KACZMARZ ALGORITHM IN HILBERT SPACE AND TIGHT FRAMES

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Abstract. We prove that any tight frame \( \{ g_n \}_{n=0}^{\infty} \), with \( \| g_0 \| = 1 \), in a Hilbert space can be obtained by the Kaczmarz algorithm. The uniqueness of the correspondence is determined.

1. Introduction

Let \( \{ e_n \}_{n=0}^{\infty} \) be a linearly dense sequence of unit vectors in a Hilbert space \( \mathcal{H} \). In 1937 Kaczmarz considered the problem of reconstructing vectors \( x \) from the data \( \langle x, e_n \rangle \). He proved that in the finite-dimensional case we have \( x_n \to x \) for any \( x \), where elements \( x_n \) are defined recursively by

\[
\begin{align*}
x_0 &= \langle x, e_0 \rangle e_0, \\
x_n &= x_{n-1} + \langle x - x_{n-1}, e_n \rangle e_n.
\end{align*}
\]

This formula is called the Kaczmarz algorithm ([1]).

It can be shown that if vectors \( g_n \) are given by the recurrence relation

\[
(1) \quad g_0 = e_0, \quad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i \rangle g_i
\]

then \( g_0 \) is orthogonal to \( g_n \), for any \( n \geq 1 \) and

\[
(2) \quad x_n = \sum_{i=0}^{n} \langle x, g_i \rangle e_i.
\]

By (1) the vectors \( \{ g_n \}_{n=0}^{\infty} \) are linearly dense in \( \mathcal{H} \). Also by definition of the algorithm the vectors \( x - x_n \) and \( e_n \) are orthogonal to each other.

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Hence
\[ \|x\|^2 = \|x - x_0\|^2 + |\langle x, g_0 \rangle|^2, \]
\[ (3) \quad \|x - x_{n-1}\|^2 = \|x - x_n\|^2 + |\langle x, g_n \rangle|^2, \quad n \geq 1. \]

For \( n \geq 1 \) let \( S_n \) denote a finite dimensional operator defined by the rule
\[ (4) \quad S_n y = \sum_{j=0}^{n} \langle y, e_j \rangle g_j, \quad y \in \mathcal{H}. \]
Observe that the formulas (1) and (2) can be restated as
\[ (5) \quad (I - S_{n-1}) e_n = g_n \]
\[ (6) \quad (I - S_n^*) x = x - x_n. \]
Moreover by (3) it follows that
\[ (7) \quad \|x - x_n\|^2 = \|(I - S_n^*) x\|^2 = \|x\|^2 - \sum_{j=0}^{n} |\langle x, g_j \rangle|^2. \]

In particular
\[ (8) \quad \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 \leq \|x\|^2, \quad x \in \mathcal{H}. \]

A sequence \( \{e_n\}_{n=0}^{\infty} \) is called effective if \( x_n \to x \) for any \( x \in \mathcal{H} \). By virtue of (7) this is equivalent to \( \|x\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 \) for any \( x \in \mathcal{H} \), which means \( \{g_n\}_{n=0}^{\infty} \) is a tight frame. We refer to [2] for more information on the Kaczmarz algorithm and to [3] for the characterization of effective sequences through the Gram matrix of the sequence \( \{e_n\}_{n=0}^{\infty} \).

2. **Bessel sequences**

**Definition 1.** A sequence of vectors \( \{g_n\}_{n=0}^{\infty} \) in a Hilbert space \( \mathcal{H} \) will be called a Bessel sequence if (8) holds. The sequence \( \{g_n\}_{n=0}^{\infty} \) will be called a special Bessel sequence if in addition \( \|g_0\| = 1 \).

Observe that if \( \{g_n\}_{n=0}^{\infty} \) is a special Bessel sequence then substituting \( x = g_0 \) into (8) implies \( g_n \perp g_0 \) for \( n \geq 1 \).

Let \( P_n \) denote the orthogonal projection onto \( e_n^\perp \), the orthogonal complement to the vector \( e_n \). By [3, (1)] we have
\[ (9) \quad I - S_n^* = P_n P_{n-1} \ldots P_0, \]
\[ (10) \quad I - S_n = P_0 \ldots P_{n-1} P_n. \]
Theorem 1. For any special Bessel sequence \( \{g_n\}_{n=0}^{\infty} \) in a Hilbert space \( \mathcal{H} \) there exists a sequence \( \{e_n\}_{n=0}^{\infty} \) of unit vectors such that (1) holds. In other words any special Bessel sequence can be obtained through the Kaczmarz algorithm.

Proof. We will construct the sequence \( \{e_n\}_{n=0}^{\infty} \) recursively. Set \( e_0 = g_0 \).

Assume the unit vectors \( e_1, \ldots, e_{N-1} \) have been constructed such that the formula (1) holds for \( n = 0, \ldots, N-1 \). We want to find \( y \) such that

\[
(I - S_{N-1})y = g_N, \quad \|y\| = 1.
\]

By (10) we have \( (I - S_{N-1})e_{N-1} = 0 \), i.e. the operator \( I - S_{N-1} \) admits nontrivial kernel. Hence the solvability of (11) is equivalent to that of

\[
(I - S_{N-1})y = g_N, \quad \|y\| \leq 1.
\]

By the Fredholm alternative the equation \( (I - S_{N-1})y = g_N \) is solvable if and only if \( g_N \) is orthogonal to \( \ker(I - S_{N-1}^*) \). We will check that this condition holds. Let \( x \in \ker(I - S_{N-1}^*) \). Then by (7) and (8) we have

\[
0 = \| (I - S_{N-1}^*)x \|^2 = \|x\|^2 - \sum_{j=0}^{N-1} |\langle x, g_j \rangle|^2 \geq \sum_{j=N}^{\infty} |\langle x, g_j \rangle|^2.
\]

In particular \( \langle x, g_N \rangle = 0 \), i.e. \( g_N \perp \ker(I - S_{N-1}^*) \).

Let \( y \) denote the unique solution to

\[
(I - S_{N-1})y = g_N, \quad y \perp \ker(I - S_{N-1}).
\]

The proof will be complete if we show that \( \|y\| \leq 1 \). Again by the Fredholm alternative we have \( y \in \operatorname{Im}(I - S_{N-1}^*) \). Let \( y = (I - S_{N-1}^*)x \) for some \( x \in \mathcal{H} \). We may assume that \( x \perp \ker(I - S_{N-1}) \). In particular \( \langle x, g_0 \rangle = 0 \), as (9) yields \( g_0 \in \ker(I - S_{N-1}^*) \). By (7) we have

\[
\|y\|^2 = \| (I - S_{N-1}^*)x \|^2 = \|x\|^2 - \sum_{j=1}^{N-1} |\langle x, g_j \rangle|^2.
\]

On the other hand

\[
\|y\|^2 = \langle x, (I - S_{N-1})y \rangle = \langle x, g_N \rangle.
\]

Therefore

\[
\|y\|^2 - \|y\|^4 = \|x\|^2 - \sum_{j=1}^{N} |\langle x, g_j \rangle|^2 \geq 0,
\]

which implies \( \|y\| \leq 1 \). \( \square \)
Corollary 1. For any special tight frame \( \{g_n\}_{n=0}^{\infty} \) in a Hilbert space \( \mathcal{H} \) there exists an effective sequence \( \{e_n\}_{n=0}^{\infty} \) of unit vectors such that (1) holds, i.e. any special tight frame can be obtained through the Kaczmarcz algorithm.

For a sequence \( \{e_n\}_{n=0}^{\infty} \) of unit vectors the special Bessel sequence \( \{g_n\}_{n=0}^{\infty} \) is determined uniquely by (1). However a given special Bessel sequence may correspond to many sequences of unit vectors due to two reasons. First of all for certain \( N \) the dimension of the space \( \ker(I - S_N) \) may exceed 1. Secondly, if we fix a unit vector \( u \) in \( \ker(I - S_N) \) the vector \( e_N \) can be defined as \( e_N = y + \lambda u \) for any complex \( \lambda \) number such that \( |\lambda|^2 + \|y\|^2 = 1 \). In what follows we will indicate properties which guarantee a one to one correspondence between \( \{e_n\}_{n=0}^{\infty} \) and \( \{g_n\}_{n=0}^{\infty} \).

Definition 2. A sequence of unit vectors \( \{e_n\}_{n=0}^{\infty} \) will be called strongly redundant if the vectors \( \{e_n\}_{n=0}^{\infty} \) are linearly dense for any \( N \). A special Bessel sequence \( \{g_n\}_{n=0}^{\infty} \) will be called strongly redundant if the vectors \( \{g_n\}_{n=0}^{\infty} \cup \{g_n\}_{n=N}^{\infty} \) are linearly dense for any \( N \).

Proposition 1. Let sequences \( \{e_n\}_{n=0}^{\infty} \) and \( \{g_n\}_{n=0}^{\infty} \) satisfy (1). The sequence \( \{g_n\}_{n=0}^{\infty} \) is strongly redundant if and only if \( \{e_n\}_{n=0}^{\infty} \) is strongly redundant and \( \langle e_n, e_{n+1} \rangle \neq 0 \) for any \( n \geq 0 \).

Proof. Assume \( \{g_n\}_{n=0}^{\infty} \) is strongly redundant. First we will show that the kernel of \( I - S_N \) is one dimensional and thus consists of the multiples of the vector \( e_N \) (see (10)). Assume for a contradiction that \( \dim \ker(I - S_N) \geq 2 \). By the Fredholm alternative we get \( \dim \ker(I - S_N^*) \geq 2 \). Hence there exists a nonzero vector \( x \) such that \( x \perp g_0 \) and \( (I - S_N^*)x = 0 \). By (3) we obtain

\[
\|x\|^2 = \sum_{n=1}^{N-1} |\langle x, g_n \rangle|^2.
\]

This and the condition (8) imply that \( x \) is orthogonal to all the vectors \( g_0 \) and \( \{g_n\}_{n=N}^{\infty} \), which contradicts the strong redundancy assumption.

Assume \( \langle e_{N-1}, e_N \rangle = 0 \) for some \( N \geq 1 \). Then by (10) we have \( e_{N-1}, e_N \in \ker(I - S_N) \) which is a contradiction as the kernel is one dimensional.

Concerning strong redundancy of \( \{e_n\}_{n=0}^{\infty} \), assume a vector \( y \) is orthogonal to all the vectors \( \{e_n\}_{n=N}^{\infty} \). In particular \( y \) is orthogonal to \( e_N \). Since \( \ker(I - S_N) = \mathbb{C} e_N \), by the Fredholm alternative \( y \) belongs to \( \text{Im}(I - S_N^*) \). Let \( y = (I - S_N^*)x \) for some \( x \in \mathcal{H} \). We may assume that \( x \perp g_0 \) as \( g_0 \in \ker(I - S_N^*) \). By (9), since \( y \) is orthogonal to \( e_n \) for
For the converse implication assume \( \{e_n\}_{n=0}^{\infty} \) is strongly redundant and \( \langle e_n, e_{n+1} \rangle \neq 0 \). By the inequality (see [2])

\[
\|x-x_n\| \geq |\langle e_{n-1}, e_n \rangle|\|x-x_{n-1}\|
\]

we get that \( x-x_n \neq 0 \) for any \( x \perp e_0 \). Since \( x-x_n = (I-S_n^*)x \), the kernel of \( I-S_n^* \) consists of the multiples of \( e_0 = g_0 \), only.

Let \( x \) be orthogonal to \( \{g_0\} \cup \{g_n\}_{n\geq N+1} \) for some \( N \geq 1 \). By (2) we obtain that \( x_n = x_N \) for \( n \geq N \). By the definition of the Kaczmarz algorithm we get \( x-x_N \perp e_n \) for \( n \geq N+1 \). Now strong redundancy of \( \{e_n\}_{n=0}^{\infty} \) implies \( x-x_N = 0 \). By (6) we obtain \( (I-S_n^*)x = 0 \). This yields \( x = 0 \) since the kernel is one dimensional and consists of the multiples of \( g_0 \).

For sequences \( \{e_n\}_{n=0}^{\infty} \) and \( \{\sigma_n e_n\}_{n=0}^{\infty} \), where \( \sigma_n \) are complex numbers of absolute value 1, the Kaczmarz algorithm coincides. Therefore we will restrict our attention to admissible sequences of unit vectors \( \{e_n\}_{n=0}^{\infty} \) such that \( \langle e_n, e_{n+1} \rangle \geq 0 \).

**Theorem 2.** Let \( \{g_n\}_{n=0}^{\infty} \) be a strongly redundant special Bessel sequence. Then there exists a unique admissible sequence \( \{e_n\}_{n=0}^{\infty} \) of unit vectors such that (1) holds. Moreover the sequence \( \{e_n\}_{n=0}^{\infty} \) is strongly redundant.

**Proof.** The proof will go by induction. The vector \( e_0 \) is determined by \( e_0 = g_0 \). Assume the vectors \( e_0, \ldots, e_{N-1} \) were determined uniquely. We have to show that the problem

\[
(I-S_{N-1})y = g_N, \quad \|y\| = 1, \quad \langle y, e_{N-1} \rangle \geq 0
\]

has a unique solution \( y \).

By the proof of Proposition 1 the kernel of \( I-S_{N-1} \) is one dimensional and thus consists of the multiples of the vector \( e_{N-1} \). By the proof of Theorem 1 there exists a unique solution \( y_N \) to the problem

\[
(I-S_{N-1})y = g_N, \quad y \perp \ker(I-S_{N-1})
\]

and \( \|y_N\| \leq 1 \). Moreover by this proof \( \|y_N\| = 1 \) if and only if

\[
\|x\|^2 - \sum_{j=1}^{N} |\langle x, g_j \rangle|^2 = 0,
\]
where $y_N = (I - S_{N-1}^*)x$ and $x \perp \ker(I - S_{N-1}^*)$. This leads to a contradiction because by inequality (8) we get that $x$ is orthogonal to all the vectors $g_0$ and $\{g_n\}_{n=N}^\infty$. Hence $\|y_N\| < 1$.

At this stage we know that any solution to the equation

$$(I - S_{N-1})y = 0$$

is of the form

$$y = y_N + \lambda e_{N-1}, \quad \lambda \in \mathbb{C}$$

because $\ker(I - S_{N-1}) = \mathbb{C}e_{N-1}$. Since $\|y_N\| < 1$ and $y_N \perp e_{N-1}$ there exists a unique solution $y$ satisfying $\|y\| = 1$ and $\langle y, e_{N-1} \rangle \geq 0$ namely the one corresponding to $\lambda = \sqrt{1 - \|y_N\|^2}$.

**Corollary 2.** Let $\{g_n\}_{n=0}^\infty$ be a strongly redundant special tight frame. Then there exists a unique admissible effective sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that (1) holds. Moreover the sequence $\{e_n\}_{n=0}^\infty$ is strongly redundant.

**References**


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