20 of [1] does not apply while our Theorem 2 does.

Proof. With no loss of generality we may assume that $0 \in C$. For $u^{(0)} \in \mathbb{R}^d$ and $n \geq 1$ let $u^{(n)} = Q_{\beta^{(n)}}(u^{(n-1)})$. By Theorem 1 we get

$$\|u^{(n)}\|^2 \leq \|u^{(n-1)}\|^2 - \min_{i \in I^{(n)}} \nu_i^{(n)} \max_{i \in I^{(n)}} d^2(u^{(n-1)}, C_i) \tag{7}$$

Iterating (7) leads to

$$\begin{aligned} \|u^{(n)}\|^2 &\leq \|u^{(0)}\|^2 - \sum_{m=1}^{n} \min_{i \in I^{(m)}} \nu_i^{(m)} \max_{i \in I^{(m)}} d^2(u^{(m-1)}, C_i) \\ &\leq \|u^{(0)}\|^2 - \min_{\substack{0 < m \leq n \\ i \in I^{(m)}}} \nu_i^{(m)} \max_{\substack{1 \leq m \leq n \\ i \in I^{(m)}}} d^2(u^{(m-1)}, C_i) \end{aligned} \tag{8}$$

In order to complete the proof of Theorem 2 we will make use of the following lemma.

Lemma 1. *Given a family of convex sets* $C_1, \ldots, C_N$ *with nonempty intersection* C *and a sequence* $\beta^{(n)} \in \mathcal{B}$. *Let* $I^{(n)}$ *denote the set of active indices for* $\beta^{(n)}$. *Assume that every index* $i \in \{1, 2, \ldots, N\}$ *occurs in*

$I^{(n)}$ for at least one n. Then for any positive number R there exists a nondecreasing and positive function $\eta_R : (0,+\infty) \to (0,+\infty)$ such that

$$\max_{\substack{n\geq 1\\ i\in I^{(n)}}} d^2(u^{(n-1)}, C_i) \geq \eta_R(d(u^{(0)}, C))$$

for any $u^{(0)}$ with $\|u^{(0)}\| \leq R$ and $u^{(0)} \notin C$. The function η_R is independent of the choice of the sequence $\beta^{(n)}$.

Proof. Fix $r > 0$ and consider the set

$$B_{r,R} = \{u^{(0)} \in \mathbb{R}^d \,:\, \|u^{(0)}\| \leq R,\ d(u^{(0)}, C) \geq r\}.$$

The proof will be completed if we show that for any $u^{(0)} \in B_{r,R}$ there exists a positive number $\eta_R(r)$ such that

$$\max_{\substack{n\geq 1\\ i\in I^{(n)}}} d^2(u^{(n-1)}, C_i) \geq \eta_R(r).$$

Suppose, by contradiction, that for any $m \in \mathbb{N}$ there exist vectors $u^{(0)}_{(m)} \in B_R$, and a sequence $\beta^{(n)}_{(m)} \in \mathcal{B}$, satisfying the assumptions of Lemma 1, such that

$$\max_{\substack{n\geq 1\\ i\in I^{(n)}_{(m)}}} d^2(u^{(n-1)}_{(m)}, C_i) \leq \frac{1}{m} \tag{9}$$

where

$$u^{(n)}_{(m)} = Q_{\beta^{(n)}_{(m)}}(u^{(n-1)}_{(m)}), \qquad n \geq 1.$$

By compactness of $B_{r,R}$ we may assume that $x_0^{(m)} \overset{m}{\to} y$ and $y \in B_{r,R}$. Consider the sets $I^{(1)}_{(m)}$. Some indices of $\{1, 2, \ldots, N\}$ occur for infinitely many m. Let $\mathcal{A}_1$ denote those indices. Clearly we may assume, eventually by restricting to large values of m, that only indices of $\mathcal{A}_1$ may occur in $I^{(1)}_{(m)}$, and each index does so infinitely many m. Therefore, fixing $n = 1$ and taking the limit in (9) when $m \to \infty$ yield $y \in C_i$ for any $i \in \mathcal{A}_1$. If $\mathcal{A}_1 = \{1, 2, \ldots, N\}$, then $y \in C$, which is a contradiction with $y \in B_{r,R}$. Otherwise we have $\mathcal{A}_1 \subsetneq \{1, 2, \ldots, N\}$. Since for any fixed m every index of $\{1, 2, \ldots, N\}$ occurs in $I^{(n)}_{(m)}$ at least for one n, there exists the least number l_m such that $I^{(l_m)}_{(m)} \setminus \mathcal{A}_1 \neq \emptyset$. Observe that $Q_\beta(y) = y$ for any $\beta \in \mathcal{B}$ such that the set of active indices I of β is contained in $\mathcal{A}_1$. Therefore

$$y = Q_{\beta^{(l_m-1)}_{(m)}} Q_{\beta^{(l_m-2)}_{(m)}} \ldots Q_{\beta^{(2)}_{(m)}} Q_{\beta^{(1)}_{(m)}}(y). \tag{10}$$

Define

$$\widetilde{u}^{(1)}_{(m)} := u^{(l_m-1)}_{(m)} = Q_{\beta^{(l_m-1)}_{(m)}} Q_{\beta^{(l_m-2)}_{(m)}} \dots Q_{\beta^{(2)}_{(m)}} Q_{\beta^{(1)}_{(m)}} (u^{(0)}_{(m)}). \tag{11}$$

Since the operators $Q_{\beta^{(k)}_{(m)}}$ are nonexpansive we obtain

$$d(\widetilde{u}^{(1)}_{(m)}, y) \le d(u^{(0)}_{(m)}, y). \tag{12}$$

Hence $\widetilde{u}^{(1)}_{(m)} \overset{m}{\to} y$.

Consider now the sets $I^{(l_m)}_{(m)}$. Each of these sets contains elements which do not belong to $\mathcal{A}_1$. Let $\mathcal{A}_2$ denote those indices outside $\mathcal{A}_1$ which occur for infinitely many values of m.. By restricting to large values of m we may assume that only indices of $\mathcal{A}_1$ and $\mathcal{A}_2$ may occur in $I^{(l_m)}_{(m)}$ and all indices of $\mathcal{A}_2$ occur for infinitely many values of m. Set $n = l(m) + 1$ in (9). Then since $\widetilde{u}^{(1)}_{(m)} = u^{(l_m-1)}_{(m)}$ we get

$$\max_{i \in I^{(l_m)}_{(m)}} d^2(\widetilde{u}^{(1)}_{(m)}, C_i) \le \frac{1}{m}. \tag{13}$$

Therefore for any $i \in \mathcal{A}_2$ we have

$$d^2(\widetilde{u}^{(1)}_{(m)}, C_i) \le \frac{1}{m}$$

for infinitely many m. As $\widetilde{u}^{(1)}_{(m)}$ tends to y, when $m \to \infty$, we obtain that $y \in C_i$ for any $i \in \mathcal{A}_2$. Therefore $y \in C_i$ for $i \in \mathcal{A}_1 \cup \mathcal{A}_2$.

By repeating this argument at most N times we get that $y \in C_i$ for any $i = 1, 2, \dots, N$. Hence $y \in C$ which contradicts the fact that $y \in B_{r,R}$. □

Let's return to the proof of Theorem 2. With no loss of generality we may assume that $0 \in C$. Fix $R > 0$ and assume that $\|x^{(0)}\| \le R$. Then since every operator Q_β is nonexpansive we obtain $\|x^{(n)}\| \le R$ for any n. Assume that

$$\{1, 2, \dots, N\} \subset \bigcup_{j=n_{k-1}}^{n_k-1} I^{(j)}.$$

Then combining Lemma 1 with $u^{(0)} = x^{(n_{k-1})}$ and formula (8) yields

$$\begin{aligned}\|x^{(n_k)}\|^2 &\le \|x^{(n_{k-1})}\|^2 - \min_{\substack{n_{k-1} < n \le n_k \\ i \in I^{(n)}}} \nu_i^{(n)} \max_{\substack{n_{k-1} < n \le n_k \\ i \in I^{(n)}}} d^2(x^{(n-1)}, C_i) \\ &\le \|x^{(n_{k-1})}\|^2 - \nu^{(k)}\, \eta_R(d(x^{(n_{k-1})}, C)).\end{aligned}$$

This implies that the series

$$\sum_{k=1}^{\infty} \nu^{(k)}\ \eta_R(d(x^{(n_{k-1})}, C))$$

is convergent. Since the operators Q_β are nonexpansive, the sequence $d(x^{(n)}, C)$ is nonincreasing. Therefore, by assumptions made on the coefficients $\nu_i^{(n)}$, we obtain that $\eta_R(d(x^{(n)}, C)) \overset{n}{\to} 0$. Hence $d(x^{(n)}, C) \overset{n}{\to} 0$. Since $x^{(n)}$ is bounded, it contains a convergent subsequence $x^{(n_m)}$. Denote its limit by c. Then $c \in C$. By Proposition 1 the sequence $\|x^{(n)} - c\|$ is nonincreasing. Therefore, it tends to zero, i.e. $x^{(n)} \overset{n}{\to} c$. □

4. Perturbation

Proposition 2. *Given a family of convex sets $C_1, \ldots, C_N$ with nonempty intersection C and a sequence $\widetilde{\beta}^{(n)} \in \mathcal{B}$. Assume that for any $m \in \mathbb{N}$ and $x \in \mathbb{R}^d$ the sequence*

$$Q_{\widetilde{\beta}^{(n)}} Q_{\widetilde{\beta}^{(n-1)}} \ldots Q_{\widetilde{\beta}^{(m)}}(x)$$

is convergent to an element of C as $n \to \infty$. Let a sequence $\beta^{(n)} \in \mathcal{B}$ satisfy

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N} |\beta_i^{(n)} - \widetilde{\beta}_i^{(n)}| < \infty.$$

Then for any $x \in \mathbb{R}^d$ the sequence

$$Q_{\beta^{(n)}} Q_{\beta^{(n-1)}} \ldots Q_{\beta^{(1)}}(x)$$

is convergent to an element of C as $n \to \infty$.

Proof. By (1) we have

$$d(Q_{\beta^{(k)}}(y), Q_{\widetilde{\beta}^{(k)}}(y)) \le \sum_{j=1}^{N} |\beta_j^{(k)} - \widetilde{\beta}_j^{(k)}|\, \|P_{C_j} y - y\| \le \sum_{j=1}^{N} |\beta_j^{(k)} - \widetilde{\beta}_j^{(k)}|\, d(y, C). \tag{14}$$

Denote for simplicity

$$Q_n := Q_{\beta^{(n)}}, \quad \widetilde{Q}_n := Q_{\widetilde{\beta}^{(n)}}$$

and

$$x^{(m)} = Q_m Q_{m-1} \ldots Q_1(x).$$

Then

$$\begin{aligned}
d(x^{(n)},\, C) &= d(Q_nQ_{n-1}\dots Q_{m+1}(x^{(m)}),\, C)\\
&\le \sum_{k=m+1}^{n} d(\widetilde{Q}_n\dots\widetilde{Q}_{k+1}\widetilde{Q}_k(x^{(k-1)}),\widetilde{Q}_n\dots\widetilde{Q}_{k+1}Q_k(x^{(k-1)}))\\
&\qquad\qquad + d(\widetilde{Q}_n\widetilde{Q}_{n-1}\dots\widetilde{Q}_{m+1}(x^{(m)}),\, C)\\
&\le \sum_{k=m+1}^{n} d(\widetilde{Q}_k(x^{(k-1)}),Q_k(x^{(k-1)})) \,+\, d(\widetilde{Q}_n\widetilde{Q}_{n-1}\dots\widetilde{Q}_{m+1}(x^{(m)}),C)\\
&\le \sum_{k=m+1}^{n}\sum_{j=1}^{N}|\beta_j^{(k)}-\widetilde{\beta}_j^{(k)}|\, d(\widetilde{x}^{(k-1)},C) \,+\, d(Q_nQ_{n-1}\dots Q_{m+1}(\widetilde{x}^{(m)}),C)\\
&\le \sum_{k=m+1}^{n}\sum_{j=1}^{N}|\beta_j^{(k)}-\widetilde{\beta}_j^{(k)}|\, d(x,C) \,+\, d(\widetilde{Q}_n\widetilde{Q}_{n-1}\dots\widetilde{Q}_{m+1}(x^{(m)}),C)
\end{aligned}$$

Now the conclusion follows from the assumptions. Indeed, we may assume that $d(x,C)>0$ as otherwise $x^{(n)}=x\in C$ for any n. Let m be large so that

$$\sum_{k=m+1}^{\infty}\sum_{j=1}^{N}|\beta_j^{(k)}-\widetilde{\beta}_j^{(k)}| < \frac{\varepsilon}{2d(x,C)}.$$

Next let n be large so that

$$d(\widetilde{Q}_n\widetilde{Q}_{n-1}\dots\widetilde{Q}_{m+1}(x^{(m)}),C) < \varepsilon/2.$$

Thus $d(x^{(n)},C)<\varepsilon$ for n large. Hence $d(x^{(n)},C)\to 0$ as $n\to\infty$. This implies that $x^{(n)}$ tends to a point in C (see the end of the proof of Theorem 2). ☐

Corollary 1. *Fix a family of convex sets $C_1,\dots,C_N$ with nonempty intersection C. Given a sequence $\beta^{(n)}\in\mathcal{B}$. Let $I^{(n)}$ denote the set of active indices for $\beta^{(n)}$ and let $J^{(n)}$ be a sequence of subsets of $I^{(n)}$ such that*

$$\sum_{n=1}^{\infty}\sum_{i\in I^{(n)}\setminus J^{(n)}}\beta_i^{(n)} < +\infty.$$

Assume that every index $i\in\{1,2,\dots,N\}$ occurs in $J^{(n)}$ for infinitely many n. Let n_k be positive integers such that $n_{k-1}<n_k$ and

$$\{1,2,\dots,N\}\subset J^{(n_{k-1})}\cup J^{(n_{k-1}+1)}\cup\dots\cup J^{(n_k-1)},$$

i.e. every index occurs at least once for n such that $n_{k-1} \le n < n_k$. For

$$\nu_i^{(n)} = \frac{2\beta_i^{(n)}\left(2 - \sum_{k=1}^N \beta_k^{(n)}\right)}{\beta_i^{(n)} + 2 - \sum_{k=1}^N \beta_k^{(n)}}$$

let

$$\nu_J^{(k)} = \min\{\nu_i^{(n)} \mid n_{k-1} < n \le n_k, i \in J^{(n)}\}.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_J^{(k)} = +\infty.$$

Then for any $x^{(0)} \in \mathbb{R}^d$ the sequence $x^{(n)}$ defined as

$$x^{(n)} = Q_{\beta^{(n)}}(x^{(n-1)}), \qquad n \ge 1$$

is convergent to a point in C.

Proof. Define

$$\widetilde{\beta}_i^{(n)} = \begin{cases} \beta_i^{(n)} & \text{if } i \in J^{(n)}, \\ 0 & \text{if } i \in I^{(n)} \setminus J^{(n)}. \end{cases}$$

Clearly we have

$$\sum_{k=1}^{N} \widetilde{\beta}_k^{(n)} \le \sum_{k=1}^{N} \beta_k^{(n)}.$$

Hence $\widetilde{\nu}_i^{(n)} \ge \nu_i^{(n)}$ for $i \in J^{(n)}$, which implies $\widetilde{\nu}^{(k)} \ge \nu_J^{(k)}$. Therefore, the sequence $\widetilde{\beta}^{(n)}$ satisfies the assumptions of Theorem 2. Consequently the sequence

$$Q_{\widetilde{\beta}^{(n)}} Q_{\widetilde{\beta}^{(n-1)}} \dots Q_{\widetilde{\beta}^{(m)}}(x)$$

is convergent to an element of C as $n \to \infty$. The sequences $\beta^{(n)}$ and $\widetilde{\beta}^{(n)}$ satisfy

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N} |\widetilde{\beta}_i^{(n)} - \beta_i^{(n)}| < \infty.$$

Thus applying Proposition 1 concludes the proof. □

Example. Consider $N = 3$ and

$$\begin{array}{llllll} \beta_1^{(2n)} & = \frac{1}{n^2}, & \beta_2^{(2n)} & = \frac{1}{n}, & \beta_3^{(2n)} & = 2 - \frac{2}{n}, \\ \beta_1^{(2n+1)} & = \frac{1}{n}, & \beta_2^{(2n+1)} & = \frac{1}{n^2}, & \beta_3^{(2n+1)} & = 2 - \frac{2}{n}. \end{array}$$

The scheme is 1-intermittent, i.e. $I^{(n)} = \{1, 2, 3\}$ for any n. Observe that the assumptions of Theorem 2 are not satisfied. Indeed, with

$n_k = k$ the series in (6) is convergent. Now, consider this scheme as 2-intermittent and let

$$J^{(2n)} = \{2,3\} \qquad J^{(2n+1)} = \{1,3\}.$$

Then we can apply Corollary 1 to obtain that this scheme leads to the convergence of the algorithm.

5. Intermittent control

The assumptions of Theorem 2 depend on the behaviour of the coefficients $\beta_i^{(n)}$ where $i \in I^{(n)}$, i.e. those coefficients which are positive. Roughly the conclusion holds if these coefficients are not too small and the sums $s^{(n)} = \sum_{i=1}^N \beta_i^{(n)}$ do not approach the value 2 too fast. By Corollary 1 we can allow some small coefficients $\beta_i^{(n)}$ by using perturbation technique. However in special case of intermittent control and when the sums $s^{(n)}$ stay away from 2 we can entirely liberate ourselves from assumptions on all positive coefficients $\beta_i^{(n)}$.

Theorem 3. *Fix a family of convex sets $C_1, \ldots, C_N$ with nonempty intersection C. Given a sequence $\beta^{(n)} \in \mathcal{B}$ such that*

$$s^{(n)} = \sum_{i=1}^N \beta_i^{(n)} \leq 2 - \varepsilon,$$

for some constant $\varepsilon > 0$. Let $I^{(n)}$ denote the set of active indices for $\beta^{(n)}$ and let $J^{(n)}$ be a sequence of subsets of $I^{(n)}$. Assume that there is a positive integer p such that for any k we have

$$\{1, 2, \ldots, N\} \subset J^{((k-1)p)} \cup J^{(k-1)p+1)} \cup \ldots \cup J^{(kp-1)}.$$

Let

$$\nu_J^{(k)} = \min\{\beta_i^{(n)} \mid (k-1)p < n \leq kp,\, i \in J^{(n)}\}.$$

Assume that

$$\sum_{k=1}^\infty \nu_J^{(k)} = +\infty.$$

Then for any $x^{(0)} \in \mathbb{R}^d$ the sequence $x^{(n)}$ defined as

$$x^{(n)} = Q_{\beta^{(n)}}(x^{(n-1)}), \qquad n \geq 1$$

is convergent to a point in C.

Proof. First observe that since $s^{(n)} \leq 2 - \varepsilon$ we have

$$\frac{2\varepsilon}{2+\varepsilon}\beta_i^{(n)} \leq \nu_i^{(n)} = \frac{2\beta_i^{(n)}(2 - s^{(n)})}{\beta_i^{(n)} + 2 - s^{(n)}} \leq 2\beta_i^{(n)}.$$

Therefore we can replace the coefficients $\nu_i^{(n)}$ with $\beta_i^{(n)}$ when applying Theorem 2 and Corollary 1.

Let $n_k = kp$. If for

$$\nu^{(k)} = \min\{\beta_i^{(n)} \mid (k-1)p < n \le kp, i \in I^{(n)}\}$$

we have

$$\sum_{k=1}^{\infty} \nu^{(k)} = +\infty \tag{15}$$

we can apply Theorem 2 to get the conclusion. Thus it suffices to consider the case when

$$\sum_{k=1}^{\infty} \nu^{(k)} < +\infty. \tag{16}$$

Let

$$A = \{k \in \mathbb{N} \mid (\exists\, n)(\exists\, i)\ (k-1)p < n \le kp,\ i \in I^{(n)} \setminus J^{(n)},\ \nu^{(k)} = \beta_i^{(n)}\}.$$

For every $k \in A$ choose n_k and i_k such that

$$(k-1)p < n_k \le kp, \quad i_k \in I^{(n)} \setminus J^{(n)}, \quad \nu^{(k)} = \beta_{i_k}^{(n_k)}.$$

By (16) we have

$$\sum_{k \in A} \beta_{i_k}^{(n_k)} < +\infty.$$

Define the new coefficients $\widetilde{\beta}_i^{(n)}$ by nullifying the coefficients $\beta_i^{(n)}$ for $i = i_k$ and $n = n_k$, i.e. let

$$\widetilde{\beta}_i^{(n)} = \begin{cases} 0 & \text{if } n = n_k,\ i = i_k \text{ for some } k \\ \beta_i^{(n)} & \text{otherwise} \end{cases}$$

By construction the sums $\widetilde{s}^{(k)}$ stay away from 2 since $\widetilde{s}^{(k)} \le s^{(k)}$. Moreover $J^{(n)} \subset \widetilde{I}^{(n)}$, where $\widetilde{I}^{(n)}$ denote the set of active indices for $\widetilde{\beta}^{(n)}$. By Corollary 1 the convergence of the algorithm for the new coefficients implies its convergence for the original ones. Thus we can restrict ourselves to the coefficients $\widetilde{\beta}_i^{(n)}$. Clearly for $i \in J^{(n)}$ we have $\widetilde{\beta}_i^{(n)} = \beta_i^{(n)}$. If the new coefficients satisfy (15) we are done by Theorem 2. If not, we can perform the same transformation as before. After at most pN iterations we will obtain a sequence to which we can apply Theorem 2 and which differs from original sequence as in Corollary 1. $\square$

Example Let $N = 3$ and

$$\begin{array}{llllll} \beta_1^{(3n)} & = 1, & \beta_2^{(3n)} & = \frac{1}{n}, & \beta_2^{(3n)} & = \frac{1}{n^2}, \\ \beta_1^{(3n+1)} & = \frac{1}{n}, & \beta_2^{(3n+1)} & = 1, & \beta_2^{(3n+1)} & = \frac{1}{n^2}, \\ \beta_1^{(3n+2)} & = \frac{1}{n^2}, & \beta_2^{(3n+2)} & = \frac{1}{n}, & \beta_2^{(3n+2)} & = 1. \end{array}$$

We have $I^{(n)} = \{1, 2, 3\}$. Let

$$J^{(3n)} = \{1\}, \quad J^{(3n+1)} = \{2\}, \quad J^{(3n+2)} = \{1\}$$

and $p = 3$. We have $\nu_J^{(k)} = 1$ for any k. Hence all the assumptions of Theorem 3 are satisfied.

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