On the isotriviality of projective iterative $\delta$-varieties

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Abstract

We study algebraic varieties $X$ over a universal iterative differential field $(K, \delta)$ (typically of positive characteristic), together with an extension of $\delta$ to an iterative derivation $D$ of the structure sheaf of $X$. Our work is motivated by the conjecture that if $X$ is projective then the pair $(X, D)$ is isotrivial, namely isomorphic over $K$ to a pair $(Y, D_0)$ where $Y$ is defined over the constants $C$ of $K$ and $D_0$ is the lifting to $K$ of the trivial iterative derivation on $Y_C$. We prove that up to isomorphism there is at most one such $D$ on $X$ extending $\delta$, thus answering the question when $X$ is defined over $C$. Other special cases are also proved, including abelian varieties, and smooth curves.

1 Introduction

In this paper we attempt to generalize results of Buium (see [Bu]) on projective $\delta$-varieties over differential fields of characteristic 0, to the positive characteristic case. In the characteristic 0-case, the ground field $K$ is equipped with a derivation $\delta$ such that $(K, \delta)$ is differentially closed. A $\delta$-structure on a variety $X$ defined over $K$ is an extension of $\delta$ to a derivation $D$ of the structure sheaf of $X$. Giving $X$ a $\delta$-structure is equivalent to equipping $X$

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with a regular section \( s : X \to T_{\delta}(X) \) (defined over \( K \)) of a certain twisted version \( T_{\delta}(X) \) of the tangent bundle of \( X \). The pair \( (X, D) \) or \( (X, s) \) is called a \( \delta \)-variety over \( K \).

If \( X \) is defined over the field of constants \( C \) of \( K \), then the structure sheaf of \( X \) over \( C \) can be equipped with the 0-derivation, which can be tensored with \( \delta \) over \( K \), to get a derivation \( D_0 \) of the structure sheaf of \( X \). This corresponds to the 0-section of the tangent bundle of \( X \). We call such a pair \( (X, D_0) \) a trivial \( \delta \)-variety.

There is a natural notion of morphism of \( \delta \)-varieties, and \( (X, D) \) is said to be isotrivial if it is isomorphic to a trivial \( \delta \)-variety.

In [Bu], Buium proves that (in this characteristic 0 context), any \( \delta \)-variety \( (X, D) \) over \( K \) such that \( X \) is projective, is isotrivial.

The work presented here is an attempt to generalize Buium’s theorem to a suitable positive characteristic context. In characteristic 0 if we equip a function field \( K = k(t) \) (say where \( k \) is algebraically closed) with the derivation \( d/dt \) then the field of constants is \( k \). However in characteristic \( p > 0 \), the field of constants of \( K \) is \( K^p \) rather than \( k \). The situation can be remedied by replacing the single derivation \( d/dt \) by a suitable sequence of maps (a Hasse derivation) whose common field of constants will be \( k \).

So we will work with such generalized derivations.

**Definition 1.1** Let \( R \) be a ring. Then

(i) A sequence \( \delta = (\delta_n : n < \omega) \) of additive maps from \( R \) to itself is called a Hasse derivation if \( \delta_0 \) is the identity, and for all \( n > 0 \) and \( x, y \in R \),

\[
\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y).
\]

(ii) We call the sequence \( \delta \) an iterative Hasse derivation on \( R \) if in addition to (i) we have for all \( i, j \)

\[
\delta_i \circ \delta_j = \binom{i+j}{i} \delta_{i+j}.
\]

We will sometimes use the expression “iterative derivation” for “iterative Hasse derivation”.

\( (K, \delta) \) will usually denote a field \( K \) of characteristic \( p > 0 \) equipped with an iterative derivation \( \delta \). Ziegler [Zi] identified a complete first order theory \( SCH_{p,1} \) (the theory of separably closed Hasse fields of characteristic \( p \) and
Ershov invariant 1) whose models are appropriate to work over in our context. In fact it will usually be appropriate to take \((K, \delta)\) to be a “universal domain” namely a saturated model of \(SCH_{p,1}\).

The field of (absolute) constants \(C\) of \(K\) consists of those \(x \in K\) such that \(\delta_i(x) = 0\) for all \(i\), which coincides with the intersection of all the \(K^{p^n}\).

In section 3 we introduce iterative \(\delta\)-schemes over \(K\). For now, an iterative \(\delta\)-variety over \(K\) is a variety \(X\) over \(K\) together with an extension \(D\) of \(\delta\) to an iterative derivation of the structure sheaf of \(X\). If \((X, D^1)\) and \((Y, D^2)\) are such then a morphism (of iterative \(\delta\)-varieties) between them is the obvious thing: namely a morphism \(f : X \to Y\) defined over \(K\) such that whenever \(f(a) = b\), and \(h\) is a regular function on \(Y\) at \(a\), then \(D^2(h)(b) = (D^1(h \circ f))(a)\). Again we have a notion of trivial \(\delta\)-variety over \(K\). Isotrivial means again isomorphic over \(K\) to a trivial object. To have “enough” isomorphisms we really need here to assume that \((K, \delta)\) is a universal domain.

Our main result, proved in section 6, is:

(*) if \(X\) is a projective iterative \(\delta\)-variety over \(K\), then \(X\) has at most one structure of a iterative \(\delta\)-variety over \(K\). Namely if \(D^1, D^2\) are iterative \(\delta\)-structures on \(X\) over \(K\), then \((X, D^1)\) and \((X, D^2)\) are isomorphic.

It follows from (*) that the analogue of Buium’s theorem (isotriviality of projective \(\delta\)-varieties) holds if we already know that \(X\) is defined over \(C\).

We also prove (section 7) the full analogue of Buium’s result in special cases, such as when \(X\) has ample canonical or anticanonical divisor. This proof does not use (*).

We will also mention the work of Benoit [Be] which is very relevant to our main conjecture. Benoit proves that if the algebraic variety \(X\) (over a model \((K, \delta)\) of \(SCH_{p,1}\)) can be equipped with the structure of an iterative \(\delta\)-variety over \(K\), then \(K\) descends to \(K^{p^n}\) for all \(n\). If \(X\) belongs to a family with a good “moduli space” one can conclude that \(X\) descends to \(C\). By this means we can also, using (*), conclude that the conjecture holds when for example \(X\) is an abelian variety.

2 More on iterative derivations

We have already defined the notion of an iterative (Hasse) derivation \(\delta\) on a ring \(R\). There are obvious notions of an iterative differential ideal of \(R\), and of a homomorphism between iterative differential rings.
Let us fix an iterative differential field \((K, \delta)\). By a \(\delta\)-algebra over \(K\), or a \(K\)-\(\delta\)-algebra, we mean a \(K\)-algebra \(R\) together with an iterative derivation \(D\) on \(R\) such that for \(x \in K\), \(y \in R\) and \(n > 0\),

\[
D_n(xy) = \sum_{i+j=n} \delta_i(x)D_j(y).
\]

Now suppose \((R, \delta)\) to be an iterative differential ring. By a \(\delta\)-module over \(R\), or an \(R\)-\(\delta\)-module, we mean an \(R\)-module \(V\) together with a sequence \(D = (D_n : n < \omega)\) of endomorphisms of the abelian group \(V\), such that \(D_0\) is the identity,

\[
D_n(rx) = \sum_{i+j=n} \delta_i(r)D_j(x)
\]

(for \(n > 0\), \(r \in R\) and \(x \in V\)), as well as the iterativity property

\[
D_i \circ D_j = \binom{i+j}{i} D_{i+j}.
\]

We may sometimes speak for example of a \(K\)-\(\delta\)-algebra \(R\) when we really mean \((R, D)\).

We will need the following well-known facts about Hasse derivations.

**Fact 2.1** Assume \(R, R'\) are \(K\)-\(\delta\)-algebras and \(V, V'\) are \(R\)-\(\delta\)-modules. Then:

(i) \(D\) uniquely extends to a Hasse derivation on any subring of the quotient field of \(R\) with image maybe outside this subfield. The image stays inside, if the subring is a localization.

(ii) \(R \otimes_K R'\) is a \(K\)-\(\delta\)-algebra.

(iii) \(V \oplus V', V \otimes_R V', \bigwedge^n V, V^*\) are \(R\)-\(\delta\)-modules.

(iv) A direct limit of \(R\)-\(\delta\)-modules is an \(R\)-\(\delta\)-module.

(v) If \(I\) is a radical \(\delta\)-ideal and \(R\) is noetherian, then primary components of \(I\) are \(\delta\)-ideals.

Denote \(\text{Spec}(K)\) by \(A\) and \(\text{Spec}(K[X]/X^n)\) by \(A^{(n)}\). If a tensor product has no subscript it is taken over \(K\) and if a product of schemes has no subscript, it is taken over \(A\). The same applies to isomorphisms.

For a \(K\)-algebra \(S\), we will denote \(S \otimes K[X]/X^n\) by \(S^{(n)}\) and more generally for an \(A\)-scheme \(X\), \(X^{(n)}\) denotes \(X \times A^{(n)}\). Let \(S^{(\infty)} := \lim S^{(n)}\) and \(X^{(\infty)} := \lim X^{(n)}\).
Note that $S^{(n)} \cong S[X]/X^n$ and $S^{(\infty)} \cong S[[X]]$, but usually $S^{(\infty)} \not\cong S \otimes K[[X]]$
eq S[X]/X^n$. As before, $S^{(\infty)} := \lim \leftarrow S^{(n)}$.

Note that $S^{(n)}$ is isomorphic as a $K$-algebra to the tensor product $S \otimes K[[X]]$, where the $K$-algebra structure comes from the following composition

$$K \xrightarrow{\delta} K[X]/X^n \xrightarrow{\rho} S \otimes K[[X]] .$$

Therefore, we can define $X^{(n)}_{\delta}$ for any scheme $X$, as the product $X \times \mathbf{A}^{(n)}$, where the $\mathbf{A}$-scheme structure comes from the following composition

$$X \times \mathbf{A}^{(n)} \longrightarrow \mathbf{A}^{(n)} \xrightarrow{\text{Spec}(\delta)} \mathbf{A} .$$

Notice that $X^{(n)}_{\delta}$ is naturally an $\mathbf{A}^{(n)}$-scheme and that we have an easy fact.

**Fact 2.2** There is an isomorphism over $X$

$$X^{(n)}_{\delta} \times_{\mathbf{A}^{(n)}} Y^{(n)}_{\delta} \cong (X \times Y)^{(n)}_{\delta}$$

given by

$$((x, v), (y, v)) \mapsto ((x, y), v).$$

Note also that we cannot replace $n$ by $\infty$ in the isomorphism above (e.g. it is not true that for $K$-algebras $R, S$, $R[[X]] \otimes_K [[X]] S[[X]]$ is isomorphic to $(R \otimes S)[[X]]$).

For any ring $L$, consider the map

$$c : L[[X]] \rightarrow L[[X, Y]], \quad c(\sum a_i X^i) = \sum \binom{i + j}{i} a_{i+j} X^i Y^j .$$

Clearly $c$ is a natural transformation and for each $n$ there is a map $c_n : L[X]/X^n \rightarrow L[X, Y]/(X^n, Y^n)$ such that $c = \lim \leftarrow c_n$.  

5
3 Extensions of iterative derivations

Given a ring $L$, we can consider a sequence of maps $D = (D_n : L \to L)$ as a map
\[ D : L \to L[[X]], \quad D(x) = \sum D_i(x)X^i, \]
and we sometimes denote by the same symbol the map into $L[X]/X^n$
\[ D : L \to L[X]/X^n, \quad D(x) = \sum_{i=0}^{n-1} D_i(x)X^i, \]

Note that $D = (D_i)$ is an iterative Hasse derivation on $L$ if and only if $D : L \to L[[X]]$ is a ring homomorphism and the following diagram is commutative
\[
\begin{array}{ccc}
L[[X,Y]] & \xrightarrow{D^{(\infty)}} & L[[X]] \\
\uparrow c & & \uparrow D \\
L[[X]] & \xleftarrow{D} & L,
\end{array}
\]
where $D^{(\infty)}$ is the image of $D : L \to L[[X]]$ under the power series functor.

Assume $R$ is a $K$-algebra. An iterative Hasse derivation $D$ on $R$ extends $\delta$ if and only if the $K$-algebra structure on $R[[X]]$ given by $D$ coincides with $R_\delta^{(\infty)}$. Hence we obtain the following:

**Fact 3.1** Extensions of $\delta$ to $R$ correspond to $K$-algebra homomorphisms
\[ D : R \to R_\delta^{(\infty)}, \quad D(x) = \sum D_i(x)X^i \]
such that the following diagram is commutative
\[
\begin{array}{ccc}
(R_\delta^{(\infty)})^{(\infty)} & \xrightarrow{D_\delta^{(\infty)}} & R_\delta^{(\infty)} \\
\uparrow c & & \uparrow D \\
R_\delta^{(\infty)} & \xleftarrow{D} & R.
\end{array}
\]

So, we need to deal with algebras of the form $(R_\delta^{(\infty)})^{(\infty)}$.

It is well-known (see e.g. [Tr, page 30]) that (not necessarily iterative) Hasse
derivations considered as maps $D : L \to L[[X]]$ extend to ring endomorphisms $e^D : L[[X]] \to L[[X]]$. One defines

$$e^D(\sum a_i X^i) = \sum_i \left( \sum_{n+m=i} D_n(a_m) \right) X^i.$$ 

Clearly $e^D$ is the inverse limit of maps

$$e_m^D : L[[X]]/X^{m+1} \to L[[X]]/X^{m+1},$$

and it is also true that each $e_m^D$ is an isomorphism, so $e^D$ is an isomorphism as well.

If $D$ is an iterative Hasse derivation on $R$ extending $\delta$, then $e^D$ is a $K$-algebra isomorphism between $R[[X]]$ with and $R^{(\infty)}_{\delta}$. Therefore $K^{(n)}_{\delta} \cong K[X]/X^n$ and $K^{(\infty)}_{\delta} \cong K[[X]]$. So, obviously $A^{(n)}_{\delta} \cong A^{(n)}$ and $A^{(\infty)}_{\delta} \cong A^{(\infty)}$.

From now on, for any ring $L$, we use the notation $L^{(\infty)}$ for $L[[X]]$.

Assume we are given an iterative Hasse derivation $D$ on $R$ extending $\delta$. As noticed above we have an isomorphism

$$e^D : R^{(\infty)} \to R^{(\infty)}_{\delta},$$

which induces isomorphisms

$$(e^D)^{\infty} : (R^{(\infty)})^{(\infty)} \to (R^{(\infty)}_{\delta})^{(\infty)},$$

$$(e^D)^{\infty}_{\delta} : (R^{(\infty)})^{\infty}_{\delta} \to (R^{(\infty)}_{\delta})^{\infty}_{\delta}.$$ 

Also,

$$D^{(\infty)}_{\delta} : R^{(\infty)}_{\delta} \to (R^{(\infty)}_{\delta})^{(\infty)}_{\delta}$$

determines an iterative Hasse derivation on $R^{(\infty)}_{\delta}$, so we obtain an another isomorphism

$$e^{D^{(\infty)}_{\delta}} : (R^{(\infty)}_{\delta})^{(\infty)}_{\delta} \to (R^{(\infty)}_{\delta})^{(\infty)}_{\delta}.$$ 

There is also one more iterative Hasse derivation coming from the composition below

$$R^{(\infty)} \xrightarrow{e^D} R^{(\infty)}_{\delta} \xrightarrow{(D_0)^{\infty}_{\delta}} (R^{(\infty)}_{\delta})^{(\infty)}.$$
(where $D_0$ is the 0-derivation on $R$) which we call (for lack of a better name) $D_{(\infty)}$. Note that $D_{(\infty)}$ coincides with
\[ D^{(\infty)} : R^{(\infty)} \to (R^{(\infty)})^{(\infty)} \]
composed with an “$\langle X, Y \rangle \mapsto (Y, X)\rangle$”-isomorphism
\[ (R^{(\infty)})_{\delta}^{(\infty)} \cong (R^{(\infty)})^{(\infty)}_{\delta}. \]
As before, $D_{(\infty)} : R^{(\infty)} \to (R^{(\infty)})^{(\infty)}_{\delta}$ gives us the forth isomorphism
\[ e^{D_{(\infty)}} : (R^{(\infty)})^{(\infty)}_{\delta} \to (R^{(\infty)})^{(\infty)}_{\delta}. \]
It is clear that the following diagram is commutative
\[ \begin{array}{ccc} (R^{(\infty)})^{(\infty)}_{\delta} & \xrightarrow{(eD)_{\delta}^{(\infty)}} & (R^{(\infty)})^{(\infty)}_{\delta} \\ e_{\delta}^{D_{(\infty)}} \uparrow & & \uparrow e^{D_{(\infty)}} \\ (R^{(\infty)})^{(\infty)}_{\delta} & \xrightarrow{(eD)_{\delta}^{(\infty)}} & (R^{(\infty)})^{(\infty)}_{\delta} \end{array} \]
By an easy diagram chase we obtain.

**Fact 3.2** Extensions of $\delta$ to $R$ correspond to $K$-algebra homomorphisms
\[ D : R \to R^{(\infty)}_{\delta}, \quad D(x) = \sum D_i(x)X^i \]
such that the following diagram is commutative
\[ \begin{array}{ccc} (R^{(\infty)}_{\delta})^{(\infty)} & \xrightarrow{(eD)_{\delta}^{(\infty)}} & (R^{(\infty)})^{(\infty)}_{\delta} \\ e_{\delta}^{D_{(\infty)}} \uparrow & & \uparrow e^{D_{(\infty)}} \\ R^{(\infty)}_{\delta} & \xrightarrow{(eD)_{\delta}^{(\infty)}} & R^{(\infty)} \end{array} \]

**Remark**
We could have picked some other diagram (and there were more natural choices) instead of the above one. But we will need precisely the diagram above in the sequel.
It is also clear that extensions of $\delta$ as above are in 1-to-1 correspondence with isomorphisms $\phi : R^{(\infty)}_{\delta} \cong R^{(\infty)}$ such that a diagram similar the one above is commutative.
4 Buium’s prolongations in the iterative case

We recall Buium’s definition of prolongations. We start from a very general set-up.
In any category $\mathcal{C}$, for $X \in \mathcal{C}$ we denote by $\mathcal{C}_X$ the category of objects over $X$, i.e. arrows $Y \to X$. If we have 2 morphisms $p_1 : T' \to T_1$, $p_2 : T' \to T_2$ and an object $X \in \mathcal{C}_{T_2}$, then we can define a functor

$$G_{X,p_1,p_2} : \mathcal{C}_{T_1} \to \text{Set}, \quad G_{X,p_1,p_2}(Y) = \text{Hom}_{T_2}(Y \times_{T_1} T', X).$$

Since the association $X \mapsto G_{X,p_1,p_2}$ is functorial, we obtain a functor

$$\{X \in \mathcal{C}_{T_2} \mid G_{X,p_1,p_2} \text{ is representable} \} \to \mathcal{C}_{T_1}.$$

In our case we have $\mathcal{C} = \text{Sch}_A$, $T_1 = T_2 = A$, $T' = A^{(n)}$, $p_1$ comes from the 0-derivation and $p_2$ comes from $\delta$.

Note that a scheme $Z$ represents the functor $G_{X,p_1,p_2}$ if and only if there is a natural bijection

$$\text{Hom}(Y^{(n)}_\delta, X) \leftrightarrow \text{Hom}(Y, Z).$$

It is easy to see that the functor on $K$-algebras $R \mapsto R^{(n)}_\delta$ has a left-adjoint, which we call $\nabla_n$ which commutes with localizations. Therefore for any scheme $X$, the object representing the functor $G_{X,p_1,p_2}$ exists and we call it $\nabla_n(X)$. Obviously, $\nabla_n$ is the right adjoint functor to $Y \mapsto Y^{(n)}_\delta$ and for a $K$-algebra $R$, we have

$$\text{Spec}(\nabla_n(R)) \cong \nabla_n(\text{Spec}(R)).$$

For $m < n$, we have quotient maps $K^{(n)} \to K^{(m)}$ which give natural maps between functors $\nabla_n \to \nabla_m$ and functors $X \to X^{(m)}_\delta$, $X \to X^{(n)}_\delta$. Therefore we get an inverse system of functors $\nabla_n \to \nabla_m$ and we define the infinite (or total) prolongation to be $\nabla := \lim \nabla_n$. Similarly, we define $X^{(\infty)}_\delta$ as $\lim X^{(n)}_\delta$.

Note that for schemes $X, Y$ we get (since inverse limits commute with representable functors)

$$\nabla X(Y) = (\lim \nabla_n X)(Y) \cong \lim \nabla_n X(Y) = \lim X(Y^{(n)}_\delta) \cong X(\lim Y^{(n)}_\delta) \cong X(Y^{(\infty)}_\delta).$$

Hence $\nabla$ represents the functor

$$\text{Sch} \ni Y \mapsto X(Y^{(\infty)}_\delta) \in \text{Sets},$$

and is right-adjoint to the functor $Y \mapsto Y^{(\infty)}_\delta$. 
5 Iterative $\delta$-schemes

$(K, \delta)$ still denotes an iterative differential field.

An iterative $\delta$-scheme (over $K$) is a scheme $X$ over $K$ together with an iterative derivation of its structure sheaf which extends $\delta$. Namely, for each open $U \subset X$, there is an iterative Hasse derivation $D_U$ on $\mathcal{O}_X(U)$ which extends $\delta$ and such that the restriction maps are iterative differential ring homomorphisms.

There is a natural notion of a morphism of iterative $\delta$-schemes, which we call a $\delta$-morphism, so we obtain the category of iterative $\delta$-schemes over $K$.

If we think about the category of schemes as an extension of the category of rings, where new object comes by gluing rings along localizations, then the same analogy holds in the iterative Hasse case – we just need to replace rings with iterative Hasse ring extensions of $(K, \delta)$ and all the gluing maps have to be $K$-$\delta$-isomorphisms.

From now on we will assume that $(K, \delta)$ is a universal iterative differential field, in the sense discussed in section 1.

Let $C$ be the field of absolute constants of $(K, \delta)$:

$$C = \{x \in K \mid \forall i \delta_i(x) = 0\}.$$  

As in the characteristic 0 case, if $X$ is a scheme defined over $C$, then there is a natural $\delta$-structure on $X$. Namely,

$$\mathcal{O}_X \cong \mathcal{O}_{X,C} \otimes K_X$$

and we put on $\mathcal{O}_{X,C}$ the 0-iterative differential structure and on $\mathcal{O}_X$ the tensor product $\delta$-structure (see 2.1(ii)). We call iterative $\delta$-schemes as above trivial and ones $\delta$-isomorphic to them isotrivial.

**Fact 5.1** The global section functor

$$\Gamma : \text{Affine iterative} \ \delta \text{-schemes} \rightarrow K-\delta\text{-algebras}$$

is an equivalence of categories.

**Proof** Since an affine scheme has an open basis consisting of the localization schemes, the result follows from 2.1(i) and an argument as in 7.3(iv) below.  \(\square\)
In other words, the fact above means that an iterative $\delta$-scheme structure on an affine scheme $\text{Spec}(R)$ is the same thing as an extension of $\delta$ to $R$. So, we can think of iterative $\delta$-schemes structures on a scheme $X$ as “extensions” of $\delta$ to $X$.

**Definition 5.2** Let $X$ be an iterative $\delta$-scheme. Then

$$X^\sharp = \{ x \in X(K) \mid x : \mathcal{O}_{X,x} \to K \text{ is a } \delta\text{-map} \} .$$

The following is the essential use of the assumption that $(K, \delta)$ is a saturated model of $SCH_{p,1}$:

**Fact 5.3** If $X$ is an iterative $\delta$-variety (namely iterative $\delta$-scheme over $K$ whose underlying scheme is a variety over $K$), then $X^\sharp$ is Zariski dense in $X$.

We will need several equivalent conditions for the existence of an iterative $\delta$-scheme structure on a given scheme. For the proof of one equivalence we need a purely category-theoretic (and probably well-known) lemma.

**Lemma 5.4** Let $\mathcal{C}$ be a category and $F, G : \mathcal{C} \to \mathcal{C}$ a pair of adjoint functors. Assume $c : F^2 \to F$ is a natural transformation, and for a morphism $f : F(X) \to Y$, let $f^\dagger : X \to G(Y)$ denote the adjoint morphism. Then

(i) There is an “adjoint” natural transformation $G \to G^2$, which we still denote by $c$.

(ii) If there is a morphism $f : F(X) \to X$ such that the following diagram is commutative

$$
\begin{array}{ccc}
F^2(X) & \xrightarrow{F(f)} & F(X) \\
\downarrow c & & \downarrow f \\
F(X) & \xrightarrow{f} & X,
\end{array}
$$

then the following diagram is commutative

$$
\begin{array}{ccc}
G^2(X) & \xleftarrow{G(f^\dagger)} & G(X) \\
\uparrow c & & \uparrow f^\dagger \\
G(X) & \xleftarrow{f^\dagger} & X.
\end{array}
$$
Proof
(i) Note that there is no natural bijection between \(\text{Hom}(F^2(X), F(X))\) and \(\text{Hom}(G(X), G^2(X))\), so we really need to use the fact that \(c\) is a natural transformation. Let \(X, Y \in C\). We will define the “adjoint” of \(c\) using “\(Y\)-points” of \(G(X)\), e.g. arrows \(Y \rightarrow G(X)\).
We have a morphism \(c : F^2(Y) \rightarrow F(Y)\), which gives a map
\[
\text{Hom}(F(Y), X) \rightarrow \text{Hom}(F^2(Y), X).
\]
By adjointness, we get a map
\[
\text{Hom}(Y, G(X)) \rightarrow \text{Hom}(Y, G^2(X)).
\]
By naturality of \(c\) and of the adjointness bijection, the above map is natural in \(X\) and \(Y\). Hence, Yoneda’s Lemma gives a natural transformation \(c : G \rightarrow G^2\).

(ii) It is clearly enough to show that
\[
(\ast) \quad (f \circ c)\uparrow = c \circ f^\dagger,
\]
\[
(\ast\ast) \quad (f \circ F(f))\uparrow = G(f^\dagger) \circ f^\dagger.
\]
\((\ast)\) clearly follows from the way \(c : G \rightarrow G^2\) was constructed.
For \((\ast\ast)\) one needs to use two commutative diagrams

\[
\begin{array}{ccc}
\text{Hom}(FX, X) & \xrightarrow{\dagger} & \text{Hom}(X, GX) \\
F(f) \downarrow & & \downarrow (f^\dagger) \\
\text{Hom}(F^2X, X) & \xrightarrow{\dagger} & \text{Hom}(FX,GX)
\end{array}
\]

plugging \(f\) in the left-upper corner of both of them.

Note that if \(X = \bigcup U_i\), then \(X^{(\infty)} = \bigcup U_i^{(\infty)}\) and \(X^{(\infty)}_{\delta} = \bigcup (U_i)_{\delta}^{(\infty)}\). In particular, a map \(D : X^{(\infty)}_{\delta} \rightarrow X\) extends to an isomorphism \(e^D : X^{(n)}_{\delta} \cong X^{(\infty)}\) as in the case of rings. Similarly, we have \(e^D_n : X^{(n)}_{\delta} \cong X^{(n)}\).

Fact 5.5 Let \(X\) be a scheme over \(K\). The following are equivalent
(i) There is an iterative \(\delta\)-scheme structure on \(X\).
(ii) There is a morphism $D : X^{(\infty)}_\delta \to X$ such that the following diagram is commutative

\[
\begin{array}{ccc}
(X^{(\infty)}_\delta)^{(\infty)}_{\delta} & \xrightarrow{D^{(\infty)}_\delta} & X^{(\infty)}_\delta \\
c & & D \\
X^{(\infty)}_\delta & \xrightarrow{D} & X.
\end{array}
\]

(iii) There is a morphism $D : X^{(\infty)}_\delta \to X$ such that the following diagram is commutative

\[
\begin{array}{ccc}
(X^{(\infty)}_\delta)^{(\infty)}_{\delta} & \xrightarrow{(e^{D})^{(\infty)}_\delta} & (X^{(\infty)})^{(\infty)}_\delta \\
c & & e^{D^{(\infty)}} \\
X^{(\infty)}_\delta & \xrightarrow{e^{D}} & X^{(\infty)}.
\end{array}
\]

(iv) There is a map $s : X \to \nabla(X)$ such that the following diagram is commutative

\[
\begin{array}{ccc}
\nabla(\nabla X) & \xleftarrow{\nabla(s)} & \nabla X \\
c & & s \\
\nabla X & \xleftarrow{s} & X.
\end{array}
\]

**Proof** Let $X = \bigcup U_i$, where $U_i = \text{Spec}(R_i)$.

(i) $\Rightarrow$ (ii)

Then clearly each $U_i$ has an iterative $\delta$-scheme structure, so we have extensions of $\delta$ to $R_i$, call them $D^i$. By 2.1(i), all these derivations are compatible with respect to the gluing maps. Therefore, the corresponding $K$-algebra maps $D_i : R_i \to R_i^{(\infty)}$ are also compatible and satisfy the condition from (ii). Therefore, they give a global morphisms as in 2.

(ii) $\Leftrightarrow$ (iii)

Obvious.

(ii) $\Leftrightarrow$ (iv)

Follows directly from 5.4

(ii) $\Rightarrow$ (i)

By 3.1 and 2.1(i) we get compatible iterative $\delta$-variety structures on each $U_i$, so we get an iterative $\delta$-variety structure on $X$. ∎
We also need the analogue of the previous fact in the case of \( \sharp \)-points and morphisms. Note that for any scheme \( X \), there is a map \( \delta_X : X(K) \to \nabla X(K) \). This map does not come from a morphism between \( X \) and \( \nabla(X) \), since it can not be completed to a natural transformation. In the language of logic, \( \delta_X \) is definable in the structure \((K, \delta)\) but not definable in the structure \( K \).

**Fact 5.6** For an iterative \( \delta \)-scheme \( X \) we have (where \( s \) as in 5.5(iv))

\[
X^\sharp = \{ x \in X(K) \mid s(x) = \delta_X(x) \}.
\]

**Fact 5.7** Let \( X, Y \) be iterative \( \delta \)-schemes with maps \( D_X \) and \( D_Y \) as in 5.5(ii) and \( s_X, s_Y \) as in 5.5(iv). Let \( f : X \to Y \) be a morphism. Then, the following are equivalent

(i) \( f \) is a \( \delta \)-morphism.

(ii) The following diagram is commutative

\[
\begin{array}{ccc}
X^{(\infty)} & \xrightarrow{f^{(\infty)}} & Y^{(\infty)} \\
\downarrow D_X & & \downarrow D_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

(iii) The following diagram is commutative

\[
\begin{array}{ccc}
X^{(\infty)} & \xrightarrow{f^{(\infty)}} & Y^{(\infty)} \\
\downarrow e^{D_X} & & \downarrow e^{D_Y} \\
X^{(\infty)} & \xrightarrow{f^{(\infty)}} & Y^{(\infty)}
\end{array}
\]

(iv) The following diagram is commutative

\[
\begin{array}{ccc}
\nabla(X) & \xleftarrow{\nabla(f)} & \nabla(Y) \\
\uparrow s_X & & \uparrow s_Y \\
X & \xleftarrow{f} & Y
\end{array}
\]

(v) \( f(X^\sharp) \subseteq Y^\sharp \).
6  The automorphism group functor and the first isotriviality theorem

We again begin in quite a general setting. Let \( C \) be a category and \( f : X \to Y \) an arrow in \( C \). Then \( f \) induces a pull-back (or fibre-product) functor

\[
f^* : C_Y \to C_X, \quad f^*(Z) = Z \times_Y X,
\]

where for a morphism over \( X \), \( \phi : Z \to Z' \),

\[
f^*(\phi) : Z \times_Y X \to Z' \times_Y X, \quad f^*(\phi) := \phi \times_Y \text{id}_X.
\]

We can extend the group of automorphisms of \( X \) to the following contravariant functor

\[
\text{Aut}_X : C^{\text{op}} \to \text{Gps}, \quad \text{Aut}_X(Y) = \text{Aut}_Y(X \times Y)
\]

where for a morphism \( f : Y \to Z \) we define

\[
\text{Aut}_X(f) : \text{Aut}_Z(X \times Z) \to \text{Aut}_Y(X \times Y), \quad \text{Aut}_X(f)(\phi) := f^*(\phi),
\]

since \((X \times Z) \times_Z Y\) is canonically isomorphic to \( X \times Y \).

In our case \( C = \text{Sch} \) and \( X \) is an iterative \( \delta \)-scheme. Let \( D \) be as in 5.5(ii).

Let us denote the functor \( \text{Aut}_X \) by \( \mathcal{G} \).

Note that for any contravariant functor (in particular \( \mathcal{G} \))

\[
\mathcal{F} : \text{Sch}^{\text{op}} \to \text{Sets},
\]

we can define contravariant functors

\[
\nabla_n(\mathcal{F}) : \text{Sch}^{\text{op}} \to \text{Sets}, \quad \nabla_n(\mathcal{F})(Y) := \mathcal{F}(Y^{(n)}_\delta).
\]

\[
\nabla(\mathcal{F}) : \text{Sch}^{\text{op}} \to \text{Sets}, \quad \nabla(\mathcal{F})(Y) := \mathcal{F}(Y^{(\infty)}_\delta).
\]

Our aim is to define a natural map

\[
\mathcal{G} \to \nabla(\mathcal{G}),
\]

which will give an iterative \( \delta \)-structure on the group scheme \( G \) representing the functor \( \mathcal{G} \) (if such a group scheme exists).
We will use the following sequence of isomorphisms. Note that these isomorphisms are both over $X$ and $Y^{(n)}_\delta$ and are natural with respect to $Y$. They are also compatible with the maps induced from the direct system $(Y^{(n)}_\delta)$. 

\[
(X \times Y)^{(n)}_\delta \cong X^{(n)}(n) \cong X^{(n)} \times Y^{(n)}_\delta \quad \text{from 2.2}
\]

\[
e^D : X^{(n)}_\delta \times A^{(n)}_\delta \cong X^{(n)}_\delta \times A^{(n)}_\delta \quad \text{from 5.5(iii)}
\]

\[
X^{(n)} \times A^{(n)}_\delta \cong (X \times A^{(n)}_\delta) \times Y^{(n)}_\delta \quad \text{since } A^{(n)}_\delta \cong A^{(n)}
\]

\[
(X \times A^{(n)}_\delta) \times Y^{(n)}_\delta \cong X \times Y^{(n)}_\delta \quad \text{obvious}
\]

(by $e^D$ in the second isomorphism we mean obviously $e^D \times A^{(n)}_\delta \text{id}_{A^{(n)}_\delta}$). Note that the existence of an iterative $\delta$-structure on $X$ was used only to get the second isomorphism.

Therefore, we have obtained an isomorphism of schemes

\[
D^{(n)}_Y : (X \times Y)^{(n)}_\delta \cong X \times Y^{(n)}_\delta.
\]

Note that we cannot pass to a direct limit, since product does not usually commute with direct limits.

Assume now that we have two iterative $\delta$-scheme structures on $X$, and let us call them $D$ and $\bar{D}$. We get an inverse system of maps

\[
\Delta_Y^{(n)} : \mathcal{G}(Y) \to \mathcal{G}(Y^{(n)}_\delta) = \nabla_n(\mathcal{G})(Y), \quad \Delta_Y^{(n)}(\phi) := \bar{D}^{(n)}_Y \circ \phi^{(n)}_\delta \circ (D^{(n)}_Y)^{-1}
\]

and therefore a map

\[
\Delta_Y = \lim \Delta_Y^{(n)} : \mathcal{G}(Y) \to \lim \nabla_n(\mathcal{G})(Y) \cong \nabla(\mathcal{G})(Y).
\]

Since $D^{(n)}_Y$, $\bar{D}^{(n)}_Y$ are natural in $Y$, $\Delta_Y$ is also natural in $Y$, so we get a natural transformation

\[
\Delta : \mathcal{G} \to \nabla(\mathcal{G}).
\]

We want to check, if $\Delta$ satisfies the condition from 5.5(iv). Note that this is the only one of the equivalent conditions in 5.5 which makes sense on the level of functors. We could define in such a way, a notion of an iterative $\delta$-structure on any contravariant functor from the category of schemes to the category of sets. However, we have no idea how we could use it, since

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the existence of ♯-points (to be defined in this context later) is problematic, when a functor is not representable. Actually, we will see later that for some automorphism functors ♯-points do not exist at all.

For our purposes, it will be enough to check 5.5(iv) on \( G(A) \) only, since later on \( G \) will be replaced by an irreducible \( K \)-variety \( G \) and the set \( G(K) \) is Zariski dense in \( G \).

**Fact 6.1** For each \( n \), the following diagram is commutative

\[
\begin{array}{ccc}
G(A) & \xrightarrow{\Delta_A^{(n)}} & \nabla_n G(A) \\
\Delta_A^{(n)} & & \downarrow \nabla_n (\Delta_A^{(n)}) \\
\nabla_n G(A) & \xrightarrow{G(c)} & \nabla_n \nabla_n G(A)
\end{array}
\]

**Proof**

We will omit the composition symbol on the level of “points” of \( G \). We start with the ”clockwise” composition.

\[
[\nabla_n (\Delta_A^{(n)}) \circ \Delta_A^{(n)}](\phi) = \Delta_A^{(n)}(\bar{D}_A^{(n)} \phi \delta \cdot (D_A^{(n)})^{-1})
\]

\[
= \bar{D}_{A_\delta}^{(n)}(\bar{D}_A^{(n)} \phi \delta \cdot (D_A^{(n)})^{-1} \delta (D_{A_\delta}^{(n)})^{-1}
\]

\[
= [\bar{D}_{A_\delta}^{(n)}(\bar{D}_A^{(n)}) \delta (\phi \delta \cdot (D_A^{(n)})^{-1} \delta (D_{A_\delta}^{(n)})^{-1}]
\]

We will now identify domains and targets, which will be useful later.

\[
D_{A_\delta}^{(n)}(D_A^{(n)}) \delta : (X_\delta^{(n)}) \delta \to X \times (A_\delta^{(n)}) \delta \cong (X^{(n)})
\]

\[
(\phi \delta )^{(n)} : (X_\delta^{(n)}) \delta \to (X_\delta^{(n)}) \delta
\]

The third map just acts in the opposite direction as the first one.

Now, we deal with the ”counterclockwise” composition. The reader is advised to recall the connection between the pull-back functor \( c^* \) and the automorphism group functor.

\[
[G(c) \circ \Delta_A^{(n)}](\phi) = c^* (\bar{D}_A^{(n)} \phi \delta \cdot (D_A^{(n)})^{-1})
\]
\[ [c^*(\bar{D}_A^n)][c^*(\phi_\delta^{(n)})][c^*(D_A^n)]^{-1}. \]

Of course we want to show that
\[ (*) \quad c^*(\phi_\delta^{(n)}) = (\phi_\delta^{(n)})^\delta, \]
\[ (**) \quad c^*(D_A^n) = D_{A_\delta^{(n)}}(D_A^n)^{(n)}. \]

There is a slight problem since domains and targets do not quite match:
\[ c^*(D_A^n) : X_\delta^{(n)} \times A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)} \to (X \times A_\delta^{(n)}) \times A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)}. \]

But we have the following isomorphism
\[ (X \times A_\delta^{(n)}) \times A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)} \cong X \times (A_\delta^{(n)})_\delta^{(n)} \cong (X^{(n)})^{(n)}, \]

and a commutative diagram (naturality of \( c \))

\[
\begin{array}{ccc}
(X_\delta^{(n)})_\delta^{(n)} & \xrightarrow{c} & X_\delta^{(n)} \\
\downarrow (\iota_\delta^{(n)})_\delta & & \downarrow \phi_\delta^{(n)} \\
(A_\delta^{(n)})_\delta & \xrightarrow{c} & A_\delta^{(n)}
\end{array}
\]

(\( \iota : X \to A \) is the structure map) giving another isomorphism
\[ c \times A_\delta^{(n)} (\iota_\delta^{(n)})_\delta^{(n)} : (X_\delta^{(n)} \setminus A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)} \to X_\delta^{(n)} \setminus A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)}. \]

So, to show \((*)\), it is enough to see that the following diagram is commutative

\[
\begin{array}{ccc}
(X_\delta^{(n)})_\delta^{(n)} & \xrightarrow{\phi_\delta^{(n)}} & (X_\delta^{(n)})_\delta^{(n)} \\
\downarrow c \times A_\delta^{(n)} (\iota_\delta^{(n)})_\delta^{(n)} & & \downarrow c \times A_\delta^{(n)} (\iota_\delta^{(n)})_\delta^{(n)} \\
X_\delta^{(n)} \times A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)} & \xrightarrow{c^*(\phi_\delta^{(n)})} & X_\delta^{(n)} \times A_\delta^{(n)} (A_\delta^{(n)})_\delta^{(n)}
\end{array}
\]

and this is just abstract nonsense – since
\[ c^*(\phi_\delta^{(n)}) = \phi_\delta^{(n)} \times A_\delta^{(n)} \text{id}_{(A_\delta^{(n)})_\delta^{(n)}}, \]

after projecting to \(X_\delta^{(n)}\), commutativity of the diagram follows from naturality of \( c \) and after projecting to \((A_\delta^{(n)})_\delta^{(n)}\) commutativity just means that
\((\phi_{\delta}^{(n)})_{\delta}^{(n)}\) is a morphism over \((A_{\delta}^{(n)})_{\delta}^{(n)}\).

To show \((**)\) we need (at last) to use the fact that \(D\) is iterative.

Again, it is enough to show that the following diagram is commutative

\[
\begin{array}{ccc}
(X_{\delta}^{(n)})_{\delta}^{(n)} & \xrightarrow{D_{\delta}^{(n)}(D_{A}^{(n)})_{\delta}^{(n)}} & X \times (A_{\delta}^{(n)})_{\delta}^{(n)} \\
\downarrow & & \downarrow \cong \\
X_{\delta}^{(n)} \times A_{\delta}^{(n)} & \xrightarrow{c^{*}(D_{A}^{(n)})} & (X \times A_{\delta}^{(n)}) \times (A_{\delta}^{(n)})_{\delta}^{(n)}
\end{array}
\]

So, again we can consider projections to \((A_{\delta}^{(n)})_{\delta}^{(n)}\) and \(X \times A_{\delta}^{(n)}\).

Since all the maps above are over \((A_{\delta}^{(n)})_{\delta}^{(n)}\), the commutativity in the first case is trivial.

For the commutativity in the second case, we first make the following identification

\[X \times A_{\delta}^{(n)} \cong X^{(n)}.\]

Then \(D_{A}^{(n)}\) becomes \(e_{n}^{D}\) and \(D_{A_{\delta}}^{(n)}\) becomes \(e_{n}^{D(\infty)}\). Therefore the commutativity in the second case is equivalent to the commutativity of the following diagram

\[
\begin{array}{ccc}
(X_{\delta}^{(n)})_{\delta}^{(n)} & \xrightarrow{(e_{n}^{D})_{\delta}^{(n)}} & (X^{(n)})_{\delta}^{(n)} \\
\downarrow c & & \downarrow e_{n}^{D(\infty)} \\
X_{\delta}^{(n)} & \xrightarrow{e_{D}} & X^{(n)}
\end{array}
\]

which is exactly 5.5(iii) with \(\infty\) replaced by \(n\) and it is clear that commutativity of the diagram in 5.5(iii) is equivalent to commutativity of the diagram above for each \(n\). \(\Box\)

We now try to define the set of “\(\sharp\)-points” of \(G\) as

\[G^{\sharp} := \{\phi \in G(A) \mid G(\delta)(\phi) = \Delta_{A}(\phi)\},\]

where \(\delta\) is understood as a morphism \(\delta : A_{\delta}^{(\infty)} \to A\).

**Fact 6.2** Let \(\phi\) be an automorphism of \(X\). Then \(\phi \in G^{\sharp}\) if and only if \(\phi\) is a \(\delta\)-isomorphism between the iterative \(\delta\)-schemes \((X, D)\) and \((X, \bar{D})\).
Proof For each $n$, let $\delta^{(n)}$ denote the morphism $\delta^{(n)} : A_{\delta}^{(n)} \rightarrow A$. We again identify $X \times A_{\delta}^{(n)}$ with $X^{(n)}$. Then

$$G(\delta^{(n)})(\phi) = \phi^{(n)}, \quad \Delta_{A}^{(n)}(\phi) = e_{n}^{D} \phi_{b}^{(n)}(e_{n}^{D})^{-1}.$$ 

Therefore

$$G(\delta)(\phi) = \Delta_{A}(\phi)$$

if and only if for each $n$ the following diagram is commutative

$$\begin{array}{ccc}
X_{\delta}^{(n)} & \xrightarrow{\phi_{\delta}^{(n)}} & Y_{\delta}^{(n)} \\
e_{n}^{D} \downarrow & & \downarrow e_{n}^{D} \\
X^{(\infty)} & \xrightarrow{\phi^{(n)}} & X^{(\infty)},
\end{array}$$

and we finish as in the proof of the previous fact using 5.7(iii). \(\square\)

By the above fact, we can give examples of functors with no $\sharp$-points – it is enough to give two non-isomorphic iterative $\delta$-structures on a given scheme $X$, which is easy to do for instance if $X$ is an affine line.

We now have the main result of this paper.

**Theorem 6.3** Let $X$ be a projective variety over $K$, and suppose $D$ and $\bar{D}$, are iterative $\delta$-structures on $X$. Then $(X, D)$ is isomorphic to $(X, \bar{D})$. In particular, if $X$ is defined over $C$, then any iterative $\delta$-structure on $X$ is isotrivial.

Proof

It is well-known that for any projective variety $X$, the functor $G = \text{Aut}_{X}$ is representable, by a group scheme $G$ whose connected component $G^{0}$ is an algebraic group. So let $G$ be such for our given $X$. Let $D$ and $\bar{D}$ be two iterative $\delta$-structures on $X$ (as in 5.5(ii)). By 6.1 and Yoneda’s Lemma, $G$ has a $\delta$-scheme structure (but not a $\delta$-group scheme one!) coming from $D$ and $\bar{D}$ and this $\delta$-scheme structure is iterative on $G(K)$. It is rather easy to see (using e.g. 2.1(v) which did not use iterativity) that each irreducible component of $G$ is a $\delta$-subscheme. In particular this equips $G^{0}$ with an iterative $\delta$-variety structure, since $G^{0}(K)$ is dense in $G^{0}$. By Fact 5.3, $(G^{0})^{\sharp}$ is nonempty. By 6.2, any $\phi \in (G^{0})^{\sharp}$ is a $\delta$-isomorphism between $(X, D)$ and
For the final clause, it is enough to take for $\bar{D}$ the trivial iterative $\delta$-structure. □

7 $\delta$-sheaves and the Second Isotriviality theorem

In this section $(K, \delta)$ is still a universal iterative differential field.

A $\delta$-vector group is a vector group together with a $\delta$-group structure. A $\delta$-vector space is a $\delta$-vector group such that the $\delta$-structure preserves also scalar multiplication.

If $V$ is a $\delta$-module, then $V^\sharp = \{ x \in V \mid \delta(x) = 0 \}$.

The following is well-known. For example (ii) is part of the theory of linear iterative differential equations.

Fact 7.1

(i) The notion of a finite dimensional $\delta$-module is equivalent to the notion of a $\delta$-vector space.

(ii) Any $\delta$-module has a basis consisting of $\sharp$-points of the correspondent $\delta$-vector space. Such a basis gives isomorphism between the given $\delta$-module (resp. $\delta$-vector space) and a trivial $\delta$-module (resp. $\delta$-vector space).

(iii) If $X$ is an iterative $\delta$-scheme, then $\mathcal{O}_X(X)^\sharp$ corresponds to $\delta$-morphisms from $X$ to the trivial $\delta$-variety $\mathbb{A}^1$.

We need to extend the notion of a $\delta$-module to the context of sheaves.

Definition 7.2

(i) $F$ is a $\delta$-(pre)sheaf, if it is a (pre)sheaf of $\delta$-$\mathcal{O}$-modules, i.e. for each $U \subseteq X$ open, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and there is $\delta_U$, a $\delta$-$\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ s.t. restriction maps are $\delta$-morphisms.

(ii) A $\delta$-sheaf is $\delta$-trivial if it is $\delta$-isomorphic to $\mathcal{O}_X^\text{en}$ (some $n$).

(iii) A $\delta$-sheaf $\mathcal{F}$ is locally $\delta$-trivial if for any $x \in X$, there is $U \ni x$ open such that $\mathcal{F}|_U$ is $\delta$-trivial.

Fact 7.3

(i) If $\mathcal{F}$ is a $\delta$-presheaf, $x \in X$, then $\mathcal{F}_x$ (a stalk of $\mathcal{F}$ at $x$) has a natural $\delta$-structure.

(ii) If $\mathcal{F}$ is a $\delta$-presheaf, then $\mathcal{F}^+$ (the sheafication of $\mathcal{F}$) is a $\delta$-sheaf.

(iii) If $\mathcal{F}, \mathcal{G}$ are $\delta$-sheaves, then $\mathcal{F} \otimes \mathcal{G}, \mathcal{F} \oplus \mathcal{G}, \mathcal{F}^*, \bigwedge^n \mathcal{F}$ are $\delta$-sheaves.

(iv) Let $(U_i)_{i \in I}$ be an open basis of $X$, $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules and assume
that for each \( U_i \) there is an iterative derivation \( \delta_i \) on \( \mathcal{F}(U_i) \) such that the restriction maps are \( \delta \)-maps. Then the \( \delta_i \)'s extend to make \( \mathcal{F} \) a \( \delta \)-sheaf.

**Proof** (i) By 2.1(iv), the direct limit of \( \delta \)-modules is a \( \delta \)-module. Therefore any \( \mathcal{F}_x \) is a \( \delta \)-module.

(ii) By (i), for each open \( U \), the module of functions from \( U \) into \( \bigcup_{x \in U} \mathcal{F}_x \) (being sections of the natural projection) has a \( \delta \)-module structure. It is easy to see that \( \mathcal{F}^+ \) is a \( \delta \)-submodule of the above \( \delta \)-module of functions (see [Ha, Section 2.1]).

(iii) By (ii), it follows from 2.1(iii).

(iv) Take \( U_i \), open in \( X \), \( s \in \mathcal{F}(U) \) and let \( U_i = \bigcup_{j \in I} U_{ij} \) (some \( I' \subseteq I \)). Let \( s_i := \delta_i(s|_{U_i}) \) (a sequence of sections over \( U_i \)). Then for each \( i, j \in I' \), and each \( x \in U_i \cap U_j \) there is \( k \in I \) such that \( x \in U_k \subseteq U_i \cap U_j \) and

\[
s_i|_{U_k} = \delta_i(s|_{U_i})|_{U_k} = \delta_k(s|_{U_k}) = \delta_j(s|_{U_j})|_{U_k} = s_j|_{U_k}.
\]

Since \( \mathcal{F} \) is a sheaf a collection \( (s_i) \) comes from a sequence of sections of \( \mathcal{F} \) over \( U \) which is our \( \delta(s) \).

Since \( \mathcal{F} \) is a sheaf, we can check all the respective axioms of iterative derivations locally, so we get a sheaf of \( \delta \)-modules (resp. \( \delta \)-algebras) indeed. \( \square \)

**Fact 7.4** If \( \mathcal{F} \) is a very ample invertible \( \delta \)-sheaf on a projective iterative \( \delta \)-variety \( X \), then

(i) \( \mathcal{F} \) is locally \( \delta \)-trivial.

(ii) \( \mathcal{F}(X) \) has a basis consisting of elements of \( \mathcal{F}(X)^\sharp \) and the map into \( \delta \)-trivial projective space defined by this basis is a \( \delta \)-map.

**Proof**

(i) Let \( X = \bigcup_{i \in I} U_i \) be an open cover of \( X \) such that for each \( i \) there is an isomorphism of sheaves of \( \mathcal{O} \)-modules

\[
f_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}.
\]

that coming from the Since \( X \) is projective, \( \mathcal{F}(X) \) is a finite-dimensional \( \delta \)-module. By 7.1(i), there is \( \{s_0, \ldots, s_n\} \), a basis of \( \mathcal{F}(X) \) contained in \( \mathcal{F}(X)^\sharp \). Let

\[
U_{ij} := U_i \setminus Z(f_i(s_j|_{U_i})), \quad f_{ij} := f_i|_{U_{ij}}, \quad s_{ij} := s_i|_{U_{ij}}.
\]

Since \( \mathcal{F} \) is very ample, \( (U_{ij}) \) is a cover of \( X \). For each \( i, j \in I, \{s_{ij}\} \) is a basis of \( \mathcal{F}(U_{ij}) \), since \( \{f_{ij}(s_{ij})\} \) is a basis of \( \mathcal{F}(U_{ij}) \) (being an invertible element).

But \( s_{ij} \in \mathcal{F}(U_{ij})^\sharp \). Therefore the map

\[
f_{ij}(s_{ij})^{-1} f_{ij} : \mathcal{F}|_{U_{ij}} \rightarrow \mathcal{O}_{U_{ij}}
\]
is a $\delta$-sheaf isomorphism.

(ii) Let $X = \bigcup U_i$ be an open cover of $X$ such that for each $i$ there is an isomorphism of $\delta$-sheaves

$$f_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$$

(such a cover exists by (i)).

If $s \in \mathcal{F}(X)$, then $f_i(s|_{U_i}) \in \mathcal{O}_{U_i}(U_i)^\sharp$. Therefore (by 7.1(iii)),

$$f_i(s|_{U_i}) : U_i \to \mathbb{A}^1$$

is a $\delta$-map (where $\mathbb{A}^1$ has the trivial $\delta$-variety structure).

Let $B = \{s_0, \ldots, s_n\} \subset \mathcal{F}(X)$ be a basis of $\mathcal{F}(X)$. Then clearly the map

$$f_B : X \to \mathbb{P}^n$$

is a $\delta$-map, where $\mathbb{P}^n$ has the trivial $\delta$-variety structure. \qed

**Fact 7.5** If $A \to B$ is a $\delta$-map of $\delta$-rings and $f \in B$, then

(i) $\Omega_{B/A}$ is naturally a $\delta$-module.

(ii) The map $\Omega_{B/A} \to \Omega_{B_f/A}$ is a $\delta$-map.

**Proof** (i) By [Ha, II.8.1A.], $\Omega_{B/A}$ is isomorphic to $I/I^2$, where

$$I = \ker(B \otimes_A B \to B).$$

$I$ has clearly the $\delta$-module structure, so has $I^2$, hence $I/I^2$ gets the quotient $\delta$-module structure.

(ii) Since the localization map $B \to B_f$ is a $\delta$-map its tensor square is a $\delta$-map as well and it clearly preserves the kernel of multiplication, so the result follows. \qed

**Proposition 7.6** If $X$ is an iterative $\delta$-scheme, then $\Omega_X$ is a $\delta$-sheaf.

**Proof** We will use 7.3(iv). Take $(U_i)_i$ the open base of $X$ consisting of open affine subvarieties. By the 7.5(i), each $\Omega_X(U_i)$ has a natural $\delta$-module structure. We need to check that the restriction map preserves the $\delta$-module structure. Since any affine variety has an open basis consisting of subsets corresponding to localizations, it is enough to use 7.5(ii). \qed
Corollary 7.7 The canonical and anticanonical sheaves are locally $\delta$-trivial invertible $\delta$-sheaves.

Proof By the previous proposition, 7.3(iii) and 7.4.

Theorem 7.8 If $V$ is a projective iterative $\delta$-variety and the canonical or the anticanonical divisor of $V$ is ample, then $V$ is $\delta$-isotrivial.

Proof By 7.7, and 7.4. (after taking a suitable tensor power), $V$ is $\delta$-isomorphic to a $\delta$-subvariety of a trivial $\delta$-variety (namely $\mathbb{P}^n$), so $V$ is isotrivial.

Corollary 7.9 If $V$ is a smooth projective $\delta$-curve, then $V$ is $\delta$-isotrivial.

Proof By [Ha, IV.3.3] a divisor $X$ on $V$ is ample if and only if $\deg(X) > 0$. Hence, by 7.8, we are done in the cases when the degree of the canonical divisor is non-zero, i.e. when $V$ is not an elliptic curve. But the case of elliptic curve is solved in [Be].

8 Further remarks

In [Be] Benoit proves the following:

Proposition 8.1 Let $X$ be an algebraic variety over $K$ (where $(K, \delta)$ is a universal iterative differential field). Then $X$ has an iterative $\partial$-structure if and only if for each $n$ there is an isomorphism $\alpha_n$ between $X$ and a variety $X_n$ defined over $K^{p^n}$ such that for each $n$ the isomorphism $\alpha_n^{-1} \circ \alpha_n$ between $X_{n+1}$ and $X_n$ is defined over $K^{p^n}$.

So we ask here whether any projective variety $X$ over $K$ satisfying the conditions of the proposition descends to $C = \bigcap_n K^{p^n}$.

On the other hand, if $X$ belongs to a family with a fine moduli space (even after adding additional structure), then simply the fact that $X$ is (isomorphic to something) defined over each $K^{p^n}$ implies that $X$ descends to the intersection. This is in particular true of abelian varieties. Together with Theorem 6.3 we obtain:

Corollary 8.2 Let $(X, D)$ be an iterative $\delta$-variety over $K$, where $X$ is an abelian variety. Then $(X, D)$ is $\delta$-isotrivial.
It is also possible to combine the results of Section 6 together with [Be] to obtain Theorem 7.8. It is just because a projective variety polarized by its canonical or anticanonical divisor belongs to a family with a fine moduli space. But note that our proof of Theorem 7.8 does not use results of Section 6 at all.

References


http://mathweb.mathsci.usna.edu/faculty/traveswn/publications/thesis.ps