Transportation problem

A company produces copy paper in 4 different factories. The paper is delivered to 5 wholesale centers. Each factory can produce a fixed amount of paper per month and each center has a fixed demand of paper per month. These figures are given in two tables below.

Factory	Production (in tons)
1	200
2	280
3	300
4	150

Wholesale center	Demand (in tons)
1	100
2	160
3	350
4	100
5	220

Observe that the total production amounts to 930 tons and is equal to the total demand of the wholesale centers. The managing director is in charge of planning delivery from the factories to the wholesale centers so as to minimize the cost of transportation. For this purpose he determined the cost of delivery (in Euro) of one ton of the paper from each factory to each wholesale center. He has put these data in the table below.

	wsc1	wsc2	wsc3	wsc4	wsc5
f1	100	120	80	150	100
f2	120	90	180	130	130
f3	100	100	120	200	120
f4	150	120	160	120	80

In order to facilitate the planning we will put all the data we have in one table. For simplicity the supplies and the demands are given in tens of tons, and the costs are given in tens of Euro. The rows of the table correspond to the factories while the columns corresponds to the wholesale centers.

10	12	8	15	10	20
12	9	18	13	13	28
10	10	12	20	12	30
15	12	16	12	8	15
10	16	35	10	22	

The director has planned the delivery of the paper according to the following scheme.

10 / 5	12 / 0	8 / 10	15 / 0	10 / 5	20
12 / 0	9 / 6	18 / 0	13 / 10	13 / 12	28
10 / 5	10 / 10	12 / 15	20 / 0	12 / 0	30
15 / 0	12 / 0	16 / 10	12 / 0	8 / 5	15
10	16	35	10	22	

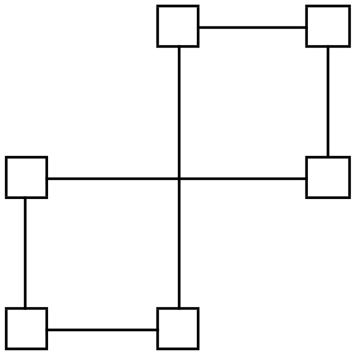
The numbers in the cells represent the amount of the paper which is to be dispatched monthly from a given factory to a given wholesale center. For example the delivery from the factory *f*3 to the center *wsc*2 will be 10. The sum of the numbers in the first row of the table is equal to the amount of the paper dispatched from the factory *f*1. That's why this sum is equal 20. On the other hand the sum of the numbers in the first column is the amount of paper delivered to the wholesale center *h*1. Hence this figure is equal to 10. The same is valid for all other rows and columns.

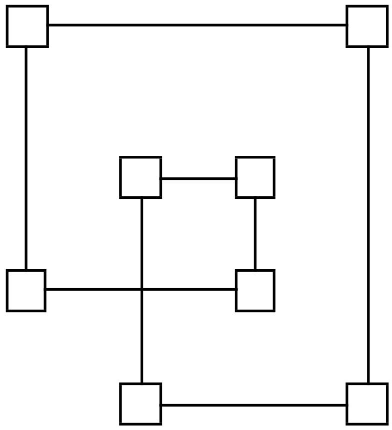
The question arises whether this plan is optimal ? The cost of monthly delivery of director's plan is 105 000 Euro. Does there exist another plan for which the cost is lower than 105 000 Euro ?

Consider the cells indicated in the table by framing the delivery numbers.

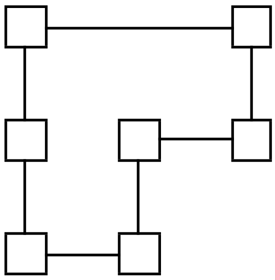
10 / **5**	12 / 0	8 / 10	15 / 0	10 / **5**	20
12 / 0	9 / 6	18 / 0	13 / 10	13 / 12	28
10 / **5**	10 / 10	12 / **15**	20 / 0	12 / 0	30
15 / 0	12 / 0	16 / **10**	12 / 0	8 / **5**	15
10	16	35	10	22	

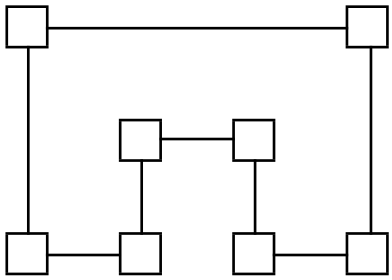
The set of cells indicated in the table forms a cycle. We have other examples of cycles in the figures below.





On the other hand configuration shown in the following figures are not cycles, because they contain more than two cells in one line.





Now we are going to modify the plan by changing the delivery numbers in the cells of the cycle as the next table shows.

10 / +5 / 5	12 / 0	8 / 10	15 / 0	10 / −5 / 5	20
12 / 0	9 / 6	18 / 0	13 / 10	13 / 12	28
10 / −5 / 5	10 / 10	12 / +5 / 15	20 / 0	12 / 0	30
15 / 0	12 / 0	16 / −5 / 10	12 / 0	8 / +5 / 5	15
10	16	35	10	22	

Before we switch to the new plan, we first determine if it is any better than the initial plan. Let's compute the change of the cost of paper delivery. This change is equal

$5\times10 + (-5)\times10 + 5\times12 + (-5)\times16 + 5\times8$
$+ (-5)\times10 = 5\times(10 - 10 + 12 - 16 + 8 - 10)$
$= 5\times(-6)=-30$

This means that modified plan reduces the cost by 3 000 Euro. If we made the opposite modification of the delivery, i.e. we added 5 to the cells for which we subtracted 5, and we subtracted 5 to the cells for which we added 5, the cost would increase by 3 000 Euro.

A cell with positive delivery number will be called a positive cell.

This example shows that if the delivery plan contains a configuration of positive cells which forms a cycle, this plan can be modified to a new plan, whose cost is lower or equal to the cost of the original plan. Moreover the new plan does not contain this cycle, because after the change the delivery number in one of the cells of the cycle will become 0. Therefore every delivery plan can be modified in such a way that the new plan does not contain a cycle of positive cells and the cost is reduced.

In order to carry it out we move the unit along the cycle and we check whether the cost decreases, increases or remains the same. In case it increases we perform the opposite change. Next we shift along the cycle as much units of the commodity as possible. In this way the cycle will be eliminated.

It is obvious that if there are many positive cells in the table the chances of getting a cycle is high. It can be shown (exercise) that if a plan does not contain a cycle then the number of positive cells can be at most 8, in the case of 4 suppliers and 5 receivers.

Let's try to find such a plan with no cycles in our example. After modification performed before the new plan looks as follows.

10 / 10	12	8 / 10	15	10	20
12	9 / 6	18	13 / 10	13 / 12	28
10	10 / 10	12 / 20	20	12	30
15	12	16 / 5	12	8 / 10	15
10	16	35	10	22	

We can still find another cycle. We eliminate it by the method described before and we arrive at another plan, whose cost is lower by yet another 50 times 10 tons times 10 Euro = 5 000 Euro.

10 / 10	12	8 / 10	15	10	20
12	9 / 11	18	13 / 10	13 / 7	28
10	10 / 5	12 / 25	20	12	30
15	12	16	12	8 / 15	15
10	16	35	10	22	

The present plan does not contain a cycle of positive cells and its cost is less than the cost of director's plan by 8 000 Euro. Is this plan optimal ? In order to determine this we will examine whether it is profitable to plan a positive delivery in one of the cells for which the present delivery number is 0.

Consider the cell indicated in the table below.

10	12	8	15	10	
10		10			20
12	9	18	13	13	
□	11		10	7	28
10	10	12	20	12	
	5	25			30
15	12	16	12	8	
				15	15
10	16	35	10	22	

This cell forms a cycle with some of the positive cells.

10 −10 [10]	12	8 +10 [10]	15	10	20
12 +10 [ ]	9 −10 [11]	18	13 10	13 7	28
10	10 +10 [5]	12 −10 [25]	20	12	30
15	12	16	12	8 15	15
10	16	35	10	22	

Let's shift 10 units of paper to this cell. In order to preserve the supply and the demands numbers in rows and columns we will have to alternately subtract and add 10 units in the consecutive cells of the cycle. The change of cost will amount to 10 tons times 10 Euro times

$$10\times(12 - 9 + 10 - 12 + 8 - 10) = -10.$$

Thus the further modification of the plan is profitable. The similar analysis should be carried out for any cell in the table for which the delivery number is at present state at the level zero. This could be too tiring. It would be desirable to perform this analysis simultaneously for all cells.

For this purpose we will make some observations. Let's modify the cost table in the following way. We add a constant number c to each cell of the first row. This number can be either negative or positive. We can think of this change, as if all of a sudden a new tax or a new bonus, depending on the sign of c, has been introduced at the level c for each unit of paper dispatched from the factory 1. Notice that if we find an optimal plan before the introduction of the additional payment, then this plan will remain optimal after the introduction of the new payment. This follows from the fact that no matter the table of costs is we have to dispatch 20 units of paper from the factory 1, anyway.

The same modification is possible for each of the remaining rows as well as the columns of the cost table.

Summarizing, if the cost table is modified by adding to its rows and columns certain constants, not necessarily equal to each other, then the optimal plan for the original cost table will remain optimal for the modified cost table, and vice versa. Thus if we are after the optimal plan, we are free to modify the cost table in the manner described above. But there are many ways to carry this out. Which one is going to be chosen ?

Let's modify the cost table in such a way, that the new cost will become zero for each positive cell. The consecutive steps are shown in the tables below.

−10	10 / 10	12	8 / 10	15	10	20
	12	9 / 11	18	13 / 10	13 / 7	28
	10	10 / 5	12 / 25	20	12	30
	15	12	16	12	8 / 15	15
	10	16	35	10	22	

			+2			
−10	10 / 10	12	8 / 10	15	10	20
	12	9 / 11	18	13 / 10	13 / 7	28
	10	10 / 5	12 / 25	20	12	30
	15	12	16	12	8 / 15	15
	10	16	35	10	22	

			+2			
−10	10 / 10	12	8 / 10	15	10	20
	12	9 / 11	18	13 / 10	13 / 7	28
−14	10	10 / 5	12 / 25	20	12	30
	15	12	16	12	8 / 15	15
	10	16	35	10	22	

Eventually we arrive at

		+4	+2			
−10	10 / 10	12	8 / 10	15	10	20
−13	12	9 / 11	18	13 / 10	13 / 7	28
−14	10	10 / 5	12 / 25	20	12	30
−8	15	12	16	12	8 / 15	15
	10	16	35	10	22	

The new cost table is thus the following.

0 / 10	6	0 / 10	5	0	20
−1	0 / 11	7	0 / 10	0 / 7	28
−4	0 / 5	0 / 25	6	−2	30
7	8	10	4	0 / 15	15
10	16	35	10	22	

The plan is optimal with respect to the new cost table if and only if it is optimal with respect to the initial cost table. Thus we may forget about the initial cost table and deal with the new one.

Observe that according to the new cost table the total delivery cost is equal to 0, since the cost numbers in the positive cells are equal to 0. We can see also that the new cost numbers are negative in several cells. Let's choose a cell with the most negative cost. This cell forms a cycle with some of the positive cells.

					Supply
0 / −10 / 10	6	0 / +10 / 10	5	0	20
−1	0 / 11	7	0 / 10	0 / 7	28
−4 / +10 /	0 / 5	0 / −10 / 25	6	−2	30
7	8	10	4	0 / 15	15
Demand: 10	16	35	10	22	

Let's shift 10 units of commodity along this cycle. In this way we obtain a new plan, cheaper by 40 times 10 tons times 10 Euro.

					Supply
0	6	0 / 20	5	0	20
−1	0 / 11	7	0 / 10	0 / 7	28
−4 / 10	0 / 5	0 / 15	6	−2	30
7	8	10	4	0 / 15	15
Demand: 10	16	35	10	22	

As before we will modify this cost table in order to nullify the costs in the positive cells. It suffices to add 4 to the first column. We obtain a new cost table.

4	6	0 / 20	5	0	20
3	0 / [11]	7	0 / 10	0 / [7]	28
0 / 10	0 / [5]	0 / 15	6	−2 / [ ]	30
11	8	10	4	0 / 15	15
10	16	35	10	22	

There is only one cell left, where the cost number is negative. By shifting 5 units of commodity along an appropriate cycle we arrive at the next delivery plan cheaper by another 10 times 10 tons times 10 Euro.

4	6	0 20	5	0	20
3	0 16	7	0 10	0 2	28
0 10	0	0 15	6	−2 5	30
11	8	10	4	0 15	15
10	16	35	10	22	

We modify this table in order to nullify the cost in the cells with positive delivery numbers.

	−2		−2			
+2	4	6	0 20	5	0	20
	3	0 16	7	0 10	0 2	28
+2	0 10	0	0 15	6	−2 5	30
	11	8	10	4	0 15	15
	10	16	35	10	22	

Eventually we obtain

4	8	0 / 20	7	2	20
1	0 / 16	5	0 / 10	0 / 2	28
0 / 10	2	0 / 15	8	0 / 5	30
9	8	8	4	0 / 15	15
10	16	35	10	22	

The cost of our plan with respect to this table is still 0. But the cost numbers in all cells are now nonnegative. Therefore for any possible delivery plan the cost is nonnegative. Thus our plan must be optimal. This plan is also optimal with respect to the initial cost table. Summing up all the profits which resulted from eliminating cycles and performing two further modifications gives 13 000 Euro. Thus the cost of the optimal plan amounts to 92 000 Euro.

The Optimal Planning Procedure

1. Find an initial delivery plan, which does not admit cycles consisting of positive cells.
2. Nullify costs in the positive cells by adding appropriate constants to the rows and columns of the cost table.
3. If all the new costs are nonnegative, the plan is optimal.
4. If there are negative costs in the table choose a cell with the most negative one.
5. Find a cycle containing this cell and positive cells.
6. Shift the amount of commodity along the cycle so as to make the delivery number in the new cell as large as possible. Then in one of the cells of the cycle the delivery number will decrease to zero.
7. Go back to 2.

This procedure is called the *simplex method.*

How can we find an initial plan without cycles ? The simplest method is due to Dantzig. We begin at the cell at the upper-left corner of the cost matrix. We allocate maximal possible delivery number in this cell. Then we move one column to the right, if there is any supply left in the first row. Otherwise we move one row down. We continue in this way until we reach a plan (by construction with no cycles).

10 10	12 10	8	15	10	20
12	9 6	18 22	13	13	28
10	10	12 13	20 10	12 7	30
15	12	16	12	8 15	15
10	16	35	10	22	

The cost of this plan is equal to 123 000 Euro, so it is more costly then the one director set up. The reason is that this method ignores completely the cost numbers of the cells.

There are other methods of determining the initial plan without cycles, which take the costs into account. The most popular is called the Vogel approximation method.

If we have an initial plan we can start applying the algorithm described before. The number of iterations essentially depends on the choice of the initial plan. The better the plan the lower the number of iterations leading to the optimal solution. That's why it is worthwhile putting more effort into determining a good initial plan, and thus reducing the number of iterations of the simplex method necessary for obtaining the optimal plan.

The transportation problem was formulated by F. L. Hitchcock in 1941 and solved by G. B. Dantzig in 1951.

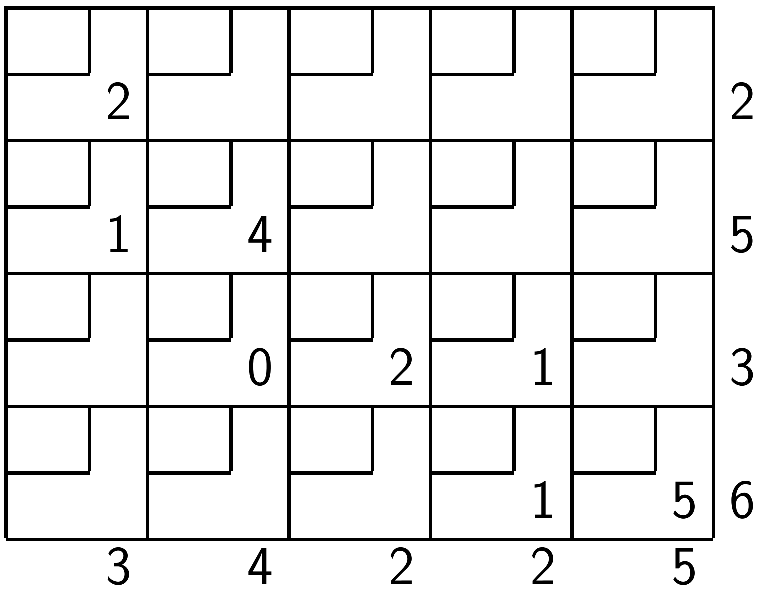
The method is still being successfully applied in practice. For example in 1997 the company Procter and Gamble redesigned the North America production and distribution system to reduce costs and improve speed to the market. The annual savings amounted to 200 mln USD.

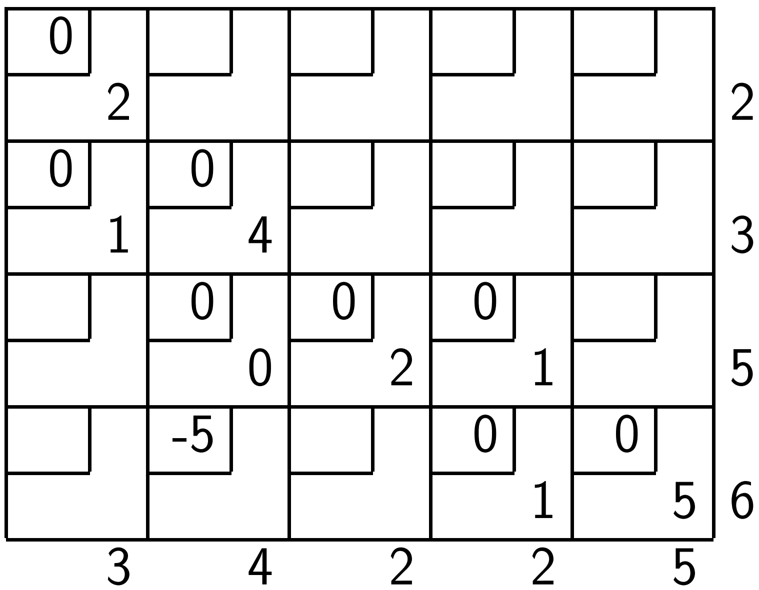
The case of degeneracy

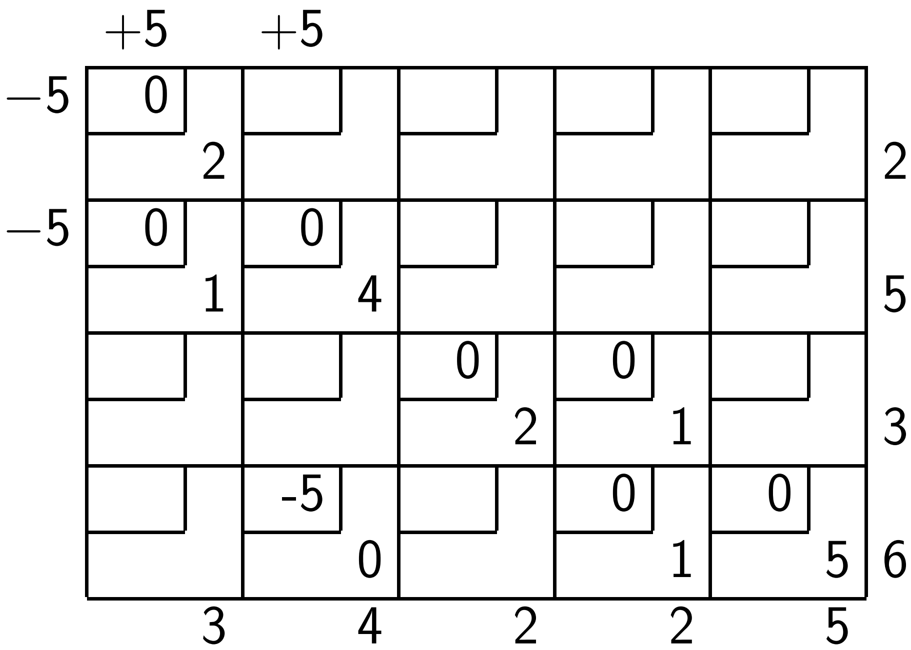
While applying the simplex procedure for the transportation problem the following cases may occur.

1. The initial solution, found by the Dantzig North-West corner rule has less than 8 positive cells.
2. At some iteration of the simplex method we obtain a solution with less than 8 positive cells.

In case 1 we add some zero cells to the set of positive cells so as to obtain 8 cells which do not form a cycle. Next we launch the simplex method. In case 2, when shifting the commodity along the cycle we nullify more than one cell, we remove only one of these cells and we add as usual the cell with negative cost. Then at certain iteration it may happen that we will not be able to shift any positive amount of the commodity, so the cost will remain the same as the cost before starting the iteration.


2 2
1 4 5
0 2 1 3
1 5 6
3 4 2 2 5


0
2 2
0 0
1 4 3
0 0 0
0 2 1 5
-5 0 0
1 5 6
3 4 2 2 5


+5 +5
−5 0 2 2
−5 0 1 0 4 5
0 2 0 1 3
-5 0 0 1 0 5 6
3 4 2 2 5

The cycling problem

In the degenerate case it may happen that after certain iteration we come back to the same cost matrix and the same delivery plan that occurred at some step before. This is called a cycling. Assume that at any iteration we choose a cell with the most negative cost to enter the solution. If there more such cells we choose one in an arbitrary way.

The W. Szwarc conjecture states that cycling can never occur. It is well known that cycling can occur in Linear Programming. The conjecture states that for particular optimization problem like the transportation problem it is impossible. The method for proving this conjecture uses so called directed weighted trees, and this is pure combinatorics. Only partial results have been obtained so far.