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NEW INTERPOLATIONS BETWEEN CLASSICAL AND FREE GAUSSIAN PROCESSES

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Abstract. In this paper we present the base of a general technique to derive new positive definite functions on pairings from already known ones. To describe this technique we use two concrete applications. The first one refers to the function depending on the number of connected components, the second one to the function depending on the number of crossings. In the first application we get a new family of functions identifying nontrivial connected components.

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1. INTRODUCTION

An interesting method for constructing non-commutative Gaussian processes was found by Bożejko and Speicher (see [9], [12], [8], [36]). The key object in this procedure is the field operator related to an appropriate Fock space. Roughly speaking, there is some underlying real Hilbert space $H_{\mathbb{R}}$ and an $\mathbb{R}$-linear transformation $G : H_{\mathbb{R}} \to *alg\,[G\,(H_{\mathbb{R}})]$ into the unital *-algebra with a state ϕ, making $G\,(H_{\mathbb{R}})$ the Gaussian field, i.e.,

$$\phi\big(G\,(f_1)\dots G\,(f_m)\big) = \phi\Big(G\big(O\,(f_1)\big)\dots G\big(O\,(f_m)\big)\Big), \quad \forall f_j \in H_{\mathbb{R}},\, m \in \mathbb{N},$$

where O is any orthogonal transformation on $H_{\mathbb{R}}$. The quantity above can be evaluated by using the so-called Wick formula. This formula claims the existence of a function $t : \mathbb{P}_2\,(\infty) \to \mathbb{C}$ on pairings (see Definition 2.1) with the property

$$\phi\big(G(f_1)\dots G(f_{2m})\big) = \sum_{\nu \in \mathbb{P}_2(2m)} t\,(\nu) \prod_{p=1}^{m} \langle f_{i_p}, f_{j_p}\rangle_{H_{\mathbb{R}}},$$

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where $\nu = \left\{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\right\}$ is a pairing on the set $\{1, \ldots, 2m\}$. We refer to the papers [31] and [30] for very interesting investigations on this topic. The functions t appearing in the formula above are called *positive definite*. The original definition of this positive definiteness was formulated in terms of tensor algebra over Hilbert space with involution (see [12]). Later Guţă and Maassen characterized in [17] the positivity above as usual positivity in the *-semigroup of broken pairings. They also proved that for any multiplicative (see Definition 2.3) positive definite function t there exists a non-commutative Gaussian process (generated by field operators) having the function t in its Wick formula.

Indeed, given a positive definite function t with underlying Hilbert space $H_{\mathbb{R}} = L_2(\mathbb{R}_+, dx)$ one can construct the Gaussian process $B(t)$ by the formula $B(t) = G(\chi_{[0,t]})$. We refer to the papers [3], [6], [15] and [35] where the hyper- and ultracontractivities for the functor of second quantization on some Gaussian fields are treated as well as some general kind of Khintchine inequalities is studied.

At the present time a number of positive definite functions are known: "number of crossings" (see [9]), "number of connected components" (see [12]), "recurrence formula for orthogonal polynomials" (see [1]), "Thoma's characters" (see [7]). See also [18] for some very interesting general considerations on this topic. The set of positive definite functions on pairings is closed under convex combinations and pointwise multiplications. There are some other interesting operations preserving the set of these functions.

Bryc et al. [13] computed the free additive convolution (see [37] and [24] for the concept and other interesting investigations of free convolution) of the classical Gaussian measure and of the Wigner measure as the limit distribution of large random Markov matrices. They wrote the moments of the measure $\gamma_M = \gamma_0 \boxplus \gamma_1$, $d\gamma_0(x) = \chi_{[-2,2]}(x)\sqrt{4-x^2}dx$, $d\gamma_1(x) = (2\pi)^{-1/2}\exp(-x^2/2)dx$ in the following way:

$$m_{2m}(\gamma_M) = \sum_{\nu \in \mathbb{P}_2(2m)} 2^m \left(\frac{1}{2}\right)^{m-h(\nu)},$$

where $h(\nu)$ is the number of connected components consisting of exactly one block (non-crossing pairings maximize the function h). We draw the pictures below for clarification:

$h(\nu) = 3$ $\quad$ $h(\nu) = 0$ $\quad$ $h(\nu) = 5$

In this paper we prove that the function

$$H_q(\nu) = q^{m-h(\nu)},$$

where m is the number of blocks in the pairing ν, i.e. $\nu \in \mathbb{P}_2(2m)$, is positive definite for any $q \in [0, 1]$. The function H_q is multiplicative (see Definition 2.3). The expectation ϕ in the algebra affiliated with Gaussian processes is a trace ($\phi\,(ab) = \phi\,(ba)$). By an adaptation of the proof given in [13] ($\gamma_M = \gamma_0 \boxplus \gamma_1$) it follows that the spectral measure (in the state ϕ) of the (normalized in the second norm) Gaussian operator arisen in this setting is equal to the free additive convolution of (properly dilated) γ_0 and γ_1,

$$\gamma_{H_q} = (D_{1/\sqrt{1-q}}\gamma_0) \boxplus (D_{1/\sqrt{q}}\gamma_1),$$

where $(D_\lambda \mu)\,(A) = \mu\,(\lambda \cdot A)$. This gives an interpolation between classical ($q = 1$) and free ($q = 0$) Gaussian processes. The measure γ_M appears in the middle of the scale, $\gamma_M = D_{1/2}\gamma_{H_{1/2}}$.

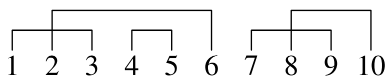
The main tool in the proof is the non-commutative Central Limit Theorem (shortly, nc-CLT). We start with a Gaussian field (or process). Then we make some perturbation, and finally we come back to the class of Gaussian fields using the nc-CLT. In the present paper we make the deformation of the Gaussian field that is the easiest in computations, depending on the number of connected components. This gives us the function H_q.

In the last section we apply the same perturbation to the q-Gaussian field depending on the number of crossings. In this case we do not get new positive definite functions; however, this result gives some kind of limit-distribution inclusion among the q-Gaussian fields. It is related to the question whether von Neumann algebras generated by field operators arising from q-commutation relations (given in Fock representation) are isomorphic for various parameters q.

2. NOTIONS AND THE nc-CLT

DEFINITION 2.1. A covering of the set $A = \{1, \ldots, 2m\}$ by two-element disjoint subsets (blocks) $\{i, j\} \subset A$ is called a *two-partition* or *pairing*. The set of all two-partitions will be denoted by $\mathbb{P}_2\,(2m)$ or, simply, by $\mathbb{P}_2$.

The usual graphical representation of a pairing $\big\{\{1,3\},\{2,6\},\{4,5\},\{7,9\},\{8,10\}\big\}$ looks like the following picture:



Big part of non-commutative probabilities is connected with pairings. In the paper [12] by Bożejko and Speicher the notion of positive definite function on pairings was introduced. In the paper [17] by Guţă and Maassen, positivity was characterized as usual positivity in the *-semigroup of broken partitions.

The most important (for this paper) observation related to pairings is the following nc-CLT by Bożejko and Speicher (see Theorem 0 in [12]).

THEOREM 2.1. *Let $\mathcal{B}$ be a unital *-algebra equipped with a state ϕ. Consider selfadjoint elements $b_i = b_i^* \in \mathcal{B}$ $(i \in \mathbb{I})$ with normalized second moments $(\phi(b_i^2) = 1)$, which fulfill for all $m \in \mathbb{N}$ the following assumptions (singleton and exchanging conditions):*

(a) $\phi(b_{i_1} b_{i_2} \dots b_{i_m}) = 0$ *whenever* $\exists r : \#\{p : i_r = i_p\} = 1$,

(b) $\phi(b_{i_1} b_{i_2} \dots b_{i_m}) = \phi(b_{\pi(i_1)} b_{\pi(i_2)} \dots b_{\pi(i_m)})$ *for any injection* $\pi : \mathbb{I} \to \mathbb{I}$.

Then for the sequence $\big(S_N(k)\big)_{k\in\mathbb{N}}$ of operators,

$$S_N(k) = \frac{1}{\sqrt{N}} \sum_{i \in A_{N,k}} b_i,$$

where for fixed N, $(A_{N,k})_{k\in\mathbb{N}}$ is an arbitrary family of disjoint sets with cardinality N, the following equalities hold for all $m, k_j \in \mathbb{N}$:

(i) $$\lim_{N\to\infty} \phi\big(S_N(k_1)\dots S_N(k_{2m+1})\big) = 0,$$

(ii) $$\lim_{N\to\infty} \phi\big(S_N(k_1)\dots S_N(k_{2m})\big) = \sum_{\nu\in\mathbb{P}_2(2m)} t(\nu) \prod_{p=1}^{m} \delta_{k_{i_p},k_{j_p}},$$

where $\nu = \big\{\{i_1, j_1\}, \dots, \{i_m, j_m\}\big\}$ and t is some positive definite function on $\mathbb{P}_2$.

The elements b_i in the above theorem are called Gaussian operators whenever the conditions (i) and (ii) hold without the limit. In fact, it is enough to prove this for $N = 1$.

DEFINITION 2.2. Let $\mathcal{B}$ and ϕ be as in the theorem above. The operators $g_i \in \mathcal{B}$ are called *Gaussian operators* whenever the following conditions hold for all $m \in \mathbb{N}$:

(i) $$\phi(g_{k_1}\dots g_{k_{2m+1}}) = 0,$$

(ii) $$\phi(g_{k_1}\dots g_{k_{2m}}) = \sum_{\nu\in\mathbb{P}_2(2m)} t(\nu) \prod_{p=1}^{m} \phi(g_{k_{i_p}} g_{k_{j_p}}),$$

where $\nu = \big\{\{i_1, j_1\}, \dots, \{i_m, j_m\}\big\}$ and t is some positive definite function on $\mathbb{P}_2$.

DEFINITION 2.3. Let t be a function on pairings. The function t is called *multiplicative* whenever for any m and any pairing $\nu \in \mathbb{P}_2(2m)$ of the form $\nu = \nu_1 \uplus \nu_2$, where ν_1 is a pairing of the subinterval $\{k, k+1, \dots, l\} \subset \{1, \dots, 2m\}$, the following condition holds:

$$t(\nu) = t\big(I(\nu_1)\big) \cdot t\big(I(\nu_2)\big),$$

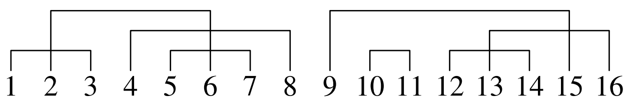
where $I(\mu)$ for μ being a pairing of even-cardinality subset of naturals is defined by order-preserving bijection of partitioned sets, e.g. $I\big(\{\{7, 17\}, \{11, 23\}\}\big) = \big\{\{1, 3\}, \{2, 4\}\big\}$.

3. q-NUMBER OF CONNECTED COMPONENTS

In this section we fix a natural number n and a real number $q \in (0,1)$. For this reason we write the function names t_c, t_{T_c}, $T_{c,N,k}$ instead of $t_{c,q,n}$, $t_{T_{c,q,n}}$, $T_{c,q,n,N,k}$. Let $(c_\alpha^{(q)})_{\alpha \in \mathbb{N} \times \{1,\ldots,n\}}$ be a sequence of Gaussian operators having the function t given by the formula

$$t_c(\nu) = q^{m-c(\nu)},$$

where $c(\nu)$ is the number of connected components in the pairing $\nu \in \mathbb{P}_2(2m)$ and $q \in (0,1)$. As an explanation of the quantity $c(\nu)$ we give the picture below and we refer to the paper [12] by Bożejko and Speicher for details. We have

$\nu =$ 

, $\quad c(\nu) = 3$.

Let us consider the operators

$$T_{c,N,k} = \frac{1}{\sqrt{N}} \sum_{i \in A_{N,k}} \frac{c_{(i,1)}^{(q)} \cdots c_{(i,n)}^{(q)} + c_{(i,n)}^{(q)} \cdots c_{(i,1)}^{(q)}}{\sqrt{2(1+q^{n-1})}},$$

where $A_{N,k}$ is as in the assertion of Theorem 2.1.

Since the operator $T_{c,N,k}$ has the form of the operators $S_{N,k}$ from Theorem 2.1, where

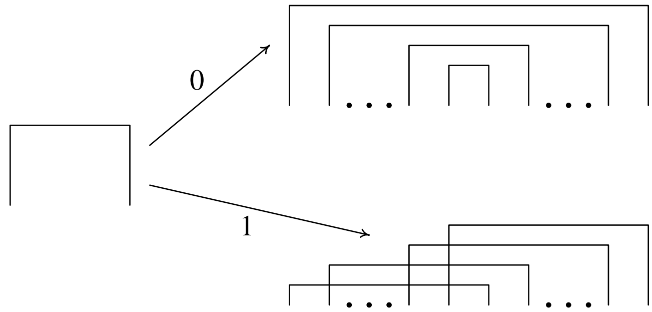
$$b_i = \frac{c_{(i,1)}^{(q)} \cdots c_{(i,n)}^{(q)} + c_{(i,n)}^{(q)} \cdots c_{(i,1)}^{(q)}}{\sqrt{2(1+q^{n-1})}},$$

we can compute the corresponding function t_{T_c}. The following formula for t_{T_c} can be easily checked:

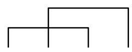
$$t_{T_c}(\nu) = \frac{1}{[2(1+q^{n-1})]^m} \sum_{\mu \in Z(\nu)} 2^m \, t_c(\mu),$$

where m is the number of blocks in the pairing ν, and $Z(\nu)$ is the set of the two-partitions of the set $\{1,\ldots,2n \cdot m\}$ which are constructed from ν by replacing any block by one of the blocks $\{\{1,n\},\{2,n-1\},\ldots,\{n/2,n/2+1\}\}$ or

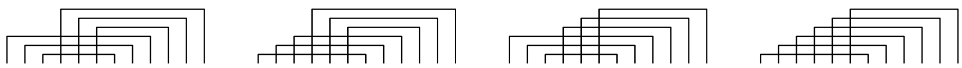
$\{\{1, n/2+1\}, \{2, n/2+2\}, \ldots, \{n/2, n\}\}$, as shown in the following picture:

(1) 

For example, when

$\nu =$ 

and $n = 3$, then $Z(\nu)$ consists of the following pairings:



In the case when $\nu \in \mathbb{P}_2(2m)$ has exactly one connected component, any $\mu \in Z(\nu)$ has also one connected component (with the exception of $m = 1$) and $\mu \in \mathbb{P}_2(2m \cdot n)$. Consequently, under the assumption that $m \geqslant 2$, we obtain

$$t_{T_c}(\nu) = q^{m-1} \cdot \left(\frac{2q^{n-1}}{1+q^{n-1}}\right)^m .$$

For $m = 1$ we have

$$\begin{aligned} t_{T_c}\big(\{\{1,2\}\}\big) &= \frac{1}{[2(1+q^{n-1})]^1}\big(2t_c(\nu_1) + 2t_c(\nu_2)\big) \\ &= \frac{1}{[2(1+q^{n-1})]^1}(2+2q^{n-1}), \end{aligned}$$

where $\nu_1 = \{\{1,n\}, \{2,n-1\}, \ldots, \{n,n+1\}\}$ and ν_2 is the partition $\nu_2 = \{\{1,n+1\}, \{2,n+2\}, \ldots, \{n,2n\}\}$.

Since the function t_c is multiplicative and the transformation above sends connected components into connected components, the function t_{T_c} is also multiplicative and, finally,

$$t_{T_c}(\nu) = t_c(\nu)\, q_1^{m-h(\nu)}, \tag{2}$$

where m is the number of blocks in ν, $h(\nu)$ is the number of connected components consisting of one block only and $q_1 = 2q^{n-1}/(1+q^{n-1})$.

Now we would like to erase the factor $t_c(\nu)$ in the formula (2). For this reason we take the limit when q goes to 1 ($q_i = 1 - 1/i$). Simultaneously, we change n according to the formula $n_i = [a \cdot i]$, where $a \in (0, \infty)$ is fixed. This gives us q_1 in formula (2) equal to $2e^{-a}/(1 + e^{-a})$. Since $2e^{-a}/(1 + e^{-a})$ runs over $(0, 1)$ when $a \in (0, \infty)$, we get the following

THEOREM 3.1. *Let $q \in [0, 1]$ and $h(\nu)$ be equal to the number of connected components in the pairing $\nu \in \mathbb{P}_2(2m)$ consisting of one block. Then the function*

$$H_q(\nu) = q^{m-h(\nu)}, \quad q \in [0, 1],$$

is positive definite.

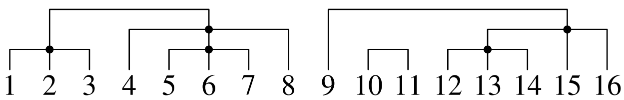
Let us note that the function above was discovered (in the one-dimensional case for $q = \frac{1}{2}$) in [13].

4. q-NUMBER OF CROSSINGS

In this section, as in the previous one, we fix n, q and we do not write indices n and q in the function names except when it is needed. Let us replace the operators $c_\alpha^{(q)}$ from the preceding section by Gaussian operators $g_\alpha^{(q)}$ related to the function

$$t_i(\nu) = q^{i(\nu)},$$

where $i(\nu)$ is the number of crossings in the pairing ν and $q \in (-1, 1)$. As an explanation of the quantity $i(\nu)$ we give the picture below and we refer to the paper [9] by Bożejko and Speicher for details. We have

$\nu =$ 

, $\quad i(\nu) = 5.$

Moreover, let us replace the scaling factor $1/\sqrt{2(1+q^{n-1})}$ by $1/\sqrt{2\big(1+q^{\binom{n}{2}}\big)}$, so that the new operators

$$T_{i,N,k} = \frac{1}{\sqrt{N}} \sum_{i \in A_{N,k}} \frac{g_{(i,1)}^{(q)} \cdots g_{(i,n)}^{(q)} + g_{(i,n)}^{(q)} \cdots g_{(i,1)}^{(q)}}{\sqrt{2\big(1+q^{\binom{n}{2}}\big)}}$$

become normalized in the second moment $\phi(T_{i,N,k}^2) = 1$.

As in the previous section we get

$$t_{T_i}(\nu) = \frac{1}{\big[2\big(1+q^{\binom{n}{2}}\big)\big]^m} \sum_{\mu \in Z(\nu)} 2^m\, t_i(\mu),$$

where $\nu \in \mathbb{P}_2(2m)$.

Let us enumerate the blocks in the pairing ν and write $\mu = \nu\,(i_1, \ldots, i_m)$, where the indices $i_1, \ldots, i_m$ are equal to 0 or 1 depending on the choice of the replacement (0 or 1 in the picture (1)) for the corresponding blocks $1, \ldots, m$ in the pairing ν. One can check that

$$t_i\big(\nu\,(i_1, \ldots, i_m)\big) = t_{i,q^{n^2}}\,(\nu) \prod_{p=1}^{m} \big(q^{\binom{n}{2}}\big)^{i_p}.$$

The factor $t_{i,q^{n^2}}\,(\nu)$ appears because any crossing in the pairing ν turns into n^2 crossings, and the factors $\big(q^{\binom{n}{2}}\big)^{i_p}$ appear because we get additional $q^{\binom{n}{2}}$ crossings if we substitute 1 for the p's block and we do not get additional crossings if we substitute 0 for this block. Finally, we get $t_{T_i}\,(\nu) = t_{i,q^{n^2}}\,(\nu) = \big(q^{n^2}\big)^{i(\nu)}$.

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