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Groups Acting on Trees and Approximation Properties of the Fourier Algebra

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Let $\mathfrak{X}$ be a tree and G a locally compact group acting on $\mathfrak{X}$ by isometries with respect to the natural metric on $\mathfrak{X}$. We construct the series of representations of G parametrized by the complex unit disc associated canonically with the distance on $\mathfrak{X}$ via the matrix coefficients. We apply this series to prove that for any group G acting on a tree in such a way that the stabilizer of a vertex is a compact subgroup of G the Fourier algebra $A(G)$ admits an approximate unit bounded in the multiplier norm on $A(G)$. For the special case of semihomogeneous trees and the full group $\mathrm{Aut}(\mathfrak{X})$ of isometries of $\mathfrak{X}$ we decompose the constructed representations obtaining finally an analytic continuation of the principal series of $\mathrm{Aut}(\mathfrak{X})$. © 1991 Academic Press, Inc.

INTRODUCTION

Let $\mathfrak{X}$ be a tree and G be a group acting on $\mathfrak{X}$ by automorphisms. One of the typical examples is the free group $\mathbb{F}_n$ on n generators which acts naturally on its Cayley graph which is the homogeneous tree of degree $2n$. In the paper [9] a construction of analytic series of uniformly bounded representations of $\mathbb{F}_n$ was given. Here we generalize that construction to any group acting on an arbitrary tree (not necessarily homogeneous). As an application we prove that if a group G acts on a tree and the stabilizer of a vertex is compact then the Fourier algebra $A(G)$ admits an approximate unit bounded in the multiplier norm on $A(G)$ (even bounded in completely bounded multiplier norm on $A(G)$ (see [5])). In particular we get the result of [3] which states that $SL(2, \mathbb{Q}_p)$ over the p-adic number field has the completely bounded approximation property. These results

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should be compared with a theorem of Julg and Valette [6] who proved that any group G acting on a tree with amenable stabilizers of vertices is so called K-amenable (see [7, 1.2. Definition]). Both notions of K-amenability and the completely bounded approximation property are generalizations of the amenability.

In Section 2 we discuss the problem of irreducibility of representations π_z, $|z|<1$, constructed in Section 1. We give the solution for the full group $\mathrm{Aut}(\mathfrak{X})$ of isometries of the semihomogeneous tree that is the tree with only two possible degrees of vertices with the property that the vertices of any edge have different degrees. In that case following the method of [11] we decompose the representations π_z into two subrepresentations, one irreducible and another equivalent to the quasiregular representation of $\mathrm{Aut}(\mathfrak{X})$. Then as in [11] we realize all irreducible components of π_z on the common Hilbert space which makes it possible to extend this series. The final result is the following

THEOREM. *Let $\mathfrak{X}_{r_1, r_2}$ be a semihomogeneous tree of degrees $r_1, r_2, r_1 < r_2$. Put $q=(r_1 r_2)^{1/2}$. There exists a series of uniformly bounded representations Π_z, $q^{-1}<|z|<1$, of the group $\mathrm{Aut}(\mathfrak{X}_{r_1, r_2})$ on the Hilbert space $\mathscr{H}$, such that*

(i) *The series Π_z is analytic in the domain $\Omega=\{z : q^{-1}<|z|<1, z \neq it, t \in (-1, -r_1^{-1/2}] \cup [-r_2^{-1/2}, r_2^{-1/2}] \cup [r_1^{-1/2}, 1)\}$.*

(ii) *$\Pi_z=\Pi_{-z}$ and $\Pi_z=\Pi_u$ where $u=(qz)^{-1}$, $z \in \Omega$.*

(iii) *$\Pi_z(i)^*=\Pi_{\bar{z}}(i)^{-1}$.*

(iv) *$\Pi_z(i)-\Pi_{z'}(i)$ has finite rank.*

(v) *Any representation $\Pi_z, z^2 \neq -r_1^{-1}, -r_2^{-1}$, is irreducible. The representations Π_z and $\Pi_{z'}$ are equivalent if and only if $z=z'$, $z=-z'$, or $z=(qz')^{-1}$.*

(vi) *Π_z is a unitary representation if and only if $|z|=q^{-1/2}$, $z \in \mathbb{R}$, or $z=it$ with $t \in [-r_1^{-1/2}, -r_2^{-1/2}] \cup [r_2^{1/2}, r_1^{1/2}]$. Otherwise the representation Π_z cannot be made unitary by introducing another equivalent scalar product in $\mathscr{H}$.*

It turns out also that if K denotes the subgroup of $\mathrm{Aut}(\mathfrak{X}_{r_1, r_2})$ which fixes a vertex e in $\mathfrak{X}_{r_1, r_2}$ then any representation Π_z, $q^{-1}<|z|<1$, admits a non-zero K-fixed vector in $\mathscr{H}$, which is unique up to scalar multiples. That is why they are called spherical representations (see [1, 8]).

1. The General Case

The results of this section are adapted from [9, Sect. 2], where the case of the free group was treated. Here we restate them in terms of an arbitrary tree and its group of all isometries. We also reprove some of them for the sake of completeness.

By a tree we mean a connected graph without circuits. A chain in the tree $\mathfrak{X}$ is a sequence $x_0, ..., x_n$ such that x_i and x_{i+1} are adjacent and $x_i \neq x_{i+2}$. For any two vertices $x, y \in \mathfrak{X}$ there exists a unique chain $x = x_0, x_1, ..., x_{n-1}, x_n = y$ connecting x and y. We denote this chain by $[x, y]$. The natural distance $d(x, y)$ between x and y is the length of the chain $[x, y]$, i.e., $d(x, y) = n$ if $[x, y] = \{x_0, ..., x_n\}$.

Let $\mathfrak{X}$ be an arbitrary tree. Fix a vertex e in $\mathfrak{X}$. For any vertex $x \neq e$ let $c(x)$ denote the penultimate vertex of the chain $[e, x]$ (when e and x are adjacent then $c(x) = e$).

Let us fix for a while another vertex e' and as above we define the operation c' on $\mathfrak{X}$ with respect to e'.

LEMMA 1. *Let x be a vertex of $\mathfrak{X}$ such that $x \notin [e, e']$. Then $c(x) = c'(x)$.*

Proof. Let x not belong to $[e, e']$. Then if we go from x towards e or e' the first steps are the same. It means $c(x) = c'(x)$. Moreover let $[e, e'] = \{e = x_0, x_1, ..., x_n = e'\}$. Then $c(x_i) = x_{i-1}$ for $i = 1, ..., n$ and $c'(x_i) = x_{i+1}$ for $0, 1, ..., n-1$. Thus the operations c and c' act on the chain $[e, e']$ as translations in opposite directions.

COROLLARY 1. *Let i be an automorphism of $\mathfrak{X}$. Then we have $i \circ c \circ i^{-1}(x) = c(x)$ for any vertex $x \in \mathfrak{X}$ such that $x \notin [e, i(e)]$.*

Proof. Put $e' = i(e)$ and observe that $c' = i \circ c \circ i^{-1}$. The operation c can be lifted in a natural way to complex functions defined on $\mathfrak{X}$. Namely let P be the linear operator defined on the space $\mathscr{F}(\mathfrak{X})$ of finitely supported complex functions on $\mathfrak{X}$ by the rule

$$P\delta_x = \begin{cases} \delta_{c(x)} & \text{if } x \neq e \\ 0 & \text{if } x = e, \end{cases} \tag{1}$$

where δ_x denotes as usual the function which admits the value 1 at x and vanishes elsewhere.

The isometries of $\mathfrak{X}$ act on the space $\mathscr{F}(\mathfrak{X})$ by compositions:

$$f \mapsto f \circ i^{-1}. \tag{2}$$

The corresponding map is denoted by $\lambda(i)$. Now the preceding corollary can be restated as follows.

COROLLARY 2. *For any isometry $i \in \mathrm{Aut}(\mathfrak{X})$ the operator $\lambda(i) P \lambda(i)^{-1} - P$ has finite rank. Moreover the operators P and $\lambda(i) P \lambda(i)^{-1}$ coincide on the functions vanishing on $[e, i(e)]$.*

For any complex number z the operator $I-zP$ is invertible on the space $\mathscr{F}(\mathfrak{X})$. This is because if $f\in\mathscr{F}(\mathfrak{X})$ then $P^nf=0$ for n sufficiently large so the series $\sum z^nP^nf$ has only finitely many non-zero terms.

For any $z\in\mathbb{C}$ define the representation π_z^0 of $\mathrm{Aut}(\mathfrak{X})$ on the space $\mathscr{F}(\mathfrak{X})$ setting

$$\pi_z^0(i)=(I-zP)^{-1}\,\lambda(i)(I-zP). \tag{3}$$

LEMMA 2. *Let $z\in\mathbb{C}$. Then π_z^0 extends by continuity to a bounded representation of* $\mathrm{Aut}(\mathfrak{X})$ *on the Hilbert space $l^2(\mathfrak{X})$. Moreover*:

(i) *The correspondence $z\mapsto\pi_z^0(i)$ is an analytic function.*

(ii) $\|\pi_z^0(i)\|\leqslant(1+|z|-2\,|z|^{d(e,i(e))+2})/(1-|z|)$.

(iii) *$\pi_z^0(i)-\lambda(i)$ has finite rank for any $i\in\mathrm{Aut}(\mathfrak{X})$.*

(iv) $\langle\pi_z^0(i)\,\delta_e,\delta_e\rangle=z^{d(e,i(e))}$.

Proof. Let $i\in\mathrm{Aut}(\mathfrak{X})$. Then

$$\begin{aligned}\pi_z^0(i)\,\lambda(i)^{-1}-I&=(I-zP)^{-1}\,\lambda(i)(I-zP)\,\lambda(i)^{-1}-I\\&=(I-zP)^{-1}\,[\lambda(i)(I-zP)\,\lambda(i)^{-1}-(I-zP)]\\&=z(I-zP)^{-1}\,[P-\lambda(i)\,P\lambda(i)^{-1}]\\&=\sum z^{k+1}P^k[P-\lambda(i)\,P\lambda(i)^{-1}].\end{aligned}$$

Hence the difference $\pi_z^0(i)\,\lambda(i)^{-1}-I$ vanishes on the orthogonal complement $\{\delta_{x_0},...,\delta_{x_n}\}^\perp$ where $[e,i(e)]=\{x_0,...,x_n\}$. In particular it implies (iii). Furthermore on the subspace span $\{\delta_{x_0},...,\delta_{x_n}\}$ the operators $\lambda(i)\,P\lambda(i)^{-1}$ and P are contractions in virtue of Lemma 1. Thus $\|\pi_z^0(i)\,\lambda(i)^{-1}-I\|\leqslant 2\sum_{k=0}^n|z|^{k+1}$ and $\|\pi_z^0(i)\|\leqslant 1+2\sum_{k=0}^n|z|^{k+1}$ which gives (ii). At the same time we have proved (i) because the function $z\mapsto\pi_z^0(i)=\lambda(i)+\sum_{k=0}^\infty z^{k+1}P^k[P\lambda(i)-\lambda(i)P]$ is a polynomial of degree $d(e,i(e))+1$. Finally

$$\begin{aligned}\langle\pi_z^0(i)\,\delta_e,\delta_e\rangle&=\langle(I-zP)^{-1}\,\delta_{i(e)},\delta_e\rangle\\&=\sum_{k=0}^{\infty}z^k\langle P^k\,\delta_{i(e)},\delta_e\rangle=z^{d(e,i(e))}.\end{aligned}$$

Let T denote the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\,\delta_e$. For any complex z with $|z|<1$ we define the linear operator T_z as

$$T_z=\sqrt{1-z^2}T+(I-T),$$

where $\sqrt{1-z^2}$ denotes the principal branch of the square root. The operator T_z is invertible on $l^2(\mathfrak{X})$ whatever z, $|z|<1$.

For any complex z, $|z|<1$, define the representation π_z of $\mathrm{Aut}(\mathfrak{X})$ by

$$\pi_z(i) = T_z^{-1}\pi_z^0(i)\,T_z. \tag{4}$$

THEOREM 1. *The representations π_z, $z \in \{|z|<1\}$, form an analytic series of uniformly bounded representations of* $\mathrm{Aut}(\mathfrak{X})$ *on the Hilbert space* $l^2(\mathfrak{X})$. *Moreover*:

(i) $\|\pi_z(i)\| \leqslant 2(|1-z^2|/(1-|z|))$.

(ii) $\pi_z(i)^* = \pi_{\bar z}(i)^{-1}$. *In particular the representations π_t, $t \in (-1, 1)$, are unitary.*

(iii) $\pi_z(i) - \lambda(i)$ *is a finite rank operator.*

(iv) $\phi_z(i) = \langle \pi_z(i)\,\delta_e, \delta_e\rangle = z^{d(e, i(e))}$.

Proof. The first part of the theorem together with (iii) and (iv) are straightforward consequences of Lemma 2. The proof of the estimate in (i) can be simply copied from [9, Theorem 1], so we will omit it. What is left is to prove (ii) only. But before doing so, we derive some auxiliary facts which we will need in the sequel.

For any vertex $x \in \mathfrak{X}$ let (n_x+1) denote the number of edges to which x belongs. The number n_x is called the degree of the vertex x. Assume that the degrees of $\mathfrak{X}$ are uniformly bounded or equivalently there are only finitely many possible degrees. In this case P becomes an invertible operator on $l^2(\mathfrak{X})$. Moreover its adjoint operator P^* is given by

$$P^*\,\delta_x = \sum_{c(y)=x} \delta_y. \tag{5}$$

Consider the sum $P+P^*$. It acts as $(P+P^*)\,\delta_x = \sum_{d(x,y)=1} \delta_y$. Hence the operator $P+P^*$ commutes with all isometries of the tree because any isometry i maps the circle $\{y \in \mathfrak{X} : d(x, y)=1\}$ onto the circle $\{y \in \mathfrak{X} : d(i(x), y)=1\}$.

LEMMA 3. *For any $z \in \mathbb{C}$ the operator $(I-zP)\,T_z^2(I-zP^*)$ commutes with all isometries of $\mathfrak{X}$.*

Proof. Define the linear operator N on $l^2(\mathfrak{X})$ as $N\,\delta_x = n_x\,\delta_x$ for $x \in \mathfrak{X}$. Clearly N commutes with isometries of $\mathfrak{X}$ because the degree of vertices is invariant under isometries. Then observe that $PP^* = N+T$ (cf. [11, (5)]). Hence

$$\begin{aligned}(I-zP)\,&T_z^2(I-zP^*)\\ &= (I-zP)(I-z^2T)(I-zP^*)\\ &= I - z^2T + z^2PP^* - z(P+P^*) = I + z^2N - z(P+P^*).\end{aligned} \tag{6}$$

This gives the desired result as we have seen before that N and $P+P^*$ commute with all $\lambda(i)$, $i \in \mathrm{Aut}(\mathfrak{X})$.

Proof of Theorem 1(iii). Using (ii) we can restrict our attention to the case when the degrees of vertices of $\mathfrak{X}$ are uniformly bounded. As P and P^* are bounded on $l^2(\mathfrak{X})$ so the operators $I-zP$ and $I-zP^*$ are invertible for $|z|$ small enough. Therefore by Lemma 3 and by $\lambda(i)^* = \lambda(i)^{-1}$ we obtain

$$\begin{aligned}\pi_z(i)\,\pi_{\bar{z}}(i)^* = T_z^{-1}(I-zP)^{-1}\,\lambda(i)(I-zP)\,T_z^2(I-zP^*)\\ \times \lambda(i)^{-1}(I-zP^*)^{-1}\,T_z^{-1} = I.\end{aligned}$$

Thus we have proved the identity $\pi_{\bar{z}}(i)^* = \pi_z(i)^{-1}$ for small values of $|z|$. But both sides of this identity are analytic functions of z. It implies that (ii) holds for any z, $|z| < 1$.

Remark 1. The estimate in (i) is not sharp. It can be proved that the optimal estimate is $(|1-z^2| + 2\,|\mathrm{Im} z|)/(1-|z|^2)$. This unpublished result is due to Uffe Haagerup.

Remark 2. Consider the case when there are finitely many possible degrees of vertices. Then as we saw before the operators P and P^* are bounded. It turns out that the subspaces $\mathrm{Ker}(I-zP)\,T_z$ and $\mathrm{Im}\,T_z(I-zP^*)$ are invariant under the representation π_z. Indeed, it follows from the formulas below

$$\begin{aligned}(I-zP)\,T_z\pi_z(i) &= \lambda(i)(I-zP)\,T_z\\ \pi_z(i)\,T_z(I-zP^*) &= T_z(I-zP^*)\,\lambda(i).\end{aligned} \tag{7}$$

The first identity is a simple transformation of (4) while the second relies on Lemma 3. The subspace $\mathrm{Ker}(I-zP)\,T_z$ is closed for any z in contrast to the second subspace $\mathrm{Im}\,T_z(I-zP^*)$. For real z these subspaces are orthogonal to each other and $l^2(\mathfrak{X})$ is a direct sum of $\mathrm{Ker}(I-zP)\,T_z$ and the closure of $\mathrm{Im}\,T_z(I-zP^*)$. In the next section we examine when $\mathrm{Im}\,T_z(I-zP^*)$ is closed and when the whole space can be split into the direct sum of these two invariant subspaces. All this will be done for the case of semihomogeneous trees.

Remark 3. Let G be a group acting on the tree $\mathfrak{X}$. It means any element g of G defines an automorphism i_g of $\mathfrak{X}$ and the correspondence $g \mapsto i_g$ is a homomorphism. Thus we can define the representations π_z for G acting on $\mathfrak{X}$ by $g \mapsto \pi_z(i_g)$.

2. Semihomogeneous Trees

Let $\imath_1$ and $\imath_2$ be two different natural numbers. Let $\mathfrak{X}_{\imath_1, \imath_2}$ denote a tree such that in any vertex there meet $\imath_1 + 1$ or $\imath_2 + 1$ edges and moreover the vertices of any edge have different degrees. Trees of this type are called semihomogeneous. They are the only trees which have the property that for any vertex x all its neighbors, i.e., $\{y: d(x, y) = 1\}$, have the same degree.

We divide vertices of $\mathfrak{X}_{\imath_1, \imath_2}$ into two disjoint subsets $\mathfrak{X}_{\imath_1}$ and $\mathfrak{X}_{\imath_2}$ with respect to the degree. Let $I_{\imath_1}$ and $I_{\imath_2}$ denote the orthogonal projections onto $l^2(\mathfrak{X}_{\imath_1})$ and $l^2(\mathfrak{X}_{\imath_2})$, respectively. Fix a vertex e in $\mathfrak{X}_{\imath_1}$ and define the operator P associated with e. We are going to identify the spectrum of P and its spectral properties. Denote $q = (\imath_1 \imath_2)^{1/2}$ and $\imath = \min(\imath_1, \imath_2)$. Clearly we have

$$P^{2n} P^{*2n} \delta_x = q^{2n} \delta_x, \qquad \text{for} \quad x \neq e$$

$$P^{2n} P^{*2n} \delta_e = (1 + \imath_1^{-1})\, q^{2n} \delta_e.$$

Thus $\|P^{2n}\|^{1/2n} = (1 + \imath_1^{-1})^{1/2n} q^{1/2}$ and the spectral radius of P amounts to $q^{1/2}$. Actually $\sigma(P) = \{z \in \mathbb{C} : |z| \leqslant q^{1/2}\}$, because the interior of the disc consists of eigenvalues of P. In fact, set $\chi_n(x)$ to be the function on $\mathfrak{X}_{\imath_1, \imath_2}$ which admits the value 1 when $d(e, x) = n$ and 0 otherwise. Put

$$h_z = (\imath_1 + 1)\, \imath_1^{-1/2} \delta_e + \sum_{n=1}^{\infty} q^{-n} z^n (\imath_1^{1/2} \chi_{2n} + \imath_2^{1/2} \chi_{2n-1}). \tag{8}$$

Then $h_z \in l^2(\mathfrak{X}_{\imath_1, \imath_2})$ for $|z| < q^{1/2}$ and $P h_z = z h_z$. It can be shown also that for any z from the circle $|z| = q^{-1/2}$ the operator $zI - P$ is a bijection (cf. [11, Corollary 1]).

Proposition 1. *Let $|z| \neq q^{-1/2}$ and $z^2 \neq -\imath^{-1}$. Then the operator $(I - zP)\, T_z^2 (I - zP^*)$ is invertible on the space $l^2(\mathfrak{X}_{\imath_1, \imath_2})$.*

Proof. First observe that the case $|z| < q^{-1/2}$ is trivial because both $I - zP$ and $I - zP^*$ are invertible. Let us introduce a notation which we apply throughout the paper:

$$u = (qz)^{-1}, \qquad a(z) = \imath_1 z^2 + 1. \tag{9}$$

Define the linear operators A_z and F_z as

$$A_z = \frac{a(u)}{u} I_{\imath_1} + \frac{a(z)}{z} I_{\imath_2}; \qquad F_z = \frac{1}{z} (I - zP)\, T_z^2 (I - zP^*). \tag{10}$$

We assert that

$$F_z A_z = A_u F_u. \tag{11}$$

Indeed, by (6) we have

$$F_z = \left(\frac{1}{z} + z\imath_1\right) I_{\imath_1} + \left(\frac{1}{z} + z\imath_2\right) I_{\imath_2} - (P + P^*)$$
$$= \frac{a(z)}{z} I_{\imath_1} + \left(\frac{\imath_2}{\imath_1}\right)^{1/2} \frac{a(u)}{u} I_{\imath_2} - (P + P^*).$$

Now we can easily get (11) using the above formula and the obvious identities $PI_{\imath_1} = I_{\imath_2}P$ and $P^*I_{\imath_1} = I_{\imath_2}P^*$. Assume that $|z| > q^{-1/2}$ and $z^2 \neq -\imath^{-1}$. Then by (9), $|u| < q^{-1/2}$, $a(z) \neq 0$, and $a(u) \neq 0$. It implies that A_z, A_u, and F_u are invertible thus by (11) it regards the operator F_z as well.

For $z \in \mathbb{C}$, $q^{-1/2} < |z| < 1$, define the operator U_z by

$$U_z = T_u^{-1}(I - uP)^{-1} A_z(I - zP) T_z. \tag{12}$$

Then we have (cf. [11, Proposition 1]).

PROPOSITION 2. *Let* $q^{-1/2} < |z| < 1$ *and* $z^2 \neq -\imath^{-1}$. *Then*

(i) $U_z U_{\bar{z}}^* = (a(z)\, a(u)/u^2) I$;

(ii) $R_z = I - (u^2/a(z)\, a(u))\, U_{\bar{z}}^* U_z$ *is a projection and* $R_z^* = R_{\bar{z}}$;

(iii) $U_z \pi_z(i) = \pi_u(i) U_z$ *and* $\pi_z(i)\, U_{\bar{z}}^* = U_{\bar{z}}^* \pi_u(i)$;

(iv) $\operatorname{Ker} U_z = \operatorname{Ker}(I - zP) T_z$ *and* $\operatorname{Im} U_{\bar{z}}^* = \operatorname{Im} T_z(I - zP^*)$;

(v) $R_z \pi_z(i) = \pi_z(i) R_z$.

The proof is rather easy and we omit it. Anyway we can refer to [11, Proposition 1].

THEOREM 2. *Let* $q^{-1/2} < |z| < 1$ *and* $z^2 \neq -\imath^{-1}$. *Then* $\operatorname{Im} T_z(I - zP^*)$ *and* $\operatorname{Ker}(I - zP) T_z$ *are invariant subspaces for the representation* π_z. *Moreover they give a decomposition of the entire space* $l^2(\mathfrak{X}_{\imath_1, \imath_2})$ *into the direct sum, i.e.,* $l^2(\mathfrak{X}_{\imath_1, \imath_2}) = \operatorname{Im} T_z(I - zP^*) \oplus \operatorname{Ker}(I - zP) T_z$. *The representation* π_z *restricted to the invariant subspace* $\operatorname{Im} T_z(I - zP^*)$ *is equivalent to the representation* λ.

As in [11, Theorems 3 and 4], the proof relies on Proposition 2 and the lemma below

LEMMA 4 [11, Lemma 1]. *Let* A *and* B *be bounded linear operators on a Hilbert space* $\mathscr{H}$ *such that their composition* AB *is an invertible operator. Then we have*

(i) *The subspace* $\operatorname{Im} B$ *is closed and* $\mathscr{H}$ *is a direct sum of the subspaces* $\operatorname{Ker} A$ *and* $\operatorname{Im} B$.

(ii) *The operator* $B(AB)A^{-1}$ *is a projection (not necessarily orthogonal) onto the subspace* Im B *along* Ker A.

(iii) *A linear operator* C *on the space* $\mathscr{H}$ *leaves the subspaces* Im B *and* Ker A *invariant if and only if* C *commutes with the projection* $B(AB)A^{-1}$.

(iv) *Let* $\mathscr{M}$ *be a subspace of* Ker A. *Then* $\mathscr{M}$ *is dense in* Ker A *if and only if for any* $v \in \operatorname{Ker} B^*$ *the condition* $v \perp \mathscr{M}$ *implies* $v = 0$ (*in other words* $\mathscr{M}$ *is a separable space for* Ker B^*).

Proof of Theorem 2. By Proposition 1 we can apply Lemma 4 to the operators $A = U_z$ and $B = U_{\bar{z}}^*$. Hence the first part of the theorem follows from Lemma 4, Proposition 2, and Remark 2 of the first section. Concerning the last statement of the theorem, by Proposition 2(i), (iv) the operator $U_{\bar{z}}^*$ maps $l^2(\mathfrak{X}_{\imath_1, \imath_2})$ onto Im $T_z(I - zP^*)$ isomorphically. Moreover by Proposition 2(iii) the operator $U_{\bar{z}}^*$ intertwines the representation $\pi_z|_{\operatorname{Im} T_z(I-zP^*)}$ with the representation π_u. But π_u is equivalent to the representation λ because $|u| < q^{-1/2}$.

From now on we discuss only the representation π_z restricted to $\operatorname{Ker}(I - zP)T_z$. In particular we are going to show the irreducibility of π_z on this subspace. Before doing so we introduce some new notation.

Let K denote the set of all automorphisms which leave the vertex e fixed. K turns out to be a compact open subgroup of $\operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})$ (see [10]). The function f on $\mathfrak{X}_{\imath_1, \imath_2}$ is called radial if it is K-invariant; that is, $\lambda(i) f = f$ for any $i \in K$. The radial functions have the property that $f(x) = f(y)$ for any $x, y \in \mathfrak{X}_{\imath_1, \imath_2}$ such that $d(x, e) = d(y, e)$. Indeed, it suffices to observe that if $d(x, e) = d(y, e)$ then there exists $i \in K$ for which $i(x) = y$. Thus the values $f(x)$ of the radial function depend only on $d(x, e)$. Moreover any radial function admits the unique representation of the form $\sum_{n=0} a_n \chi_n$ with complex coefficients a_n, $n = 0, 1, \ldots$. Clearly all the operators $I_{\imath_1}, I_{\imath_2}$, P, P^*, T_z, R_z leave invariant the space of radial functions, as well as its orthogonal complement.

LEMMA 5. *Let* $q^{-1/2} < |z| < 1$. *Then the subspace of radial function in* $\operatorname{Ker}(I - zP)T_z$ *is one-dimensional.*

Proof. If $f = \sum_{n=0} a_n \chi_n$ then this leads to a recurrent formula for the sequence $\{a_n\}$ which for given a_0 has the unique solution.

By (8) the unique, up to a constant multiple, function in $\operatorname{Ker}(I - zP)T_z$ is

$$f_z = \delta_e + \frac{\sqrt{1 - z^2}}{\imath_1 + 1} \sum_{n=1}^{\infty} (qz)^{-n} [\chi_{2n} + q\chi_{2n-1}]. \tag{13}$$

We can express f_z in terms of R_z as well. In fact, observe that $R_z\delta_e$ is also a radial function which belongs to $\mathrm{Ker}(I-zP)T_z$. By (12) and by Proposition 2(ii) we can compute

$$\begin{aligned}\langle R_z\,\delta_e,\delta_e\rangle &= 1-\frac{u^2}{a(z)\,a(u)}\langle U_z\,\delta_e, U_{\bar z}\,\delta_e\rangle\\ &= 1-\frac{u^2}{a(z)\,a(u)}\frac{1-z^2}{1-u^2}\frac{a(u)^2}{u^2}\\ &= 1-\frac{(1-z^2)\,a(u)}{(1-u^2)\,a^z}=\frac{(\imath_1+1)(z^2-u^2)}{(1-u^2)\,a(z)}.\end{aligned}$$

Thus $R_z\,\delta_e\neq 0$. Hence by Lemma 5

$$f_z=\frac{1}{\langle R_z\,\delta_e,\delta_e\rangle}R_z\,\delta_e. \tag{14}$$

Let $k\in K$. Since $\lambda(k)$ commutes with P and with T_z then $\pi_z(k)=\lambda(k)$ for any $|z|<1$. Let $\mathscr{P}$ denote the operator defined as $\mathscr{P}=\int_K\lambda(k)\,dk$ where dk is a normalized Haar measure on K. Then $\mathscr{P}$ is the orthogonal projection onto the radial functions in $l^2(\mathfrak{X}_{\imath_1,\imath_2})$. Furthermore $\mathscr{P}$ commutes with P and with T_z so $\mathscr{P}$ leaves $\mathrm{Ker}(I-zP)T_z$ and $\mathrm{Im}\,T_z(I-zP^*)$ invariant.

THEOREM 3. *Let $q^{-1/2}<|z|<1$ and $z^2\neq -\imath^{-1}$. Then the representation π_z restricted to the invariant subspace $\mathscr{H}_z=\mathrm{Ker}(I-zP)T_z$ is irreducible. Moreover representations $\pi_z|_{\mathscr{H}_z}$ and $\pi_{z'}|_{\mathscr{H}_{z'}}$ are equivalent if and only if $z^2=z'^2$. For $z\notin\mathbb{R}\cup i\mathbb{R}$ the representation π_z is not unitarizable.*

Proof. First we prove that f_z or $R_z\,\delta_e$ is a cyclic vector of $\pi_z|_{\mathscr{H}_z}$. We will base this on the formula below which can be easily derived from the definition of π_z:

$$\pi_z(i)\,\delta_e=z^{d(i(e),e)}\,\delta_e+\sqrt{1-z^2}\sum_{n=0}^{d(i(e),e)-1}z^n\,\delta_{c^n(i(e))}. \tag{15}$$

Assume on the contrary that $\mathscr{M}=\mathrm{span}\{\pi_z(i)\,R_z\,\delta_e : i\in\mathrm{Aut}(\mathfrak{X}_{\imath_1,\imath_2})\}$ is not dense in $\mathscr{H}_z$. By Lemma 4(iv) applied to $A=(I-zP)T_z$ and $B=T_z(I-zP^*)$ there exists a function $f\in\mathrm{Ker}(I-\bar zP)T_{\bar z}$ such that for any $i\in\mathrm{Aut}(\mathfrak{X}_{\imath_1,\imath_2})$

$$\begin{aligned}0&=\langle\pi_z(i)\,R_z\,\delta_e,f\rangle=\langle R_z\pi_z(i)\,\delta_e,f\rangle=\langle\pi_z(i)\,\delta_e,R_{\bar z}f\rangle\\ &=\langle\pi_z(i)\,\delta_e,f\rangle=\langle\delta_e,\pi_{\bar z}(i^{-1})f\rangle.\end{aligned}$$

In particular $f(e)=0$. Let x be a vertex belonging to the support of f for which the distance $d(x,e)$ is minimal. We consider two cases:

(a) $x \in \mathfrak{X}_{\imath_1}$. Then there exists $i \in \operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})$ with $i(e) = x$. Hence by (15) we have $0 = \langle \delta_e, \pi_{\bar{z}}(i^{-1}) f \rangle = \langle \pi_z(i)\, \delta_e, f \rangle = \sqrt{1 - z^2} f(x)$. Thus $f(x) = 0$ which gives a contradiction.

(b) $x \in \mathfrak{X}_{\imath_2}$. Let $y \in \mathfrak{X}_{\imath_1}$ be such that $c(y) = x$. Then there exists $i \in \operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})$ with $i(e) = y$. Again by (15),

$$0 = \langle \delta_e, \pi_{\bar{z}}(i^{-1}) f \rangle = \langle \pi_z(i)\, \delta_e, f \rangle = \sqrt{1 - z^2}\, [\bar{f}(y) + z\bar{f}(x)].$$

It means that $f(y) = -\bar{z} f(x)$ for any y satisfying $c(y) = x$. On the other hand the condition $(I - \bar{z}P) f(x) = 0$ implies $f(x) = z \sum_{c(y) = x} f(y) = -\bar{z}^2 \imath_2 f(x)$. Therefore $(1 + \bar{z}^2 \imath_2) f(x) = 0$ which contradicts $f(x) \neq 0$.

In order to prove irreducibility let $\mathcal{M}$ be an invariant subspace of π_z contained in $\mathcal{H}_z$. Then $\mathcal{P}\mathcal{M} \subseteq \mathcal{M}$. By Lemma 5 there are two possible cases:

(a) $\mathcal{P}\mathcal{M} = \mathbb{C} f_z$. Then $\mathcal{H}_z \subseteq \mathcal{M}$ because f_z is a cyclic vector of $\pi_z|_{\mathcal{H}_z}$.

(b) $\mathcal{P}\mathcal{M} = (0)$. Hence $\mathcal{M}$ consists of functions orthogonal to all radial functions. In particular for any $i \in \operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})$ and any $f \in \mathcal{M}$, $0 = \langle \pi_z(i) f, f_{\bar{z}} \rangle = \langle f, \pi_{\bar{z}}(i^{-1}) f_{\bar{z}} \rangle$. This implies that f is orthogonal to $\operatorname{Ker}(I - \bar{z}P) T_{\bar{z}}$. Hence by Lemma 4(iv) applied to $A = (I - \bar{z}P) T_{\bar{z}}$, $B = T_{\bar{z}}(I - \bar{z}P^*)$, and $\mathcal{M} = \operatorname{Ker} A$ we get $f = 0$. Because f was an arbitrary function in $\mathcal{M}$ thus we have proved $\mathcal{M} = 0$.

Let ϕ_z be the matrix coefficient of the representation $\pi_z|_{\mathcal{H}_z}$ associated with the unique K-fixed function $R_z\, \delta_e$, i.e.,

$$\phi_z(i) = \frac{1}{\langle R_z\, \delta_e, \delta_e \rangle} \langle \pi_z(i)\, R_z\, \delta_e, \delta_e \rangle. \tag{16}$$

Applying the explicit formula expressing R_z we get

$$\phi_z(i) = \frac{1}{\imath_1 + 1} \left[\frac{(\imath_1 z^2 + 1)(1 - u^2)}{z^2 - u^2} z^{d(i(e), e)} - \frac{(\imath_1 u^2 + 1)(1 - z^2)}{z^2 - u^2} u^{d(i(e), e)} \right]. \tag{17}$$

Next observe that since $\mathcal{P}$ restricted to $\mathcal{H}_z$ is the orthogonal projection onto $R_z\, \delta_e$ then

$$\mathcal{P} \pi_z(i) \mathcal{P} = \phi_z(i) \mathcal{P} \qquad \text{on } \mathcal{H}_z.$$

Fix any automorphism i of $\mathfrak{X}_{\imath_1, \imath_2}$ such that $d(i(e), e) = 2$ and $i^2 = Id$. If two representations $\pi_z|_{\mathcal{H}_z}$ and $\pi_{z'}|_{\mathcal{H}_{z'}}$ are equivalent then the spectra of the operators $\mathcal{P}\pi_z(i)\, \mathcal{P}|_{\mathcal{H}_z}$ and $P\pi_{z'}(i)\, \mathcal{P}|_{\mathcal{H}_{z'}}$ should coincide (because $\mathcal{P}$ can be expressed as $\mathscr{P}=\int_K \pi_z(k)\,dk$). But by (17), $\phi_z(i)=\phi_{z'}(i)$ if and only if $z^2=z'^2$. Moreover because $i=i^{-1}$ then if $\pi_z|_{\mathscr{H}_z}$ is equivalent to a unitary representation then the spectrum of the operator $\mathscr{P}\pi_z(i)\,\mathscr{P}|_{\mathscr{H}_z}$ must be real. But $\phi_z(i)$ is a real number if and only if $z^2\in\mathbb{R}$.

To complete the proof assume that $z=-z'$ and check that the operator $I_{\imath_1}-I_{\imath_2}$ intertwines the representations $\pi_z|_{\mathscr{H}_z}$ and $\pi_{z'}|_{\mathscr{H}_{z'}}$.

Remark 4. The functions ϕ_z defined in (16) are the spherical functions corresponding to the compact subgroup K of $\mathrm{Aut}(\mathfrak{X}_{\imath_1,\imath_2})$. Indeed

$$\begin{aligned}\int_K \phi_z(ikj)\,dk\\ &=\frac{1}{\langle R_z\,\delta_e,\delta_e\rangle}\langle \mathscr{P}\pi_z(j)\,R_z\,\delta_e,\pi_{\bar z}(i^{-1})\,\delta_e\rangle\\ &=\frac{1}{\langle R_z\,\delta_e,\delta_e\rangle^2}\langle \pi_z(j)\,R_z\,\delta_e,\delta_e\rangle\langle R_z\,\delta_e,\pi_{\bar z}(i^{-1})\,\delta_e\rangle=\phi_z(j)\,\phi_z(i).\end{aligned}$$

Remark 5. We have shown that for z unreal and non-purely imaginary the representation $\pi_z|_{\mathscr{H}_z}$ cannot be made unitary. In the sequel we prove that also for $z=it$ with $t\in(-1,-\imath^{-1/2})\cup(\imath^{-1/2},1)$ it is nonunitarizable while for $t\in[-\imath^{-1/2},-q^{-1/2})\cup(q^{-1/2},\imath^{-1/2}]$ the representations are equivalent to unitary ones.

3. Analytic Continuation of the Principal Series

In the previous section we have constructed the series of representations parametrized by the annulus $q^{-1/2}<|z|<1$, $z^2\neq -\imath^{-1}$, having the spherical functions ϕ_z as its unique $bi-K$-invariant matrix coefficients. On the other hand the formula (17) which expresses explicitly ϕ_z can be extended by analycity to the annulus $q^{-1}<|z|<1$. In this way we obtain the family of functions ϕ_z with property $\phi_z=\phi_u$ for $q^{-1}<|z|<1$. Moreover by [1] we know that the series $|z|=q^{-1/2}$ consists of positive definite functions which together with ϕ_{z_0}, $z_0=-\imath^{-1}$, give the decomposition of the regular representation of $\mathrm{Aut}(\mathfrak{X}_{\imath_1,\imath_2})$ into irreducible ones. That is why it is called the principal series. In this section we are going to extend the series of representations $\pi_z|_{\mathscr{H}_z}$, $q^{-1/2}<|z|<1$, to the annulus $q^{-1}<|z|<1$ to get the analytic continuation of the principal series. But before doing so we will realize all representations on a common Hilbert space. Our method is analogous to that of [11].

Let $\mathscr{H}_\infty=\operatorname{Ker} P$. Notice that δ_e is the unique K-fixed vector in $\mathscr{H}_\infty$. We are going to map the subspaces $\mathscr{H}_z$ onto $\mathscr{H}_\infty$. This mapping will be defined separately between the radial one-dimensional subspaces and between the nonradial parts of $\mathscr{H}_z$ and $\mathscr{H}_\infty$.

For a linear subspace $\mathscr{M}$ of $l^2(\mathfrak{X}_{\imath_1,\imath_2})$ let $\mathscr{M}^0$ denote the subspace of $\mathscr{M}$ consisting of all functions orthogonal to radial functions, i.e., $\mathscr{M}^0=\{f\in\mathscr{M} : \mathscr{P}f=0\}$. We also introduce two linear operators

$$A_\infty=\left(\frac{\imath_2}{\imath_1}\right)^{1/2} I_{\imath_1}+I_{\imath_2}, \qquad A_0=I_{\imath_1}+\left(\frac{\imath_1}{\imath_2}\right)^{1/2} I_{\imath_2}.$$

LEMMA 6. *Let $z\neq 0$. Then $(A_u-(u/z)\,A_z)\,A_\infty=q((z^2-u^2)/z)I$.*

Proof.

$$\begin{aligned}\left(A_u-\frac{u}{z}A_z\right)A_\infty
&=\left(\frac{\imath_2}{\imath_1}\right)^{1/2}\left(\frac{a(z)}{z}-\frac{u}{z}\frac{a(u)}{u}\right)I_{\imath_1}+\left(\frac{a(u)}{u}-\frac{u}{z}\frac{a(z)}{z}\right)I_{\imath_2}\\
&=\left(\frac{\imath_2}{\imath_1}\right)^{1/2}\frac{a(z)-a(u)}{z}I_{\imath_1}+\frac{z^2a(u)-u^2a(z)}{z^2u}I_{\imath_2}\\
&=\left(\frac{\imath_2}{\imath_1}\right)^{1/2}\imath_1\frac{z^2-u^2}{z}I_{\imath_1}+q\frac{z^2-u^2}{z}I_{\imath_2}=q\frac{z^2-u^2}{z}I.\end{aligned}$$

PROPOSITION 3. *Let $|z|>q^{-1/2}$ and $z^2\neq -\imath^{-1}$. Then the operator $A_\infty^{-1}(I-uP^*)^{-1}A_u^{-1}$ maps $\mathscr{H}_\infty^0$ onto $\mathscr{H}_z^0$ isomorphically; $u=(qz)^{-1}$.*

Proof. The claim follows from the formulas

$$PA_u(I-uP^*)A_\infty=-\frac{1}{z}A_0A_z(I-zP) \tag{18}$$

$$(I-zP)\,A_\infty^{-1}(I-uP^*)^{-1}\,A_u^{-1}=-zA_0^{-1}A_z^{-1}P \tag{19}$$

valid on $l^2(\mathfrak{X}_{\imath_1,\imath_2})^0$.

Concerning the first identity

$$\begin{aligned}PA_u(I-uP^*)A_\infty&=A_zP(I-uP^*)A_\infty=A_z(PA_\infty-uPP^*A_\infty)\\
&=A_z(A_0P-uPP^*A_\infty)=A_zA_0(P-uA_0^{-1}PP^*A_\infty)\\
&=A_zA_0(P-uqI)=-\frac{1}{z}A_zA_0(I-zP).\end{aligned}$$

The second identity is just a simple transformation of the first one.

LEMMA 7. *Let* $|z| > q^{-1/2}$ *and* $z^2 \neq -\imath^{-1}$. *Then* $uR_zA_u(I-uP^*)A_\infty = ((z^2-u^2)/z^2)R_z$. *Moreover if* $f, g \in \mathscr{H}^0_\infty$ *then*

$$\langle R_z f, R_{\bar{z}} g\rangle = \frac{z^2-u^2}{z^2}\langle A_\infty^{-1}(uA_u)^{-1} f, g\rangle. \tag{20}$$

Proof. Observe that in all computations we may omit the operator T_z because it coincides with the identity on $l^2(\mathfrak{X}_{\imath_1,\imath_2})^0$. By Proposition 2(ii), (iv) and Lemma 4(ii) we have $R_z(I-zP^*)=0$. Thus $R_zP^* = (1/z)R_z$. Then by virtue of Lemma 6

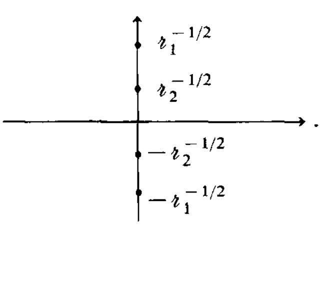
$$\begin{aligned} uR_zA_u(I-uP^*)A_\infty &= uR_zA_uA_\infty - u^2R_zA_uP^*A_\infty \\ &= uR_zA_uA_\infty - \frac{u^2}{z}R_zA_zA_\infty = uR_z\left(A_u - \frac{u}{z}A_z\right)A_\infty \\ &= \frac{uq}{z}(z^2-u^2)R_z = \frac{z^2-u^2}{z^2}R_z. \end{aligned}$$

The above implies that $uR_zA_u(I-uP^*)A_\infty = ((z^2-u^2)/z^2)I$ on $\mathscr{H}^0_z$. Thus by Proposition 3 we obtain $R_z = ((z^2-u^2)/z^2)\,A_\infty^{-1}(I-uP^*)^{-1}\,(uA_u)^{-1}$ on the space $\mathscr{H}^0_\infty$. Thus if $f, g \in \mathscr{H}^0_\infty$ then

$$\begin{aligned} \langle R_z f, R_{\bar{z}} g\rangle = \langle R_z f, g\rangle &= \frac{z^2-u^2}{z^2}\langle A_\infty^{-1}(I-uP^*)^{-1}\,(uA_u)^{-1} f, g\rangle \\ &= \frac{z^2-u^2}{z^2}\langle (uA_u)^{-1} f, (I-uP)^{-1}\,A_\infty^{-1} g\rangle \\ &= \frac{z^2-u^2}{z^2}\langle (uA_u)^{-1} f, A_\infty^{-1} g\rangle = \frac{z^2-u^2}{z^2}\langle A_\infty^{-1}(uA_u)^{-1} f, g\rangle. \end{aligned}$$

From now on we restrict ourselves to the case $\imath_1 \leqslant \imath_2$. The case $\imath_1 > \imath_2$ can be treated in a similar way. Consider two functions $z \mapsto a(z) = \imath_1 z^2 + 1$ and $z \mapsto a(u) = (\imath_2 z^2)^{-1} + 1$. The first function admits the analytic square root in the domain $\{z\colon |z| < \imath_1^{-1/2}\}$ while the second in the domain $\{z\colon |z| > \imath_2^{-1/2}\}$. Both square roots can be continued analytically to the common domain

$$\Omega = \{z \in \mathbb{C} : z \neq it,\ t \in (-\infty, -\imath_1^{-1/2}] \cup [\imath_1^{-1/2}, \infty) \cup [-\imath_2^{-1/2}, \imath_2^{-1/2}]\}$$



Moreover $a(z)^{1/2}$, $a(u)^{1/2}$ are even functions symmetric with respect to the real line, i.e.,

$$a(z)^{1/2}=a(-z)^{1/2}, \qquad a(u)^{1/2}=a(-u)^{1/2}$$
$$a(\bar{z})^{1/2}=\overline{a(z)^{1/2}}, \qquad a(\bar{u})^{1/2}=\overline{a(u)^{1/2}}.$$

Let us define the square root of the operator uA_u as

$$(uA_u)^{1/2}=a(u)^{1/2}\, I_{\imath_1}+q^{-1/2}z^{-1}a(z)^{1/2}\, I_{\imath_2}. \tag{21}$$

Observe that due to this definition we have

$$(-uA_{-u})^{1/2}=(uA_u)^{1/2}\,(I_{\imath_1}-I_{\imath_2}). \tag{22}$$

For any complex number z, $|z|>q^{-1/2}$ and $z^2\neq -\imath_1^{-1}$, define the operator V_z^0 as

$$V_z^0=(1-q^{-2}z^{-4})^{-1/2}\, R_z A_\infty^{1/2}(uA_u)^{1/2}. \tag{23}$$

Then by Lemma 7 and Proposition 3 we have

PROPOSITION 4. *Let $q^{-1/2}<|z|$ and $z^2\neq -\imath_1^{-1}$. Then the operator V_z^0 maps $\mathscr{H}_\infty^0$ onto $\mathscr{H}_z^0$ isomorphically. For real z this mapping is an isometry.*

What is left is to extend the operator V_z^0 to the one-dimensional radial part of $\mathscr{H}_\infty$. For $q^{-1/2}<|z|<1$ and $z^2\neq -\imath_1^{-1}$ let us define the operator V_z on $\mathscr{H}_\infty$ as

$$\begin{aligned} V_z f &= V_z^0 f \qquad \text{if } f\in \mathscr{H}_\infty \\ V_z\,\delta_e &= \imath_1^{1/4}(1-u^2)^{1/2}\, V_z^0\,\delta_e. \end{aligned} \tag{24}$$

The constant by $V_z^0\,\delta_e$ is chosen to satisfy $\langle V_z\,\delta_e, V_{\bar z}\,\delta_e\rangle=1$.

THEOREM 4. *Let $q^{-1/2}<|z|<1$ and $z^2\neq -\imath_1^{-1}$. Then the operator V_z maps the space $\mathscr{H}_\infty$ onto the space $\mathscr{H}_z$ isomorphically. If z is real then V_z is an isometry. Moreover*

$$\langle V_z f, V_{\bar z}\, g\rangle=\langle f, g\rangle, \qquad f, g\in \mathscr{H}_\infty. \tag{25}$$

Proof. The only thing we should prove is (25). But it holds for real z by Proposition 4. Then by analycity of the function $z\mapsto\langle V_z f, V_{\bar z}\, g\rangle$ it holds also for other z.

The isomorphisms V_z allow us to settle all the representations $\pi_z|_{\mathscr{H}_z}$ on the common Hilbert space $\mathscr{H}_\infty$. To see how they act on the new space we are going to compute the matrix coefficients. To simplify the notation set

$$B_z = A_\infty^{1/2}(uA_u)^{1/2}. \tag{26}$$

Due to the definition of $(uA_u)^{1/2}$ the function $z \mapsto B_z$ is analytic in the domain $\mathscr{D} = \{z \in \mathbb{C} : z \neq it, t \in (-\infty, -\imath_1^{1/2}] \cup [-\imath_2^{1/2}, \imath_2^{1/2}] \cup [\imath_1^{1/2}, +\infty]\}$. Moreover

$$B_z^* = B_{\bar{z}}, \qquad B_{-z} = B_z(I_{\imath_1} - I_{\imath_2}) \qquad \text{for} \quad z \in \mathscr{D}, \tag{27}$$

$$\lim_{\varepsilon \to 0^+} B_{\varepsilon + it} = (I_{\imath_1} - I_{\imath_2}) \lim_{\varepsilon \to 0^+} B_{-\varepsilon + it},$$
$$t \in (-\infty, -\imath_1^{1/2}] \cup [-\imath_2^{1/2}, \imath_2^{1/2}] \cup [\imath_1^{1/2}, +\infty]\}. \tag{28}$$

Assume $f, g \in \mathscr{H}_\infty^0$. Then applying (25), Proposition 2(ii), and (12) gives

$$\begin{aligned}
&\langle V_z^{-1} \pi_z(i) V_z f, g \rangle \\
&= \langle \pi_z(i) V_z f, V_{\bar{z}} g \rangle = \frac{z^2}{z^2 - u^2} \langle \pi_z(i) R_z B_z f, R_{\bar{z}} B_{\bar{z}} g \rangle \\
&= \frac{z^2}{z^2 - u^2} \langle R_z \pi_z(i) B_z f, B_{\bar{z}} g \rangle = \frac{z^2}{z^2 - u^2} \langle \pi_z(i) B_z f, B_{\bar{z}} g \rangle \\
&\quad - \frac{z^2}{z^2 - u^2} \frac{u^2}{a(z)\, a(u)} \langle U_{\bar{z}}^* U_z \pi_z(i) B_z f, B_{\bar{z}} g \rangle \\
&= \frac{z^2}{z^2 - u^2} \langle \pi_z(i) B_z f, B_{\bar{z}} g \rangle \\
&\quad - \frac{z^2}{z^2 - u^2} \frac{u^2}{a(z)\, a(u)} \langle \pi_u(i) U_z B_z f, U_{\bar{z}} B_{\bar{z}} g \rangle.
\end{aligned}$$

Since $U_z f = A_z f$ and $U_{\bar{z}} g = A_{\bar{z}} g$ for $f, g \in \mathscr{H}_\infty^0$, then

$$\begin{aligned}
\langle V_z^{-1} \pi_z(i) V_z f, g \rangle &= \frac{z^2}{z^2 - u^2} \langle \pi_z(i) B_z f, B_{\bar{z}} g \rangle \\
&\quad - \frac{z^2}{z^2 - u^2} \frac{u^2}{a(z)\, a(u)} \langle \pi_u(i) A_z B_z f, A_{\bar{z}} B_{\bar{z}} g \rangle.
\end{aligned}$$

Now observing that $(a(z)\, a(u))^{-1/2} (zA_z) B_z = B_u$ gives

$$\langle V_z^{-1} \pi_z(i) V_z f, g \rangle = \frac{1}{z^2 - u^2} \langle [z^2 B_z \pi_z(i) B_z - u^2 B_u \pi_u(i) B_u] f, g \rangle. \tag{29}$$

In a similar way we can compute the remaining formulas:

$$\langle V_z^{-1}\pi_z(i)\, V_z\, \delta_e, g\rangle = \left(\frac{\imath_1 - 1}{\imath_1}\right)^{1/2} \frac{1}{z^2 - u^2} \langle [z^2(1-u^2)^{1/2}\, B_z \pi_z(i) B_z - u^2(1-z^2)^{1/2}\, B_u \pi_u(i)\, B_u]\, \delta_e, g\rangle \tag{30}$$

$$\langle V_z^{-1}\pi_z(i)\, V_z\, \delta_e, \delta_e\rangle = \frac{\imath_1 - 1}{\imath_1} \frac{1}{z^2 - u^2} \langle [z^2(1-u^2)\, B_z \pi_z(i)\, B_z - u^2(1-z^2)\, B_u \pi_u(i)\, B_u]\, \delta_e, \delta_e\rangle. \tag{31}$$

THEOREM 5. *Let $\mathfrak{X}_{\imath_1, \imath_2}$ be a semihomogeneous tree of degrees $\imath_1, \imath_2, \imath_1 < \imath_2$. Put $q = (\imath_1 \imath_2)^{1/2}$. There exists a series of uniformly bounded representations $\Pi_z, q^{-1} < |z| < 1$, of the group* $\operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})$ *on the Hilbert space $\mathscr{H}_\infty = \operatorname{Ker} P$, such that*

(i) *The series Π_z is analytic in the domain $\Omega = \{z; q^{-1} < |z| < 1, z \neq it, t \in (-1, -\imath_1^{-1/2}] \cup [-\imath_2^{-1/2}, \imath_2^{-1/2}] \cup [\imath_1^{-1/2}, 1)\}$.*

(ii) *If $q^{-1/2} < |z| < 1$, $z^2 \neq -\imath_1^{-1}$, then Π_z is equivalent to $\pi_z|_{\mathscr{H}_z}$.*

(iii) *$\Pi_z = \Pi_{-z}$ and $\Pi_z = \Pi_u$ where $u = (qz)^{-1}$, $z \in \Omega$.*

(iv) *$\Pi_z(i)^* = \Pi_{\bar{z}}(i)^{-1}$.*

(v) *$\Pi_z(i) - \Pi_{z'}(i)$ has finite rank.*

(vi) *Any representation $\Pi_z, z^2 \neq -\imath_1^{-1}, -\imath_2^{-1}$, is irreducible. The representations Π_z and $\Pi_{z'}$ are equivalent if and only if $z = z'$, $z = -z'$, or $z = (qz')^{-1}$.*

(vii) *Π_z is a unitary representation if and only if $|z| = q^{-1/2}$, $z \in \mathbb{R}$, or $z = it$ with $t \in [-\imath_1^{-1/2}, -\imath_2^{-1/2}] \cup [\imath_2^{1/2}, \imath_1^{1/2}]$. Otherwise the representation Π_z cannot be made unitary by introducing another equivalent scalar product in $\mathscr{H}_\infty$.*

Proof. By the formulas (28), (29), (30) the family $V_z^{-1}\pi_z(i)\, V_z$, $q^{-1} < |z| < 1$, $z^2 \neq -\imath_1^{-1}$, extends to the analytic series of representations $\Pi_z, z \in \Omega$, satisfying (i), (ii), and the second part of (iii).

For $z = it$ with $t \in [-\imath_2^{-1/2}, -q^{-1}) \cup [\imath_1^{-1/2}, 1)$ we define Π_{it} by

$$\Pi_{it}(i) = \lim_{\varepsilon \to 0^+} \Pi_{\varepsilon + it}(i),$$

and for $z = it$ with $t \in (-1, -\imath_1^{-1/2}] \cup (q^{-1}, \imath_2^{-1/2}]$ we put

$$\Pi_{it}(i) = \lim_{\varepsilon \to 0^+} \Pi_{-\varepsilon + it}(i).$$

In this way we obtain the series of representations $\Pi_z, q^{-1} < |z| < 1$, still satisfying (i), (ii), and the second part of (iii).

We turn now to the proof of (v). Let $z \in \Omega$, then

$$\begin{aligned}
&\frac{1}{z^2-u^2}\left[z^2 B_z \pi_z(i) B_z - u^2 B_u \pi_u(i) B_u\right] \\
&\quad = \frac{1}{z^2-u^2}\left[z^2 B_z^2 - u^2 B_u^2\right] \lambda(i) \\
&\qquad + \frac{1}{z^2-u^2}\left[z^2 B_z(\pi_z(i) - \lambda(i)) B_z - u^2 B_u(\pi_u(i) - \lambda(i)) B_u\right].
\end{aligned}$$

The second term, now denoted by K_z, is a finite-dimensional operator by Theorem 3(iii). On the other hand by the definition of B_z and by Lemma 6 it is easy to check that

$$\frac{1}{z^2-u^2}\left[z^2 B_z^2 - u^2 B_u^2\right] = I.$$

Thus

$$\frac{1}{z^2-u^2}\left[z^2 B_z \pi_z(i) B_z - u^2 B_u \pi_u(i) B_u\right] = \lambda(i) + K_z,$$

where K_z is a finite-dimensional operator. Combined with (28) it implies (v).

In order to prove the first part of (iii) note that for any z, $|z| < 1$, we have

$$\pi_{-z}(i) = (I_{\imath_1} - I_{\imath_2})\, \pi_z(i)(I_{\imath_1} - I_{\imath_2})$$

(see the last part of the proof of Theorem 3). Then by (27) we get $B_z \pi_z(i) B_z = B_{-z} \pi_{-z}(i) B_{-z}$ for any z, $q^{-1} < |z| < 1$. Now the claim in (iii) follows easily from the formulas (29), (30), (31).

As regards to (iv) for real z in Ω the representation Π_z is unitary because in this case $\pi_z|_{\mathscr{H}_z}$ is a unitary representation and V_z is an isometry (Theorem 4). Then by the Riemann–Schwarz Reflection Principle, (iv) holds for all $z \in \Omega$. Furthermore if $|z| = q^{-1/2}$ then $u = \bar{z}$. Hence by (iii) and (iv) we have

$$\Pi_z(i)^* = \Pi_{\bar{z}}(i)^{-1} = \Pi_u(i)^{-1} = \Pi_z(i)^{-1}.$$

It means Π_z is unitary for $|z| = q^{-1/2}$. Assume now $z = it$ where $z \in [-\imath_1^{-1/2}, -\imath_2^{-1/2}] \cup [\imath_2^{-1/2}, \imath_1^{-1/2}]$. Then $\bar{z} = -z$. Hence by (iii) and (iv) we derive

$$\Pi_z(i)^* = \Pi_{\bar{z}}(i)^{-1} = \Pi_{-z}(i)^{-1} = \Pi_z(i)^{-1}.$$

To complete the proof of (vii) in view of (i) and Theorem 3 it suffices to show that Π_z is not unitarizable only for $z = it$ where $t \in (-1, -\imath_1^{-1/2}) \cup (-\imath_2^{-1/2}, -q^{-1}) \cup (q^{-1}, \imath_2^{-1/2}) \cup (\imath_1^{-1/2}, 1)$. Because of (iii) it is enough to consider $t \in (\imath_1^{-1/2}, 1)$. We claim that for such t there holds

$$\Pi_{it}(i)^* = (I_{\imath_1} - I_{\imath_2})\, \Pi_{it}(i)^{-1}\, (I_{\imath_1} - I_{\imath_2}). \tag{32}$$

Indeed, it follows directly from the definition of Π_{it} (recall $\Pi_{it}(i) = \lim_{\varepsilon \to 0^+} \Pi_{\varepsilon + it}(i)$) and the formulas (28), (29), (30), (31). Now the fact that Π_z cannot be made unitary relies on the following

PROPOSITION 5. *Let π be a bounded irreducible representation of a locally compact group G in a Hilbert space $\mathscr{H}$. Assume there exists a unitary operator U such that*

$$\pi(g)^* = U\pi(g)^{-1}\, U^{-1}.$$

Then π is equivalent to a unitary representation if and only if U is a constant multiple of the identity operator.

Proof. Assume that there exists an invertible linear operator A on $\mathscr{H}$ and a unitary representation σ of G in the space $\mathscr{H}$ such that $\pi(g) = A^{-1}\sigma(g)A$. Thus

$$A^*\sigma(g)^{-1}\,(A^*)^{-1} = \pi(g)^* = U\pi(g)^{-1}\, U^{-1} = UA^{-1}\sigma(g)^{-1}\, U^{-1}A.$$

It implies

$$(A^*)^{-1}\, UA^{-1}\sigma(g)^{-1} = \sigma(g)^{-1}\,(A^*)^{-1}\, UA^{-1}.$$

Since σ is also an irreducible representation of G then by the Schur lemma $(A^*)^{-1}\, UA^{-1} = cI$ for some complex constant c. Therefore $U = cA^*A$. The last is possible only if $U = \lambda I$ with $|\lambda| = 1$.

What is left is to prove (vi). For $z, q^{-1} < |z| < 1$, let $\mathscr{P}_K = \int_K \Pi_z(k)\, dk$ where (cf. the proof of Theorem 3) K is a stabilizer of the vertex e. Observe that for any $k \in K$ and any $z, q^{-1/2} < |z| < 1$, we have $\Pi_z(k) = \lambda(k)$. Indeed, if $k \in K$ then $\lambda(k)$ commutes with P, P^*, T_z, A_z, and A_∞. Thus by (23) and (24) it commutes with V_z. Therefore

$$\Pi_z(k) = V_z^{-1}\pi_z(k)\, V_z = V_z^{-1}\lambda(k)\, V_z = \lambda(k).$$

By analycity we have $\Pi_z(k) = \lambda(k)$ for any $z, q^{-1/2} < |z| < 1$. Furthermore $\mathscr{P}_K = \int_K \lambda(k)\, dk$ is the orthogonal projection onto radial functions in $\mathscr{H}_\infty = \operatorname{Ker} P$, i.e., onto $\mathbb{C}\,\delta_e$.

Let i be an isometry of $\mathfrak{X}_{\imath_1, \imath_2}$ such that $d(i(e), e) = 2$. Let $q^{-1} < |z| < 1$. Then by (31) (cf. the proof of Theorem 3) we obtain $\mathscr{P}_K \Pi_z(i) \mathscr{P}_K = \phi_z(i) \mathscr{P}_K$.

It implies that two representations Π_z and $\Pi_{z'}$ are equivalent if and only if $\phi_z(i) = \phi_{z'}(i)$. By (17) it is equivalent to $z^2 = z'^2$ or $z = 1/qz'$.

Now we turn to the irreducibility. By Theorem 3 and by (ii), (iii) we have to prove the irreducibility only for $|z| = q^{-1/2}$. Fix z, $|z| = q^{-1/2}$. Let $\mathcal{M} = \overline{\operatorname{span}}\{\Pi_z(i)\,\delta_e : i \in \operatorname{Aut}(\mathfrak{X}_{\imath_1, \imath_2})\}$. $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\infty$ invariant for Π_z. We show that $\mathcal{M}$ does not contain a proper closed subspace invariant for Π_z. Assume that $\mathcal{M}_0 \subseteq \mathcal{M}$ is a closed subspace invariant for Π_z. Thus $\mathcal{P}_K \mathcal{M}_0 \subseteq \mathcal{M}_0$. If $\mathcal{P}_K \mathcal{M}_0 \neq (0)$ then $\mathcal{P}_K \mathcal{M}_0 = \mathbb{C}\,\delta_e \subseteq \mathcal{M}_0$, which by the definition of $\mathcal{M}$ implies $\mathcal{M}_0 = \mathcal{M}$. Let $\mathcal{P}_K \mathcal{M}_0 = (0)$. Then the subspace $\mathcal{M}'_0 = \mathcal{M} \ominus \mathcal{M}_0$ is also a closed invariant subspace of $\mathcal{M}$ and $\mathcal{P}_K \mathcal{M}'_0 \neq (0)$ (recall that Π_z is unitary by (vii)). Thus $\mathcal{M}'_0 = \mathcal{M}$ which yields $\mathcal{M}_0 = (0)$. To complete the proof of irreducibility we need only to show that $\mathcal{M} = \mathcal{H}_\infty$, i.e., that δ_e is a cyclic vector for Π_z. On the contrary assume that $f \in \mathcal{H}_\infty$ is orthogonal to $\mathcal{M}$. In particular $\langle f, \delta_e \rangle = f(e) = 0$. Next, as in the proof of Theorem 3, it is not hard to show by induction on $d(x, e)$ that $f(x) = 0$ for any x. This completes the proof of Theorem 5.

4. Approximate Units of the Fourier Algebra

In this section we apply the series of representations defined in Section 1 to derive approximation properties for groups acting on trees.

Let G be a locally compact group and $A(G)$ its Fourier algebra. It is known that if G is an amenable group then there exists a net ϕ_α of functions in $A(G)$ such that:

(i) ϕ_α has compact support for any α;

(ii) $\|\phi_\alpha \phi - \phi\|_{A(G)} \to 0$ for any $\phi \in A(G)$;

(iii) $\|\phi_\alpha\|_{A(G)} \leqslant 1$.

A net ϕ_α which satisfies (i) and (ii) we call an approximate unit of $A(G)$. If G is nonamenable then there is no approximate unit bounded in $A(G)$-norm. However, in many cases it is possible to construct an approximate unit unbounded in $A(G)$-norm but bounded in the multiplier norm on $A(G)$, i.e., such that

$$\|\phi_\alpha\|_{M\mathcal{A}(G)} = \sup\{\|\phi_\alpha \phi\|_{A(G)} : \|\phi\|_{A(G)} \leqslant 1\} \leqslant 1.$$

Any function ϕ in $A(G)$ defines a linear multiplier operator m_ϕ on $A(G)$ by $A(G) \ni \psi \mapsto \phi\psi \in A(G)$. Its transposed operator M_ϕ is a σ-weakly continuous operator on the von Neumann algebra $\mathfrak{M}(G)$ of G which is determined uniquely by $M_\phi \lambda(x) = \phi(x)\,\lambda(x)$ where $\lambda(x)$ is the left translation by the element x in G. Following de Canniere and Haagerup if M_ϕ is a completely bounded map of $\mathfrak{M}(G)$ then we say that ϕ is a completely bounded multiplier of $A(G)$. The space of completely bounded multipliers is denoted by $M_0A(G)$. It is a Banach algebra with norm $\|\phi\|_{M_0(G)}=\|M_\phi\|_{CB}$ where $\| \ \|_{CB}$ denotes the completely bounded norm of the operator M_ϕ. We refer the reader to the work by de Canniere and Haagerup where a detailed exposition of this subject is given.

We say that a locally compact group G has the completely bounded approximation property if there exists an approximate unit ϕ_a in $A(G)$ such that $\|\phi_\alpha\|_{M_0A(G)} \leqslant 1$. There are many equivalent ways to define $M_0A(G)$. We refer to the work by Bożejko and Fendler [2] concerning this subject and also to the work of de Canniere and Haagerup [5] who proved approximation properties for various simple Lie groups like $SO_0(n, 1)$ and its discrete subgroups. Finally Cowling and Haagerup [4] proved that all simple Lie groups of real rank one admit completely bounded approximate units in the Fourier algebra. We will need the following fact concerning multipliers

PROPOSITION 6 (Bożejko, Fendler, Gilbert). *Let ϕ be a complex function on a locally compact group G. Assume there exist a Hilbert space $\mathscr{H}$ and two continuous bounded maps $u, v\colon G \mapsto \mathscr{H}$ such that*

$$\phi(y^{-1}x) = \langle u(x), v(y)\rangle, \qquad x, y \in G.$$

Then ϕ is a completely bounded multiplier of $A(G)$ and

$$\|\phi\|_{M_0A(G)} \leqslant \sup_{x\in G} \|u(x)\| \sup_{y\in G} \|v(y)\|.$$

EXAMPLE. Let G be a group acting on a tree $\mathfrak{X}$. Fix a vertex e in $\mathfrak{X}$ and define the representations π_z with respect to e. Let $\phi_z(g) = z^{d(ge,e)}$. By Proposition 6 the function ϕ_z belongs to $M_0A(G)$ because

$$\phi_z(h^{-1}g) = \langle \pi_z(g)\,\delta_e, \pi_{\bar z}(h)\,\delta_e\rangle.$$

Moreover by the formula (15) we have

$$\sup_{g\in G} \|\pi_z(g)\,\delta_e\| = \sup_{h\in G} \|\pi_{\bar z}(h)\,\varphi_e\| = |1-z^2|^{1/2}\,(1-|z|^2)^{-1/2} \tag{33}$$

hence

$$\|\phi_z\|_{M_0A(G)} \leqslant \frac{|1-z^2|}{1-|z|^2}.$$

The main result of this section is the following.

THEOREM 6. *Let G be a locally compact group acting on a tree $\mathfrak{X}$. Assume that there exists a vertex $e \in \mathfrak{X}$ such that the stability subgroup $G_e \in \{g \in G : ge = e\}$ is compact. Then the Fourier algebra $A(G)$ of G admits an approximate unit $\{\phi_n\}_{n=0}^{\infty}$ bounded in $M_0A(G)$-norm.*

Before the proof we derive two immediate corollaries.

COROLLARY 3. *For any closed subgroup G of* $\mathrm{Aut}(\mathfrak{X})$ *the Fourier algebra $A(G)$ has the completely bounded approximation property.*

For a prime number p let $\mathbb{Q}_p$ denote the field of p-adic numbers. By Serre [10] we know that the group $SL(2, \mathbb{Q}_p)$ acts on the homogeneous tree of degree $p+1$, and it has compact stabilizers of any vertex of the tree.

COROLLARY 4 (Bożejko and Picardello [3]). *For any prime number p the group $SL(2, \mathbb{Q}_p)$ has the completely bounded approximation property.*

In the proof of Theorem 6 we will follow the method developed in [9]. Let γ be a piecewise smooth curve contained in the unit disc. For a group G acting on a tree $\mathfrak{X}$ consider the representation π_γ of G defined as

$$\pi_\gamma = \oplus \int_\gamma \pi_z \, |dz|$$

which acts on the Hilbert space $\mathscr{H}_\gamma = \oplus \int_\gamma l^2(\mathfrak{X}) \, |dz|$. It is again a uniformly bounded representation with

$$\sup_{g \in G} \|\pi_\gamma(g)\| \leqslant 2 \max_{z \in \gamma} \frac{|1-z^2|}{1-|z|}.$$

The following proposition is just a reformulation of Proposition 2 from [9].

PROPOSITION 7. *Let f be a function holomorphic in a neighbourhood of γ. Then the complex function ϕ on G defined by*

$$\phi(g) = \int_\gamma z^{d(ge, e)} f(z) \, dz$$

is a matrix coefficient of the representation π_γ. In particular ϕ belongs to $M_0A(G)$ and

$$\|\phi\|_{M_0A(G)} \leqslant \int_\gamma |f(z)| \frac{|1-z^2|}{1-|z|^2} \, |dz|.$$

EXAMPLE. Keeping the same notation let χ_n, $n=0, 1, 2, \ldots$, denote the functions on G defined as

$$\chi_n(g)=\begin{cases}1 & \text{when} \quad d(ge, e)=n\\ 0 & \text{otherwise.}\end{cases}$$

We can express χ_n as

$$\chi_n(g)=\frac{1}{2\pi i}\int_{|z|=r} z^{d(ge,e)}z^{-(n+1)}\,dz.$$

Thus applying Proposition 7 the function χ_n belongs to $M_0A(G)$ and

$$\|\chi_n\|_{M_0A(G)}\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|1-r^2e^{2it}|}{1-r^2}\,r^{-n}\,dt\leqslant\frac{2}{r^n(1-r^2)}.$$

A simple computation of the minimum of the right hand side gives

$$\|\chi_n\|_{M_0A(G)}\leqslant\frac{e}{2}(n+2).$$

Proof of Theorem 6. Let G be a group acting on a tree $\mathfrak{X}$. Let e be a vertex of $\mathfrak{X}$ such that $G_e=\{g\in G : ge=e\}$ is a compact subgroup of G. Then the functions χ_n, $n=0, 1, 2, \ldots$ (see Example above) have compact support. Indeed, let $E_n=\{ge : g\in G, d(ge, e)=n\}$. For any x in E_n choose an element g_x in G such that $g_xe=x$. Therefore

$$\operatorname{supp}(\chi_n)\subseteq\bigcup_{x\in E_n} g_xG_e.$$

Consider the set of functions $\phi_t(g)=t^{d(ge,e)}$, $0\leqslant t<1$. By Theorem 1(ii), (iv) the functions ϕ_t are positive definite. In particular by Proposition 6 the functions ϕ_t belong to $M_0A(G)$ and $\|\phi_t\|_{M_0A(G)}\leqslant 1$. Moreover when t tends to 1 then ϕ_t tends to the function constantly 1 on G uniformly on compact sets (by compactness of G_e). Thus we can apply the theorem by Nielson (see [5, Appendix]) and we get

$$\|\phi_t\phi-\phi\|_{A(G)}\mapsto 0 \qquad \text{when} \quad t\mapsto 1^- \text{ for any } \phi\in A(G).$$

In order to complete the proof it suffices to show that the multiplier operators m_{ϕ_t} on $A(G)$ lie in the $M_0A(G)$-norm closure of multiplier operators associated with compactly supported functions. For a fixed $t<1$ and $n=0, 1, 2, \ldots$, let

$$\phi_{t,n}(g)=\begin{cases}\phi_t(g) & \text{when} \quad d(ge, e)\leqslant n\\ 0 & \text{otherwise.}\end{cases}$$

The functions $\phi_{t,n}$ have compact support because $\phi_{t,n} = \sum_{k=0}^{n} t^k \chi_k$. Furthermore by (9) we have

$$\|\phi_{t,n} - \phi_t\|_{M_0A(G)} \leqslant \frac{e}{2} \sum_{k=n+1} t^k(k+2) \to 0 \qquad \text{when} \quad n \to \infty.$$

This completes the proof.

Remark 6. If $\mathfrak{X}$ is a nonhomogeneous tree then clearly there is no group which acts transitively on $\mathfrak{X}$. It means that the stabilizers of vertices are not all conjugate to each other. However, in the proof of Theorem 6 we need only the compactness of one of the stabilizers. We conjecture that as in the theorem of Julg and Valette amenability of a stabilizer should suffice.

Remark 7. In the paper [6] it is proved that if a locally compact group G acts on a tree such that the stabilizers of the vertices are amenable then the group G is so-called K-amenable. In particular the group $SL(2, \mathbb{Q}_p)$ shares this property.

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