Symmetric moment problems and a conjecture of Valent

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Abstract

In 1998 G. Valent made conjectures about the order and type of certain indeterminate Stieltjes moment problems associated with birth and death processes having polynomial birth and death rates of degree $p \geq 3$. Romanov recently proved that the order is $1/p$ as conjectured, see [14]. We prove that the type with respect to the order is related to certain multi-zeta values and that this type belongs to the interval

$$[\pi/(p\sin(\pi/p)), \pi/(p\sin(\pi/p)\cos(\pi/p))],$$

which also contains the conjectured value. This proves that the conjecture about type is asymptotically correct as $p \to \infty$.

The main idea is to obtain estimates for order and type of symmetric indeterminate Hamburger moment problems when the orthonormal polynomials P_n and those of the second kind Q_n satisfy $P_{2n}^2(0) \sim c_1 n^{-1/\beta}$ and $Q_{2n-1}^2(0) \sim c_2 n^{-1/\alpha}$, where $0 < \alpha, \beta < 1$ can be different, and c_1, c_2 are positive constants. In this case the order of the moment problem is majorized by the harmonic mean of α, β. Here $\alpha_n \sim \beta_n$ means that $\alpha_n/\beta_n \to 1$. This also leads to a new proof of Romanov's Theorem that the order is $1/p$.

1 Introduction and results

To every indeterminate moment problem is associated an order $\rho \in [0,1]$, namely the common order of the four entire functions in the Nevanlinna matrix. If

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$0 < \rho < 1$ there is also associated a type $\tau \in [0, \infty]$ which is the common type of the same functions, see [4]. When $\rho = 1$ the type is necessarily $\tau = 0$ by a theorem of M. Riesz.

Since a moment problem is characterized by two sequences $(a_n)_{n\geq0}$ and $(b_n)_{n\geq0}$ of respectively real and positive numbers (called the Jacobi parameters or recurrence coefficients) via the three term recurrence relation

$$zr_n(z) = b_n r_{n+1}(z) + a_n r_n(z) + b_{n-1} r_{n-1}(z), \quad n \geq 0, \tag{1}$$

satisfied by the orthonormal polynomials $P_n(z)$ and those of the second kind $Q_n(z)$, it is of some interest to be able to calculate the order and type directly from the sequences a_n, b_n without calculating the functions of the Nevanlinna matrix. For P_n we have the initial conditions $P_{-1} = 0, P_0 = 1$, while we have $Q_{-1} = -1, Q_0 = 0$ with the convention that $b_{-1} := 1$.

Such results were obtained about order in the paper [5] by the authors. In particular we obtained the following result:

Theorem 1.1. *Assume that the coefficients a_n, b_n satisfy*

$$\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty, \tag{2}$$

and that (b_n) is either eventually log-convex or eventually log-concave.

Then the order ρ of the moment problem is equal to the exponent of convergence $\mathcal{E}(b_n)$ of the sequence (b_n), where

$$\mathcal{E}(b_n) = \inf\left\{\alpha > 0 \mid \sum_{n=0}^{\infty} b_n^{-\alpha} < \infty\right\}.$$

This result is well suited for applications to symmetric moment problems characterized by $a_n = 0$ for all n. In this case and when the log-convexity/log-concavity condition is satisfied, the condition (2) is equivalent to $\sum 1/b_n < \infty$, which in turn is equivalent to indeterminacy.

However, if the log-convexity/log-concavity is not satisfied, then the condition $\sum 1/b_n < \infty$ can hold also for determinate moment problems, see e.g. [5, Remark 4.5].

It turns out that for a number of the classical indeterminate Stieltjes moment problems, where $a_n > 0$, the condition (2) is not satisfied. This is in particular true for the Stieltjes moment problem associated with the birth and death rates (6) and (7), where $(1 + |a_n|)/\sqrt{b_n b_{n-1}}$ converges to 2 for $n \to \infty$. To handle these cases we need some refinements of our results.

The idea is to use the well-known one-to-one correspondence between Stieltjes moment problems and symmetric Hamburger moment problems as recalled in Section 3. If the order and type of an indeterminate symmetric Hamburger

moment problem can be calculated, then the order and type of the corresponding Stieltjes moment problem are known too, see Proposition 3.1 for details.

Before announcing our first main result we recall that a sequence of complex numbers (x_n) belongs to ℓ^α, where $\alpha > 0$, if $\sum |x_n|^\alpha < \infty$. The result reads:

Theorem 1.2. *Consider a symmetric indeterminate Hamburger moment problem and introduce the notation*

$$v_n = P_{2n}^2(0), \quad u_n = Q_{2n-1}^2(0), \quad n \geq 1. \tag{3}$$

Assume that

(i) $(u_n) \in \ell^\alpha$, $(v_n) \in \ell^\beta$, *where* $0 < \alpha, \beta \leq 1$.

Assume also that there exist constants $C, D > 0$ *such that*

(ii) $u_n \leq Cv_n^{\beta/\alpha}$, $n \geq 1$.

(iii) $v_j \leq Dv_i$ *for* $i \leq j$.

Let γ *be the harmonic mean of* α *and* β, *i.e.,*

$$\gamma^{-1} = \frac{1}{2}\left(\alpha^{-1} + \beta^{-1}\right). \tag{4}$$

Then the order ρ *of the moment problem is* $\leq \gamma$.

The proof of Theorem 1.2 is given in Section 5.

Remark 1.3. Note that condition (iii) is fulfilled, if (v_n) is eventually decreasing.

As the proof will show, a similar result is true if the conditions (ii) and (iii) are replaced by the following conditions, where the roles of u_n and v_n are interchanged

(ii)' $v_n \leq C'u_n^{\alpha/\beta}$, $n \geq 1$,

(iii)' $u_j \leq D'u_i$ for $i \leq j$.

Here $C', D' > 0$ are suitable constants.

Birth and death processes lead to Stieltjes moment problems according to a theory developed by Karlin and McGregor, see [12] and [11]. The theory depends on two sequences $(\lambda_n), (\mu_n)$ of birth and death rates with the constraints $\lambda_n, \mu_{n+1} > 0, n \geq 0$ and $\mu_0 \geq 0$. For simplicity we shall always assume $\mu_0 = 0$.

The recurrence coefficients $(a_n), (b_n)$ of the corresponding Stieltjes moment problem are given by

$$a_n = \lambda_n + \mu_n, \quad b_n = \sqrt{\lambda_n \mu_{n+1}}, \quad n \geq 0. \tag{5}$$

In a series of papers Valent and his co-authors studied Stieltjes moment problems coming from birth and death processes with polynomial rates, see [18] and

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references therein. In [17] Valent formulated some conjectures in case of polynomial birth and death rates $(\lambda_n), (\mu_n)$ of the form

$$\lambda_n = (pn + e_1) \ldots (pn + e_p), \quad n \geq 0, \tag{6}$$

and

$$\mu_n = (pn + d_1) \ldots (pn + d_p), \quad n \geq 0, \tag{7}$$

where it is assumed that

$$0 < e_1 \leq e_2 \leq \ldots \leq e_p, \quad -p < d_1 \leq d_2 \leq \ldots \leq d_p, \quad d_1 \ldots d_p = 0, \tag{8}$$

so one has $\lambda_n, \mu_{n+1} > 0$ for $n \geq 0$ and $\mu_0 = 0$.

By introducing the quantities

$$E = e_1 + \cdots + e_p, \quad D = d_1 + \cdots + d_p, \tag{9}$$

Valent proved that the Stieltjes problem is indeterminate if and only if

$$1 < \frac{E - D}{p} < p - 1. \tag{10}$$

This will also follow from the considerations in Section 4.

In the special cases $p = 3, 4$ and for special values of $e_j, d_j, j = 1, \ldots, p$, it has been possible to calculate the Nevanlinna matrices using elliptic functions, see [6],[9],[10],[18]. On the basis of these calculations Valent [17] formulated the following conjecture:

Valent's conjecture *For the indeterminate Stieltjes moment problem with the rates* (6) *and* (7) *satisfying the condition* (10) *(hence $p \geq 3$), the order, type and Phragmén-Lindelöf indicator function are given as*

$$\rho = 1/p, \quad \tau = \int_0^1 \frac{du}{(1 - u^p)^{2/p}}, \quad h(\theta) = \tau \cos((\theta - \pi)/p), \ \theta \in [0, 2\pi]. \tag{11}$$

Recently, Romanov [14] has proved Valent's conjecture concerning order using the powerful theory of canonical systems.

Theorem 1.4 (Romanov [14])**.** *The indeterminate Stieltjes moment problem with the rates* (6) *and* (7) *satisfying* (10) *has order $\rho = 1/p$.*

Based on Theorem 1.2 we can give a new proof of Theorem 1.4. Our method also leads to an estimate for the type of the moment problem in the previous theorem. The methods in [14] do not immediately lead to an upper bound for the type. Our estimates of the type depend on the following quantity.

For real $p > 2$ and $n \geq 1$ define the multi-zeta value[1]

$$\gamma_n(p) = \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} (k_1 k_2 \ldots k_{2n-1} k_{2n})^{-p/2}. \tag{12}$$

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[1]Observe that the inequalities between the indices k_j are alternating between $\leq$ and $<$: $k_{2j-1} \leq k_{2j} < k_{2j+1} \leq k_{2j+2}$

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Theorem 1.5. *The type τ of the indeterminate Stieltjes moment problem with the rates* (6) *and* (7) *satisfying* (10) *is equal to τ_p/p, where τ_p is the type of the entire function*

$$G_p(z) = \sum_{n=1}^{\infty} \gamma_n(p) z^n \qquad (13)$$

with coefficients $\gamma_n(p)$ given by (12).

Remark 1.6. The function (13) has order $1/p$ like the Stieltjes problem. This follows from the proof of Theorem 1.5.

Theorem 1.7. *The type τ of the indeterminate Stieltjes moment problem with the rates* (6) *and* (7) *satisfying* (10) *fulfils the inequalities*

$$\frac{\pi}{p\sin(\pi/p)} \leq \tau \leq \frac{\pi}{p\sin(\pi/p)\cos(\pi/p)}. \qquad (14)$$

We have not been able to prove that

$$\tau = \int_0^1 \frac{du}{(1-u^p)^{2/p}}$$

as conjectured by Valent, but we prove in Section 6 that the value of the integral lies between the bounds of (14). This shows that the conjecture about type is asymptotically correct as $p \to \infty$.

There is a modification of the multi-zeta value (12) which also determines the type of the moment problem in question. The modified multi-zeta value in Theorem 1.8 has the advantage in comparison with $\gamma_n(p)$ in (12) that the indices of summation are strictly increasing as in the classical multiple zeta values, cf. [19].

Theorem 1.8. *For real $p > 2$ and $n \geq 2$ define*

$$\zeta_n(p) = \sum_{1 \leq k_1 < k_2 < ... < k_n} (k_2 - k_1)(k_3 - k_2)\ldots(k_n - k_{n-1})(k_1 k_2 \ldots k_n)^{-p}. \qquad (15)$$

Then we have

$$\zeta_n(p) \leq \gamma_n(p) \leq \zeta(p-1)\frac{2^{p/2-1}}{p/2-1}\zeta_{n-1}(p), \quad n \geq 3, \qquad (16)$$

where

$$\zeta(x) = \sum_{k=1}^{\infty} k^{-x}, \quad x > 1$$

is the Riemann zeta-function.

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Corollary 1.9. *For real $p > 2$ the function*

$$Z_p(z) = \sum_{n=2}^{\infty} \zeta_n(p) z^n \tag{17}$$

has the same order $1/p$ and the same type τ_p as the function G_p given in (13).

Remark 1.10. The previous results show that the order and type of the indeterminate Stieltjes moment problem associated with the rates (6) and (7) are independent of the special values of the parameters e_j, d_j. It is therefore enough to consider the special case where $e_j = p/2, d_j = 0, j = 1, \ldots, p$. In this case $(E - D)/p = p/2$ so the indeterminacy condition (10) holds when $p \geq 3$. In this case the recurrence coefficients (b_n) of the corresponding symmetric indeterminate Hamburger problem are given as

$$b_n = (p/2)^{p/2}(n+1)^{p/2}.$$

In the formula for b_n above we can let p can be any positive real number, and the corresponding Hamburger moment problem is indeterminate if and only if $p > 2$, cf. [5, Example 4.12]. This leads to the following result:

Theorem 1.11. *Let $c > 1$. The indeterminate symmetric Hamburger moment problem associated with $b_n = n^c$ for $n \geq 1$ has order $1/c$ and type $T_c/2$, where T_c is the common type of the functions $G_{2c}(z)$ and $Z_{2c}(z)$.*

Using (27) we have

$$T_c = \frac{c}{e} \limsup_{n \to \infty} \left(n(\gamma_n(2c))^{1/(cn)} \right) = \frac{c}{e} \limsup_{n \to \infty} \left(n(\zeta_n(2c))^{1/(cn)} \right), \tag{18}$$

but we do not know how to evaluate T_c. In accordance with Valent's conjecture we believe that T_c is given in terms of the Beta-function as

$$T_c = B(1/(2c), 1 - 1/c). \tag{19}$$

2 Preliminaries

Let $\mathbb{M}^*(\mathbb{R})$ denote the set of positive measures on $\mathbb{R}$ with moments of any order and infinite support. For $\mu \in \mathbb{M}^*(\mathbb{R})$ we define the moment sequence

$$s_n = s_n(\mu) = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0. \tag{20}$$

It is called normalized if $s_0 = \mu(\mathbb{R}) = 1$. The theory of the indeterminate moment problem is treated in the classical monographs [1], [15] and in the survey paper [16], which introduced important new ideas. Here we follow the terminology and notation from Akhiezer [1]. We shall also rely on concepts and notation introduced in our paper [5]. The following is a classical result which can be found in [1, Chap.1, Sect. 3]:

Theorem 2.1. *For (s_n) as in* (20) *the following conditions are equivalent:*

(i) $\sum_{n=0}^{\infty} (P_n^2(0) + Q_n^2(0)) < \infty$,

(ii) $P(z) = (\sum_{n=0}^{\infty} |P_n(z)|^2)^{1/2} < \infty, \quad z \in \mathbb{C}.$

If (i) *and* (ii) *hold (the indeterminate case), then* $Q(z) = (\sum_{n=0}^{\infty} |Q_n(z)|^2)^{1/2} < \infty$
for $z \in \mathbb{C}$, *and the series for* P, Q *are uniformly convergent on compact subsets of* $\mathbb{C}$.

We now consider a normalized indeterminate moment sequence $(s_n)_{n \geq 0}$.
The following polynomials will be used, cf. [1, p.14]

$$A_n(z) = z \sum_{k=0}^{n-1} Q_k(0)Q_k(z),$$

$$B_n(z) = -1 + z \sum_{k=0}^{n-1} Q_k(0)P_k(z),$$

$$C_n(z) = 1 + z \sum_{k=0}^{n-1} P_k(0)Q_k(z),$$

$$D_n(z) = z \sum_{k=0}^{n-1} P_k(0)P_k(z).$$

Because of Theorem 2.1 these polynomials tend to entire functions denoted A, B, C, D, when n tends to infinity.
Following ideas of Simon [16], we can write the polynomial equations as

$$\begin{pmatrix} A_{n+1}(z) & B_{n+1}(z) \\ C_{n+1}(z) & D_{n+1}(z) \end{pmatrix} =$$
$$\left[I + z \begin{pmatrix} -P_n(0)Q_n(0) & Q_n^2(0) \\ -P_n^2(0) & P_n(0)Q_n(0) \end{pmatrix} \right] \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix}, \quad (21)$$

hence

$$\begin{pmatrix} A_{n+1}(z) & B_{n+1}(z) \\ C_{n+1}(z) & D_{n+1}(z) \end{pmatrix} = (I + zT_n)(I + zT_{n-1}) \ldots (I + zT_0) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$T_n = \begin{pmatrix} -P_n(0)Q_n(0) & Q_n^2(0) \\ -P_n^2(0) & P_n(0)Q_n(0) \end{pmatrix}. \quad (22)$$

Letting $n \to \infty$ we get

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \left[\prod_{n=1}^{\infty} (I + zT_n) \right] \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (23)$$

where matrix products are supposed to expand to the left. The matrix on the left of (23) is called the Nevanlinna matrix of the indeterminate moment problem.

For a non-constant entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{24}$$

the maximum modulus is the increasing function

$$\mathcal{M}_f(r) = \max_{|z|=r} |f(z)|, \quad r \geq 0. \tag{25}$$

The order $\rho = \rho_f$ of f is given by the formulas

$$\rho_f = \limsup_{n \to \infty} \frac{\log n}{\log(\frac{1}{\sqrt[n]{|c_n|}})} = \limsup_{r \to \infty} \frac{\log \log \mathcal{M}_f(r)}{\log r}. \tag{26}$$

If $0 < \rho < \infty$ the type $\tau = \tau_f$ is given as

$$\tau_f = \frac{1}{e\rho} \limsup_{n \to \infty} (n|c_n|^{\rho/n}) = \limsup_{r \to \infty} \frac{\log \mathcal{M}_f(r)}{r^\rho}. \tag{27}$$

If $0 < \tau < \infty$ we define the Phragmén-Lindelöf indicator function

$$h_f(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho}, \quad \theta \in \mathbb{R}. \tag{28}$$

For details see [7].

For two entire functions f, g it is easy to see that if

$$\mathcal{M}_f(r) \leq r^c \mathcal{M}_g(Kr) \tag{29}$$

for r sufficiently large, and c, K are suitable constants, then $\rho_f \leq \rho_g$. If in addition $0 < \rho = \rho_f = \rho_g < \infty$, then $\tau_f \leq K^\rho \tau_g$.

Changes of finitely many of the parameters $(a_n), (b_n)$ do not change the indeterminate moment problem in an essential way as expressed in the following classical result:

Proposition 2.2. *Consider an indeterminate moment problem corresponding to sequences $(a_n), (b_n)$ from (1). If another moment problem is given in terms of $(\tilde{a}_n), (\tilde{b}_n)$, and if $\tilde{a}_n = a_n, \tilde{b}_n = b_n$ for $n \geq n_0$, then the second problem is also indeterminate and the two problems have the same order, type and Phragmén-Lindelöf indicator function.*

For a proof one can refer to results about the abbreviated Jacobi matrix, cf. [1, p. 28] given by the sequences $(a_{n+1}), (b_{n+1})$. This problem is indeterminate with the same order, type and Phragmén-Lindelöf indicator function as the original problem. This follows from simple linear algebra considerations about solutions to difference equations of order 2. A complete proof can be found in [13].

We end with a simple but useful result about how the order, type and indicator function change, when the Jacobi parameters are multiplied by a constant $c > 0$.

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Proposition 2.3. *Consider an indeterminate Hamburger moment problem of order ρ and type τ corresponding to the sequences $(a_n), (b_n)$ from the three term recurrence relation. For $c > 0$ the moment problem corresponding to the sequences $(ca_n), (cb_n)$ is also indeterminate with order $\rho(c) = \rho$, type $\tau(c) = \tau/c^\rho$ and indicator $h(c)(\theta) = h(\theta)/c^\rho$.*

Proof. Let (P_n) and $(P_n(\cdot; c))$ denote the orthonormal polynomials corresponding to the two sets of Jacobi parameters. Then it is easy to see that $P_n(x; c) = P_n(x/c)$. This means that the D-functions from the Nevanlinna matrices satisfy $D(x; c) = cD(x/c)$, and from this the relations follow. $\qquad\square$

3 Symmetric Hamburger moment problems versus Stieltjes moment problems and birth and death processes

In this section we have collected some well-known facts about the relation between Stieltjes moment problems and symmetric Hamburger moment problems. For details see [8] and [3].

Let

$$\mathbb{M}^*(\mathbb{R}_+) = \{\sigma \in \mathbb{M}^*(\mathbb{R}) \mid \mathrm{supp}(\sigma) \subseteq [0, \infty[\,\} \,.$$

A non-degenerate Stieltjes moment sequence is a sequence of the form

$$s_n = \int_0^\infty x^n d\sigma(x), \quad n \geq 0, \tag{30}$$

where $\sigma \in \mathbb{M}^*(\mathbb{R}_+)$. It is called normalized if $s_0 = 1$.

The sequence (s_n) can be determinate or indeterminate in the sense of Stieltjes, denoted det(S) and indet(S) respectively, meaning that there is exactly one measure or more than one measure from $\mathbb{M}^*(\mathbb{R}_+)$ satisfying (30).

In the study of a Stieltjes moment problem it is useful to consider an accompanying symmetric Hamburger moment problem. Let $\mathbb{M}_s^*(\mathbb{R})$ denote the set of *symmetric* measures $\mu \in \mathbb{M}^*(\mathbb{R})$, i.e., $\mu(-B) = \mu(B)$ for Borel sets $B \subseteq \mathbb{R}$.

The map $\psi(x) = x^2$ induces a bijection of $\mathbb{M}_s^*(\mathbb{R})$ onto $\mathbb{M}^*(\mathbb{R}_+)$ given by $\sigma = \psi(\mu)$ (the image measure of μ under ψ), i.e., if $f : \mathbb{R}_+ \to \mathbb{C}$ is a bounded Borel function then

$$\int f(t) d\sigma(t) = \int f(x^2) d\mu(x).$$

A non-degenerate symmetric Hamburger moment sequence is of the form

$$t_n = \int_{-\infty}^\infty x^n \, d\mu(x), \quad n \geq 0, \tag{31}$$

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where $\mu \in \mathbb{M}_s^*(\mathbb{R})$. Clearly the odd moments vanish and $t_{2n} = s_n$ from (30), where $\sigma = \psi(\mu)$. Conversely, any non-degenerate Stieltjes moment sequence (30) arises in this way from a unique symmetric Hamburger moment sequence.

The Stieltjes moment problem is indet(S) if and only if the corresponding symmetric Hamburger problem is indeterminate. In case of indeterminacy there is a simple relation between the Nevanlinna matrices of the two problems. Let us just mention the following relation between the "C"-functions C_s and C from the Nevalinna matrix (23), namely: $C_s(z) = C(z^2)$, where C_s refers to the symmetric Hamburger problem and C to the Stieltjes problem. This yields the following result:

Proposition 3.1. *In the indeterminate case let ρ, τ, h respectively ρ_s, τ_s, h_s denote the order, type and Phragmén-Lindelöf indicator function of the Stieltjes problem respectively symmetric Hamburger problem. Then we have*

$$\rho = \rho_s/2, \ \tau = \tau_s, \ h(\theta) = h_s(\theta/2).$$

Let us recall the connection between the three term recurrence relation (in the notation of [1])

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1}P_{n-1}(z), \quad n \geq 0, \tag{32}$$

for a Stieltjes moment problem, where $a_n, b_n > 0$ for $n \geq 0$, and the three term recurrence relation

$$zS_n(z) = \beta_n S_{n+1}(z) + \beta_{n-1}S_{n-1}(z) \tag{33}$$

for the corresponding symmetric Hamburger moment problem. We have

$$a_n = \beta_{2n}^2 + \beta_{2n-1}^2, n \geq 1, \ a_0 = \beta_0^2, \quad b_n = \beta_{2n}\beta_{2n+1}, n \geq 0. \tag{34}$$

From this equation it is easy to calculate $(a_n), (b_n)$ from (β_n). Conversely, if $(a_n), (b_n)$ are given, then (β_n) is uniquely determined by the same equation.

If we compare (34) and (5), we see that any Stieltjes moment problem comes from a birth and death process with rates

$$\lambda_n = \beta_{2n}^2, \ \mu_{n+1} = \beta_{2n+1}^2, \ n \geq 0, \quad \mu_0 = 0.$$

From this we get

Proposition 3.2. *The symmetric Hamburger moment problem corresponding to the Stieltjes problem associated with the birth and death rates (6) and (7) has the recurrence coefficients*

$$b_{2n} = \sqrt{(pn + e_1)\ldots(pn + e_p)}, \quad b_{2n+1} = \sqrt{(p(n+1) + d_1)\ldots(p(n+1) + d_p)}. \tag{35}$$

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4 Some technical Lemmas

We are going to analyse the symmetric moment problem of Proposition 3.2 and for this we need some lemmas. By $\alpha_n \sim \beta_n$ is meant that $\alpha_n/\beta_n \to 1$ for $n \to \infty$.

Lemma 4.1. *For $x > -1$ and $n \in \mathbb{N}$ define*

$$\delta_n(x) = n \left(\frac{(x+1)\ldots(x+n)\Gamma(x+1)}{n!n^x} - 1 \right). \tag{36}$$

Then we have

$$\prod_{k=1}^{n} \left(1 + \frac{x}{k} \right) = \frac{n^x}{\Gamma(x+1)} \left(1 + \frac{\delta_n(x)}{n} \right) \tag{37}$$

and the following estimates hold for all $n \in \mathbb{N}$:

(i) $|\delta_n(x)| \leq 1$ *for* $-1 < x \leq 1$

(ii) *For any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that*

$$0 \leq \delta_n(x) \leq C_N \text{ for } 0 \leq x \leq N.$$

In particular for $x > -1$

$$\prod_{k=1}^{n} \left(1 + \frac{x}{k} \right) \sim \frac{n^x}{\Gamma(x+1)}, \quad n \to \infty. \tag{38}$$

Proof. For $0 < x \leq 1, n \in \mathbb{N}$ we have the following inequalities

$$\frac{n^x n!}{x(x+1)\ldots(x+n)} \leq \Gamma(x) \leq \frac{n^x n!}{x(x+1)\ldots(x+n)} \frac{x+n}{n},$$

as a consequence of the log-convexity of $\Gamma(x)$. (See the proof of the Bohr-Mollerup characterization of the Gamma function in [2, p. 14].) From this inequality we immediately get

$$1 \leq \frac{(x+1)\ldots(x+n)\Gamma(x+1)}{n^x n!} \leq 1 + \frac{x}{n}, \tag{39}$$

hence $0 \leq \delta_n(x) \leq x \leq 1$, which trivially holds for $x = 0$ also.

For $-1 < x \leq 0$ we apply (39) to $x + 1$ and get

$$1 \leq \frac{(x+1)\ldots(x+n)\Gamma(x+1)}{n^x n!} \frac{x+n+1}{n} \leq 1 + \frac{x+1}{n}.$$

After multiplication by $n/(n+x+1)$, subtraction of 1 and multiplication by n we get

$$0 \geq \delta_n(x) \geq -\frac{n(x+1)}{n+x+1} \geq -(x+1) \geq -1.$$

This shows (i).

Suppose that (ii) holds for some $N \in \mathbb{N}$ and let $N \leq x \leq N + 1$. Applying (ii) to $x - 1$ we get

$$0 \leq n \left(\frac{(x+1)\ldots(x+n)\Gamma(x+1)}{n^x n!} \frac{n}{x+n} - 1 \right) \leq C_N,$$

and then

$$0 \leq \delta_n(x) - x \leq \frac{x+n}{n} C_N,$$

showing that $0 \leq \delta_n(x) \leq C_{N+1} := N + 1 + (N+2)C_N.$ $\qquad \square$

Remark 4.2. The result of Lemma 4.1 can be improved, since it can be shown that $\delta_n(x) \to x(x+1)/2$ for $n \to \infty$. More precisely we have

$$\delta_n(x) = \frac{x(x+1)}{2} + O_x(n^{-1}),$$

where $nO_x(n^{-1})$ is bounded for x in a compact subset of $]-1,\infty[$ as $n \to \infty$.

The three term recurrence relation

$$z r_n(z) = b_n r_{n+1}(z) + b_{n-1} r_{n-1}(z)$$

for a symmetric moment problem is satisfied by $r_n(z) = P_n(z)$ and $r_n(z) = Q_n(z)$, and putting $z = 0$ and noting that $P_{2n+1}(0) = Q_{2n}(0) = 0$ we find

$$P_{2n}(0) = (-1)^n \frac{b_0 b_2 \ldots b_{2n-2}}{b_1 b_3 \ldots b_{2n-1}}, \quad Q_{2n+1}(0) = (-1)^n \frac{b_1 b_3 \ldots b_{2n-1}}{b_0 b_2 \ldots b_{2n}}, \quad n \geq 0, \quad (40)$$

where empty products are defined as 1.

When (b_n) is given by (35) we find

$$\frac{b_{2k-1}^2}{b_{2k}^2} = \prod_{j=1}^p \frac{1 + \frac{d_j}{pk}}{1 + \frac{e_j}{pk}}. \qquad (41)$$

This gives

$$Q_{2n+1}^2(0) = \frac{1}{b_0^2} \prod_{k=1}^n \frac{b_{2k-1}^2}{b_{2k}^2} = \frac{1}{e_1 \ldots e_p} \prod_{k=1}^n \frac{(1 + \frac{d_1}{pk}) \ldots (1 + \frac{d_p}{pk})}{(1 + \frac{e_1}{pk}) \ldots (1 + \frac{e_p}{pk})}$$

and

$$P_{2n}^2(0) = \prod_{k=1}^n \frac{b_{2k-2}^2}{b_{2k-1}^2} = \prod_{k=1}^n \frac{(1 + \frac{e_1 - p}{pk}) \ldots (1 + \frac{e_p - p}{pk})}{(1 + \frac{d_1}{pk}) \ldots (1 + \frac{d_p}{pk})}.$$

By Lemma 4.1 and (9) we find for $n \in \mathbb{N}$

$$Q_{2n+1}^2(0) = c_2 n^{-(E-D)/p} \prod_{j=1}^{p} \frac{1 + \delta_n(d_j/p)/n}{1 + \delta_n(e_j/p)/n}, \quad c_2 := p^{-p} \prod_{j=1}^{p} \frac{\Gamma(e_j/p)}{\Gamma(1 + d_j/p)} \quad (42)$$

and

$$P_{2n}^2(0) = c_1 n^{-[p-(E-D)/p]} \prod_{j=1}^{p} \frac{1 + \delta_n(e_j/p - 1)/n}{1 + \delta_n(d_j/p)/n}, \quad c_1 := \prod_{j=1}^{p} \frac{\Gamma(1 + d_j/p)}{\Gamma(e_j/p)}. \quad (43)$$

Note that

$$c_1 c_2 = p^{-p}, \quad (44)$$

and

$$P_{2n}^2(0) \sim c_1 n^{-[p-(E-D)/p]}, \quad Q_{2n+1}^2(0) \sim c_2 n^{-(E-D)/p}. \quad (45)$$

We need more precise information about the asymptotic behaviour in (45) and claim the following:

Lemma 4.3. *Given the rates (6) and (7) there exists a constant $K > 0$ and numbers τ_n, ρ_n such that*

$$P_{2n}^2(0) = c_1 n^{-[p-(E-D)/p]}(1 + \tau_n), \quad Q_{2n+1}^2(0) = c_2 n^{-(E-D)/p}(1 + \rho_n) \quad (46)$$

and

$$|\tau_n|, |\rho_n| \le K/n, \quad n \ge 1. \quad (47)$$

Proof. Define

$$\tau_n = \prod_{j=1}^{p} \frac{1 + \delta_n(e_j/p - 1)/n}{1 + \delta_n(d_j/p)/n} - 1, \quad \rho_n = \prod_{j=1}^{p} \frac{1 + \delta_n(d_j/p)/n}{1 + \delta_n(e_j/p)/n} - 1.$$

Then (46) holds. To see that (47) holds we write

$$x_{n,j} = \delta_n(e_j/p - 1), \ y_{n,j} = \delta_n(d_j/p), \quad j = 1, \dots, p$$

and by Lemma 4.1 the quantities $x_{n,j}, y_{n,j}$ are bounded independent of n. Putting

$$A_n = n \left(\prod_{j=1}^{p}(1 + x_{n,j}/n) - \prod_{j=1}^{p}(1 + y_{n,j}/n) \right), \quad B_n = \prod_{j=1}^{p}(1 + y_{n,j}/n),$$

we have $n\tau_n = A_n/B_n$ and

$$A_n = \sum_{j=1}^{p}(x_{n,j} - y_{n,j}) + \frac{1}{n} \sum_{j_1 < j_2} (x_{n,j_1} x_{n,j_2} - y_{n,j_1} y_{n,j_2})$$

$$+ \dots + \frac{1}{n^{p-1}}(x_{n,1} \dots x_{n,p} - y_{n,1} \dots y_{n,p}).$$

Since A_n is bounded and $B_n \to 1$, we see that $|\tau_n| \le K/n$ for a suitable constant $K > 0$ independent of n. A similar argument applies to ρ_n. $\square$

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A moment problem is indeterminate if and only if $(P_n(0)), (Q_n(0)) \in \ell^2$, cf. Theorem 2.1, and by (45) this is equivalent to

$$1 < (E - D)/p < p - 1,$$

which proves Valent's result in [17]: The Stieltjes problem with rates (6),(7) is indeterminate precisely when (10) holds.

5 Results about order and type for symmetric moment problems

We consider a symmetric indeterminate Hamburger moment problem given by the three term recurrence relation

$$z r_n(z) = b_n r_{n+1}(z) + b_{n-1} r_{n-1}(z),$$

for a sequence of positive numbers $b_n > 0$. As usual (P_n) and (Q_n) denote the orthonormal polynomials and those of the second kind. By the symmetry assumption $P_{2n+1}(0) = Q_{2n}(0) = 0$. Therefore (23) can be rewritten as

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \left[\prod_{n=1}^{\infty} (I + zT_{2n})(I + zT_{2n-1}) \right] \begin{pmatrix} 0 & -1 \\ 1 & z \end{pmatrix}. \tag{48}$$

For $n \geq 1$ we introduce

$$v_n = P_{2n}^2(0), \quad u_n = Q_{2n-1}^2(0) \tag{49}$$

and

$$V_n = \begin{pmatrix} 0 & 0 \\ v_n & 0 \end{pmatrix} = -T_{2n}, \quad U_n = \begin{pmatrix} 0 & u_n \\ 0 & 0 \end{pmatrix} = T_{2n-1}. \tag{50}$$

We then have

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \left[\prod_{n=1}^{\infty} (I - zV_n)(I + zU_n) \right] \begin{pmatrix} 0 & -1 \\ 1 & z \end{pmatrix}. \tag{51}$$

We observe that $U_m U_n = V_m V_n = 0$ and

$$U_n V_m = \begin{pmatrix} u_n v_m & 0 \\ 0 & 0 \end{pmatrix}, \quad V_n U_m = \begin{pmatrix} 0 & 0 \\ 0 & v_n u_m \end{pmatrix}. \tag{52}$$

We want to analyse the infinite product and its power series

$$M(z) = \begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix} := \prod_{n=1}^{\infty} (I - zV_n)(I + zU_n) = I + \sum_{n=1}^{\infty} M_n z^n.$$

Clearly

$$M_1 = \begin{pmatrix} 0 & \sum_{k=1}^{\infty} u_k \\ -\sum_{k=1}^{\infty} v_k & 0 \end{pmatrix}$$

and for $n \geq 1$

$$\begin{aligned} M_{2n} &= (-1)^n \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} V_{k_{2n}} U_{k_{2n-1}} \cdots V_{k_2} U_{k_1} \\ &+ (-1)^n \sum_{1 \leq k_1 < k_2 \leq \ldots \leq k_{2n-1} < k_{2n}} U_{k_{2n}} V_{k_{2n-1}} \cdots U_{k_2} V_{k_1}, \end{aligned}$$

because we shall consider all possible choices of $2n$ parentheses, where we select the term containing z, and because the product of two consecutive $U's$ or $V's$ gives zero, we have to alternate between $U's$ and $V's$, and this also determines the inequalities between the indices. From (52) we see that M_{2n} is a diagonal matrix. There is a similar expression for M_{2n+1} showing that the diagonal elements vanish.

Let us focus on the lower right corner of M_{2n}. This matrix entry is equal to

$$a_{2n} = (-1)^n \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} u_{k_1} v_{k_2} \cdots u_{k_{2n-1}} v_{k_{2n}}, \tag{53}$$

and by (51) we have

$$M(z) = \begin{pmatrix} A(z)z - B(z) & A(z) \\ C(z)z - D(z) & C(z) \end{pmatrix}$$

hence

$$m_{22}(z) = 1 + \sum_{n=1}^{\infty} a_{2n} z^{2n} = C(z).$$

Summing up we have proved:

Proposition 5.1. *Consider a symmetric indeterminate Hamburger moment problem and let $v_n = P_{2n}^2(0), u_n = Q_{2n-1}^2(0)$. The order, type and Phragmén-Lindelöf indicator of the moment problem is equal to the order, type and Phragmén-Lindelöf indicator of the function*

$$m_{22}(z) = 1 + \sum_{n=1}^{\infty} a_{2n} z^{2n}, \tag{54}$$

where a_{2n} is given in (53).

Proof of Theorem 1.2.
If $\alpha = \beta$ the result follows from [5, Theorem 4.7].
Suppose next that $\alpha > \beta$.

For $i \leq j$ we have

$$u_i v_j \leq C v_i^{\beta/\alpha} v_j \leq C D^{\beta(1/\gamma - 1/\alpha)} v_i^{\beta/\gamma} v_j^{\beta/\gamma},$$

because the last inequality is equivalent to

$$v_j^{\beta(1/\beta - 1/\gamma)} \leq (D v_i)^{\beta(1/\gamma - 1/\alpha)},$$

which is true by (iii) because $\beta(1/\gamma - 1/\alpha) = \beta(1/\beta - 1/\gamma) > 0$ as a consequence of (4). Therefore

$$|a_{2n}| = \sum_{1 \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} u_{k_1} v_{k_2} \ldots u_{k_{2n-1}} v_{k_{2n}}$$

$$\leq (C D^{\beta(1/\gamma - 1/\alpha)})^n \sum_{1 \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} \left(v_{k_1} v_{k_2} \cdots v_{k_{2n-1}} v_{k_{2n}} \right)^{\beta/\gamma}.$$

The last sum is majorized by the power series coefficient to z^{2n} in

$$F(z) = \prod_{n=1}^{\infty} \left(1 + v_n^{\beta/\gamma} z \right)^2. \tag{55}$$

This shows that

$$\mathcal{M}_{m_{22}}(r) = 1 + \sum_{n=1}^{\infty} |a_{2n}| r^{2n} \leq \mathcal{M}_F(Kr),$$

where $K = (C D^{\beta(1/\gamma - 1/\alpha)})^{1/2}$, and therefore $\rho = \rho_{m_{22}} \leq \rho_F$, see the explanation after (29).

Since $(v_n^{\beta/\gamma}) \in \ell^\gamma$, the zeros of $F(z)$ has exponent of convergence $\leq \gamma$ and therefore $\rho_F \leq \gamma$, cf. [7], and we conclude that $\rho \leq \gamma$.

Suppose finally that $\alpha < \beta$. Introducing $D_u = \sum_{k=1}^{\infty} u_k$, $D_v = \sum_{k=1}^{\infty} v_k$ we get

$$|a_{2n}| \leq D_u D_v \sum_{1 \leq k_2 < k_3 \leq ... \leq k_{2n-2} < k_{2n-1}} v_{k_2} u_{k_3} \ldots v_{k_{2n-2}} u_{k_{2n-1}}.$$

By (ii) and (iii) we similarly get for $i < j$

$$v_i u_j \leq C v_i v_j^{\beta/\alpha} \leq C D^{\beta/\gamma - 1} v_i^{\beta/\gamma} v_j^{\beta/\gamma},$$

so

$$|a_{2n}| \leq D_u D_v (C D^{\beta/\gamma - 1})^{n-1} \sum_{1 \leq k_2 < k_3 \leq ... \leq k_{2n-2} < k_{2n-1}} \left(v_{k_2} v_{k_3} \ldots v_{k_{2n-2}} v_{k_{2n-1}} \right)^{\beta/\gamma}.$$

The last sum is majorized by the power series coefficient to z^{2n-2} in $F(z)$ given by (55). Again this shows that $\rho \leq \gamma$. $\quad\square$

Lemma 5.2. *Consider a symmetric indeterminate moment problem of order $0 < \rho < 1$ and type τ, and assume that the recurrence coefficients (b_n) satisfy*

$$b_n \leq C(n+1)^\kappa$$

for some $C > 0$ and κ (necessarily > 1 by Carleman's criterion). Then $\rho \geq 1/\kappa$ and if $\rho = 1/\kappa$ then $\tau \geq \kappa C^{-1/\kappa}$.

Proof. We need the coefficients of the orthonormal polynomials

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k$$

and the quantity

$$c_n = \left(\sum_{j=n}^{\infty} b_{n,j}^2 \right)^{1/2}$$

studied in [5, p. 112], where it is proved that

$$\sum_{n=0}^{\infty} r^{2n} c_n^2 = \frac{1}{2\pi} \int_0^{2\pi} P^2(re^{it}) \, dt, \quad r \geq 0.$$

Here P is the function from Theorem 2.1 and it has order ρ and type τ by assumption, cf. [4]. Using that $c_n > b_{n,n} = 1/(b_0 \ldots b_{n-1})$, we get

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{b_0^2 b_1^2 \ldots b_{n-1}^2} \leq \frac{1}{2\pi} \int_0^{2\pi} P^2(re^{it}) \, dt,$$

hence by the assumption about (b_n)

$$\sum_{n=0}^{\infty} \frac{(r/C)^{2n}}{(n!)^{2\kappa}} \leq \frac{1}{2\pi} \int_0^{2\pi} P^2(re^{it}) \, dt.$$

The entire function

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^{2\kappa}}$$

has order $1/\kappa$ and type 2κ by (26) and (27) and the result follows. □

Remark 5.3. In a previous version of Lemma 5.2 we needed the assumption that b_n is eventually increasing. The referee informed us that this condition is not necessary and suggested a variational proof of the simplified result, also indicating that one can obtain results about type in this way. After this remark we realized that our proof could be modified to the present version.

We now return to the symmetric indeterminate moment problem with Nevanlinna matrix (51) and recall the notation

$$v_n = P_{2n}^2(0), \ u_n = Q_{2n-1}^2(0), \quad n \geq 1.$$

We assume that

$$v_n = c_1 n^{-1/\beta}(1 + \tau_n), \quad u_n = c_2 n^{-1/\alpha}(1 + \rho_n), \tag{56}$$

for constants $c_1, c_2 > 0, 0 < \alpha, \beta < 1$, and assume further that there exists a constant $K > 0$ such that $|\tau_n|, |\rho_n| \leq K/n$ for $n \geq 1$.

Note that by Lemma 4.3 all this is satisfied for the symmetric Hamburger moment problem of Proposition 3.2.

For $a \in \mathbb{N}$ consider

$$\sigma_n(a) = \sum_{a \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} u_{k_1} v_{k_2} \ldots u_{k_{2n-1}} v_{k_{2n}} \tag{57}$$

and

$$s_n(a) = \sum_{a \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} k_1^{-1/\alpha} k_2^{-1/\beta} \ldots k_{2n-1}^{-1/\alpha} k_{2n}^{-1/\beta}. \tag{58}$$

Note that $|a_{2n}| = \sigma_n(1)$, cf. (53).
Under these assumptions we have:

Lemma 5.4. *For $a \in \mathbb{N}$*

$$\lim_{n \to \infty} \sqrt[n]{\frac{\sigma_n(1)}{s_n(a)}} = c_1 c_2, \tag{59}$$

so the entire functions

$$m_{22}(z) = 1 + \sum_{n=1}^{\infty}(-1)^n \sigma_n(1)z^{2n}, \quad T_a(z) = \sum_{n=1}^{\infty} s_n(a)(c_1 c_2)^n z^{2n} \tag{60}$$

have the same order and type.

Proof. We have

$$s_n(1) \geq s_n(a)$$

$$= \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} (k_1 + a - 1)^{-1/\alpha}(k_2 + a - 1)^{-1/\beta} \ldots (k_{2n} + a - 1)^{-1/\beta}$$

$$= \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} k_1^{-1/\alpha} k_2^{-1/\beta} \ldots k_{2n}^{-1/\beta}$$

$$\times (1 + (a-1)/k_1)^{-1/\alpha}(1 + (a-1)/k_2)^{-1/\beta} \ldots (1 + (a-1)/k_{2n})^{-1/\beta}$$

$$\geq \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} k_1^{-1/\alpha} k_2^{-1/\beta} \ldots k_{2n}^{-1/\beta} \prod_{j=1}^{n}(1 + (a-1)/j)^{-1/\alpha - 1/\beta}$$

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since $k_2 \geq k_1 \geq 1$, $k_4 \geq k_3 \geq 2$, ..., $k_{2n} \geq k_{2n-1} \geq n$.

However, by Lemma 4.1

$$\prod_{j=1}^{n}(1 + (a-1)/j) = \frac{n^{a-1}}{\Gamma(a)}\left(1 + \frac{\delta_n(a-1)}{n}\right) \leq L(a)n^{a-1}$$

for some constant $L(a)$ depending on a. We therefore get

$$s_n(1) \geq s_n(a) \geq s_n(1)\left(L(a)n^{a-1}\right)^{-2/\gamma}, \tag{61}$$

showing that for $a \in \mathbb{N}$

$$(s_n(1))^{1/n} \sim (s_n(a))^{1/n}, \quad n \to \infty. \tag{62}$$

We also have

$$\sigma_n(1) \geq \sigma_n(a)$$
$$= (c_1 c_2)^n \sum_{a \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} k_1^{-1/\alpha}k_2^{-1/\beta}\ldots k_{2n-1}^{-1/\alpha}k_{2n}^{-1/\beta}$$
$$\times (1 + \rho_{k_1})(1 + \tau_{k_2})\ldots(1 + \rho_{k_{2n-1}})(1 + \tau_{k_{2n}}).$$

Given $\varepsilon > 0$ there exists $a \in \mathbb{N}$ such that $|\tau_k|, |\rho_k| \leq \varepsilon$ for $k \geq a$. For such a we then have

$$\sigma_n(1) \geq \sigma_n(a) \geq (c_1 c_2)^n s_n(a)(1 - \varepsilon)^{2n},$$

so by (61)

$$\sigma_n(1) \geq (c_1 c_2)^n s_n(1)\left(L(a)n^{a-1}\right)^{-2/\gamma}(1 - \varepsilon)^{2n}.$$

Since $\varepsilon > 0$ is arbitrary we find

$$\liminf_{n\to\infty}\left(\frac{\sigma_n(1)}{s_n(1)}\right)^{1/n} \geq c_1 c_2. \tag{63}$$

On the other hand we have

$$\sigma_n(1) \leq (c_1 c_2)^n \sum_{1 \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} k_1^{-1/\alpha}k_2^{-1/\beta}\ldots k_{2n-1}^{-1/\alpha}k_{2n}^{-1/\beta}$$
$$\times (1 + K/k_1)(1 + K/k_2)\ldots(1 + K/k_{2n-1})(1 + K/k_{2n})$$
$$\leq (c_1 c_2)^n s_n(1) \prod_{j=1}^{n}(1 + K/j)^2 = (c_1 c_2)^n s_n(1)\left(\frac{n^K}{\Gamma(K+1)}(1 + \delta_n(K)/n)\right)^2,$$

where we have used Lemma 4.1.

We then get

$$\limsup_{n\to\infty}\left(\frac{\sigma_n(1)}{s_n(1)}\right)^{1/n} \leq c_1 c_2,$$

which together with (63) proves that

$$\sigma_n(1)^{1/n} \sim s_n(1)^{1/n} c_1 c_2. \tag{64}$$

Combining (62) and (64) we get (59). Using (26) and (27) it follows from (59) that the functions (60) have the same order and type. $\square$

Under the same assumptions as in Lemma 5.4 we have:

Lemma 5.5. *For $n \in \mathbb{N}$ define*

$$\gamma_n = \sum_{1 \le k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} (k_1 k_2 \ldots k_{2n-1} k_{2n})^{-1/\gamma}, \tag{65}$$

where γ is the harmonic mean of α and β, i.e., $\gamma_n = \gamma_n(2/\gamma)$ from (12). Then

$$\sqrt[n]{\gamma_n} \sim \sqrt[n]{s_n(1)} \tag{66}$$

so the entire functions

$$H(z) = \sum_{n=1}^{\infty} s_n(1) z^{2n}, \quad G(z) = \sum_{n=1}^{\infty} \gamma_n z^{2n}$$

have the same order and type.

Proof. If $\alpha = \beta$ there is nothing to prove.

Suppose next that $\alpha > \beta$, hence $1/\alpha < 1/\gamma < 1/\beta$. From the inequalities

$$k_{2j-1}^{-1/\alpha} k_{2j}^{-1/\beta} \le k_{2j-1}^{-1/\gamma} k_{2j}^{-1/\gamma}, \quad j = 1, \ldots, n,$$

which hold because

$$k_{2j-1}^{1/\gamma - 1/\alpha} \le k_{2j}^{1/\beta - 1/\gamma}, \quad j = 1, \ldots, n,$$

we get $s_n(1) \le \gamma_n$.

We also have

$$
\begin{aligned}
s_n(1) &\ge \zeta(1/\alpha)^{-1} \sum_{1 \le k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n} < k_{2n+1}} k_1^{-1/\alpha} k_2^{-1/\beta} \ldots k_{2n-1}^{-1/\alpha} k_{2n}^{-1/\beta} k_{2n+1}^{-1/\alpha} \\
&\ge \zeta(1/\alpha)^{-1} \sum_{1 \le k_2 < \ldots < k_{2n-1} \le k_{2n} < k_{2n+1}} k_2^{-1/\beta} \ldots k_{2n-1}^{-1/\alpha} k_{2n}^{-1/\beta} k_{2n+1}^{-1/\alpha},
\end{aligned}
$$

where we have added the extra summation variable k_{2n+1} and then put $k_1 = 1$. Using

$$k_{2j}^{-1/\beta} k_{2j+1}^{-1/\alpha} \ge k_{2j}^{-1/\gamma} k_{2j+1}^{-1/\gamma}, \quad j = 1, \ldots, n,$$

we get

$$s_n(1) \geq \zeta(1/\alpha)^{-1} \sum_{1 \leq k_2 < ... < k_{2n-1} \leq k_{2n} < k_{2n+1}} (k_2 k_3 \ldots k_{2n} k_{2n+1})^{-1/\gamma}.$$

Applying the same procedure once more we get

$$s_n(1) \geq (\zeta(1/\alpha)\zeta(1/\gamma))^{-1} \sum_{1 < k_3 \leq k_4 < ... < k_{2n+1} \leq k_{2n+2}} (k_3 \ldots k_{2n+2})^{-1/\gamma}.$$

Substituting $k_{j+2} = l_j + 1, j = 1, \ldots, 2n$ and then renaming l_j to k_j we get

$$
\begin{aligned}
s_n(1) \;\geq\;& (\zeta(1/\alpha)\zeta(1/\gamma))^{-1} \sum_{1 \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} ((k_1 + 1) \ldots (k_{2n} + 1))^{-1/\gamma} \\
=\;& (\zeta(1/\alpha)\zeta(1/\gamma))^{-1} \sum_{1 \leq k_1 \leq k_2 < ... < k_{2n-1} \leq k_{2n}} (k_1 \ldots k_{2n})^{-1/\gamma} \\
& \times \;\; ((1 + 1/k_1) \ldots (1 + 1/k_{2n}))^{-1/\gamma} \\
\geq\;& (\zeta(1/\alpha)\zeta(1/\gamma))^{-1}(4n^2)^{-1/\gamma} \gamma_n,
\end{aligned}
$$

where we have used that

$$\prod_{j=1}^{n} (1 + 1/k_{2j-1})(1 + 1/k_{2j}) \leq \prod_{j=1}^{n} (1 + 1/j)^2 = (n + 1)^2 \leq 4n^2.$$

Summing up we have

$$(\zeta(1/\alpha)\zeta(1/\gamma))^{-1}(4n^2)^{-1/\gamma} \gamma_n \leq s_n(1) \leq \gamma_n, \tag{67}$$

and (66) follows.

Suppose finally that $\alpha < \beta$, hence $1/\beta < 1/\gamma < 1/\alpha$.
We then get

$$
\begin{aligned}
s_n(1) \;\leq\;& \zeta(1/\alpha)\zeta(1/\beta) \sum_{1 \leq k_2 < k_3 \leq ... < k_{2n-1}} k_2^{-1/\beta} k_3^{-1/\alpha} \ldots k_{2n-1}^{-1/\alpha} \\
\leq\;& \zeta(1/\alpha)\zeta(1/\beta) \sum_{1 \leq k_2 < k_3 \leq ... < k_{2n-1}} (k_2 k_3 \ldots k_{2n-1})^{-1/\gamma}
\end{aligned}
$$

because
$$k_{2j}^{-1/\beta} k_{2j+1}^{-1/\alpha} \leq (k_{2j} k_{2j+1})^{-1/\gamma}, \quad j = 1, \ldots, n - 1.$$

Applying the same procedure once more we get

$$
\begin{aligned}
s_n(1) \;\leq\;& \zeta(1/\alpha)\zeta(1/\beta)\zeta^2(1/\gamma) \sum_{1 < k_3 \leq k_4 < ... < k_{2n-3} \leq k_{2n-2}} (k_3 \ldots k_{2n-2})^{-1/\gamma} \\
\leq\;& \zeta(1/\alpha)\zeta(1/\beta)\zeta^2(1/\gamma) \gamma_{n-2}.
\end{aligned}
$$

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On the other hand, it is easy to see, by the same methods as before, that $s_n(1) \geq \gamma_n$, so summing up we have

$$\gamma_n \leq s_n(1) \leq \zeta(1/\alpha)\zeta(1/\beta)\zeta^2(1/\gamma)\gamma_{n-2}, \tag{68}$$

and (66) follows. Using (26) and (27) it follows that the two functions have the same order and type. $\qquad\square$

6   On Valent's conjecture about order and type

In this section we shall apply our results about symmetric moment problems.

Proof of Theorem 1.4.

By Proposition 3.1 it suffices to prove that the order ρ of the symmetric Hamburger moment problem of Proposition 3.2 is equal to $2/p$.

By (45) we have
$$v_n \sim c_1 n^{-1/\beta}, \quad u_n \sim c_2 n^{-1/\alpha}$$

with
$$\beta = (p - (E - D)/p)^{-1}, \quad \alpha = p/(E - D),$$

and c_1, c_2 are given by (43),(42). The harmonic mean of α and β is $\gamma = 2/p$.

For $\varepsilon > 0$ we define $\alpha_\varepsilon = \alpha + \varepsilon, \beta_\varepsilon = (1 + \varepsilon/\alpha)\beta$ and see that $(u_n) \in \ell^{\alpha_\varepsilon}, (v_n) \in \ell^{\beta_\varepsilon}$. Note that $\beta_\varepsilon/\alpha_\varepsilon = \beta/\alpha$, so the condition (ii) of Theorem 1.2 is satisfied for $\alpha_\varepsilon, \beta_\varepsilon$, while (iii) holds because of the asymptotic behaviour of (v_n).

We then get that $\rho \leq \gamma_\varepsilon$, where γ_ε is the harmonic mean of α_ε and β_ε. Letting $\varepsilon \to 0$ we get $\rho \leq \gamma = 2/p$. That $\rho \geq 2/p$ follows from Lemma 5.2 because the recurrence coefficients (b_n) given by (35) have the asymptotic behaviour

$$b_n \sim (p/2)^{p/2} n^{p/2}.$$

$\qquad\square$

Remark 6.1. The type part of Lemma 5.2 can be applied to yield $\tau \geq (p/2)C^{-2/p}$ for any $C > (p/2)^{p/2}$, hence $\tau \geq 1$. This estimate is however weaker than the lower estimate of Theorem 1.7.

The special case $p = 3$ and $(e_j) = (1, 2, 2), (d_j) = (0, 0, 1)$ of the rates (6), (7) was considered in [17] and with more details in [9]. In this case

$$\lambda_n < \sqrt{\mu_n \mu_{n+1}}, n \geq 2, \quad \mu_{n+1} > \sqrt{\lambda_n \lambda_{n+1}}, n \geq 1$$

so (b_n) in (35) is alternately log-convex and log-concave.

There is a special case where the condition of log-concavity is true, and in this case the Valent conjecture concerning order follows from the results in [5].

Proposition 6.2. *Consider the special case of the rates (6), (7), where $d_j = e_j - p/2$, $j = 1, \ldots, p$, $p \geq 3$. Then the sequence (b_n) in (35) is eventually log-concave.*

Proof. Note that $E - D = p^2/2$ and therefore (10) holds. The function

$$f(x) = \prod_{j=1}^{p}(px + e_j)$$

satisfies $f(n) = \lambda_n$ and $f(n - 1/2) = \mu_n$, and since $f(x)$ is log-concave for $x > -e_1/p$, it follows that (b_n) is log-concave. Using that

$$b_n \sim (p/2)^{p/2} n^{p/2},$$

we see that the exponent of convergence of (b_n) is $2/p$, so the symmetric Hamburger moment problem has order $2/p$ by Theorem 1.1. $\qquad\square$

The special case $p = 4$ and $(e_j) = (1, 2, 2, 3)$, $(d_j) = (-1, 0, 0, 1)$ satisfies the conditions of Proposition 6.2 and was studied in [6].

Proof of Theorem 1.5.
We apply Lemma 5.4 and Lemma 5.5 to the symmetric moment problem from Proposition 3.2. In this case

$$\beta = (p - (E - D)/p)^{-1}, \quad \alpha = p/(E - D), \quad c_1 c_2 = p^{-p}, \quad \gamma = 2/p.$$

By Theorem 1.4 we see that the common order of the functions (60) is $\rho = 2/p$.
Since $T_1(z) = H((c_1 c_2)^{1/2} z)$ the type τ of the functions (60) is

$$\tau = \tau_{T_1} = (c_1 c_2)^{\rho/2} \tau_H = \tau_H/p.$$

Finally, by Lemma 5.5 we have that $\tau = \tau_H/p = \tau_G/p$.
The type of the Stieltjes problem with rates (6) and (7) is by Proposition 3.1 equal to $\tau_H/p = \tau_G/p$. Finally note that G_p given by (13) and $G(z) = \sum_{n=1}^{\infty} \gamma_n z^{2n}$ from Lemma 5.5 have orders and types $\rho_G = 2\rho_{G_p}$ and $\tau_G = \tau_{G_p}$. $\qquad\square$

Proof of Theorem 1.7. In the definition (12) of $\gamma_n(p)$ we choose $k_1 = k_2 = l_1, k_3 = k_4 = l_2, \ldots, k_{2n-1} = k_{2n} = l_n$ and get

$$\gamma_n(p) > \sum_{1 \leq l_1 < l_2 < \ldots < l_n} (l_1 l_2 \ldots l_n)^{-p},$$

showing that for $r > 0$

$$\prod_{n=1}^{\infty}\left(1 + \frac{r^2}{n^p}\right) < 1 + \sum_{n=1}^{\infty} \gamma_n(p) r^{2n} < \prod_{n=1}^{\infty}\left(1 + \frac{r}{n^{p/2}}\right)^2. \qquad (69)$$

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The three expressions above are the maximum moduli for three entire functions of the same order $2/p$, since it is known from [7] that the canonical product

$$c(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{1/\rho}}\right), \quad 0 < \rho < 1,$$

has order $\rho_c = \rho$ and type $\tau_c = \pi/\sin(\pi\rho)$. From (69) we then get the following inequalities between the types of the three functions

$$\frac{\pi}{\sin(\pi/p)} \le \tau_p \le \frac{2\pi}{\sin(2\pi/p)} = \frac{\pi}{\sin(\pi/p)\cos(\pi/p)},$$

and dividing with p yields the inequality (14). $\quad\square$

Valent conjectured that the type is

$$T = \int_0^1 \frac{du}{(1 - u^p)^{2/p}} = \frac{1}{p} B(1/p, 1 - 2/p)$$

We have

$$T = \frac{(1/p)\Gamma(1/p)\Gamma(1 - 2/p)}{\Gamma(1 - 1/p)} = \frac{\pi/p}{\sin(\pi/p)} \frac{\Gamma(1 - 2/p)}{\Gamma^2(1 - 1/p)}, \tag{70}$$

where we have used Euler's reflection formula for Γ.

We claim that T lies between the bounds for the type τ given in (14).

The upper bound comes from

$$T < \int_0^1 \frac{du}{(1 - u^{p/2})^{2/p}} = \frac{2\pi}{p\sin(2\pi/p)},$$

where the integral is evaluated by transforming it to a Beta-integral and using the reflection formula.

Using (70) the lower bound is equivalent to

$$\Gamma^2(1 - 1/p) \le \Gamma(1 - 2/p), \quad p = 3, 4, \ldots,$$

which follows from the log-convexity of Γ:

$$\Gamma(\alpha x + (1 - \alpha)y) \le \Gamma^\alpha(x)\Gamma^{1-\alpha}(y)$$

for $\alpha = 1/2$, $x = 1, y = 1 - 2/p$.

Proof of Theorem 1.8. We have

$$\gamma_n(p) \le \sum_{1 \le k_1 < k_3 < \ldots < k_{2n-3}} (k_1 k_3 \ldots k_{2n-3})^{-p}(k_3 - k_1)\ldots(k_{2n-3} - k_{2n-5})$$

$$\times \sum_{k_{2n-3} < k_{2n-1} \le k_{2n}} (k_{2n-1} k_{2n})^{-p/2},$$

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where we have used that
$$(k_{2j-1}k_{2j})^{-p/2} \le k_{2j-1}^{-p}$$
for the $k_{2j+1} - k_{2j-1}$ values of $k_{2j} \in [k_{2j-1}, k_{2j+1}[$, $j = 1, \ldots, n-2$.

We next use
$$\sum_{k=a}^{\infty} k^{-p/2} \le \int_{a-1}^{\infty} x^{-p/2}\,dx = \frac{(a-1)^{-p/2+1}}{p/2-1} \le \frac{2^{p/2-1}}{p/2-1} a^{-p/2+1},$$

where $a \ge 2$, hence $a - 1 \ge a/2$, so for $a = k_{2n-1}$ we get

$$\gamma_n(p) \le \frac{2^{p/2-1}}{p/2-1} \sum_{1 \le k_1 < k_3 < \ldots < k_{2n-1}} (k_1 k_3 \ldots k_{2n-1})^{-p}(k_3 - k_1) \ldots (k_{2n-3} - k_{2n-5})k_{2n-1}$$

and by changing symbols

$$\gamma_n(p) \le \frac{2^{p/2-1}}{p/2-1} \sum_{1 \le k_1 < k_2 < \ldots < k_n} (k_1 k_2 \ldots k_n)^{-p}(k_2 - k_1) \ldots (k_{n-1} - k_{n-2})k_n$$

$$\le \zeta(p-1) \frac{2^{p/2-1}}{p/2-1} \sum_{1 \le k_1 < k_2 < \ldots < k_{n-1}} (k_1 k_2 \ldots k_{n-1})^{-p}(k_2 - k_1) \ldots (k_{n-1} - k_{n-2}).$$

This shows the right-hand side of (16).

On the other hand we have

$$\gamma_n(p) \ge \sum_{1 \le k_2 < k_4 < \ldots < k_{2n}} (k_2 k_4 \ldots k_{2n})^{-p} k_2 (k_4 - k_2) \ldots (k_{2n} - k_{2n-2}),$$

where we have used that
$$(k_{2j-1}k_{2j})^{-p/2} \ge k_{2j}^{-p}$$
for the $k_{2j} - k_{2j-2}$ values of $k_{2j-1} \in]k_{2j-2}, k_{2j}]$, $j = 2, \ldots, n$, respectively k_2 values of k_1 when $j = 1$.

By changing symbols we can rewrite the last inequality as

$$\gamma_n(p) \ge \sum_{1 \le k_1 < k_2 < \ldots < k_n} (k_1 k_2 \ldots k_n)^{-p} k_1 (k_2 - k_1) \ldots (k_n - k_{n-1}),$$

which shows the left-hand side of (16). $\square$

Proof of Theorem 1.11 From the proof of Proposition 2.2 we see that changing (b_n) from n^c to $(n+1)^c$ will not change the order and type. For $b_n = (n+1)^c$ we find from (40)

$$P_{2n}^2(0) = \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^{2c}, \quad Q_{2n+1}^2(0) = \left(\frac{2^{2n}(n!)^2}{(2n+1)!}\right)^{2c}.$$

Using Stirling's formula we then get

$$v_n = \pi^{-c} n^{-c} (1 + O(1/n)), \quad u_n = (\pi/4)^c n^{-c} (1 + O(1/n)),$$

so we see that v_n, u_n satisfy the conditions of (56) with $\alpha = \beta = \gamma = 1/c$ and $c_1 c_2 = 2^{-2c}$.

From Lemma 5.4 we see that the type of the problem is equal to the type of the function

$$T_1(z) = \sum_{n=1}^{\infty} s_n(1)(2^{-c}z)^{2n},$$

where $s_n(1) = \gamma_n(2c)$. Since $T_1(z)$ has order $1/c$, we see that the type of the function $\sum \gamma_n(2c) z^{2n}$ is the double of the type of $T_1(z)$, and the result follows.
$\square$

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