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STRONG EXPONENTIAL INTEGRABILITY OF SUMS OF INDEPENDENT *B*-VALUED RANDOM VECTORS

BY

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Abstract. An exponential inequality for sums of independent uniformly bounded B-valued random vectors is proved. It is applied to obtain results of the form

$$\sup_n \mathrm{E}\{\exp(\alpha \|S_n\| \log(1+\|S_n\|))\} < \infty$$

for uniformly bounded row-wise independent triangular arrays and independent series. A sharp integrability result for Poisson measures on spaces of cotype 2 follows as a corollary. Some integrability results of the form

$$\sup_n \mathrm{E}\{\exp(\alpha \|S_n\|^p)\} < \infty \quad (1 < p \leqslant 2)$$

for certain triangular arrays and series are proved, generalizing some recent work of Kuelbs. As an application some results on convergence of exponential moments in the central limit theorem are obtained.

1. Introduction. The object of this paper* is to study conditions under which row-wise independent triangular arrays or independent series of Banach space valued random vectors have very strong integrability properties: explicitly, we prove the finiteness of certain exponential moments of order higher than one under various assumptions.

Section 2 contains a generalization of an exponential inequality proved by Bennett [4] for uniformly bounded real-valued random variables to the case of uniformly bounded B-valued random vectors. This inequality plays an essential role in Section 3.

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In Section 3 we prove results of the form

$$\sup_n \mathrm{E}\{\exp(\alpha\|S_n\| \log(1+\|S_n\|))\} < \infty$$

for certain uniformly bounded triangular arrays or series in a general Banach space. The results in this section refine, in a particular case, several integrability theorems obtained in [1] and [3]. As a corollary we obtain a result on convergence of exponential moments of the above form in the central limit theorem.

The integrability results of Section 3 take a particularly satisfactory aspect in the case of spaces of cotype 2. We have isolated these results in Section 4 because of their seemingly final form. As a corollary we obtain a sharp integrability result for Poisson measures on spaces of cotype 2. At the end of this section we pose some open questions.

Section 5 contains results of the form

$$\sup_n \mathrm{E}\{\exp(\alpha\|S_n\|^p)\} < \infty, \quad p \in (1, 2],$$

with

$$S_n = \sum_j b_{nj} X_{nj},$$

where $\{b_{nj}\}$ are real numbers and $\{X_{nj}\}$ are B-valued random vectors. We obtain generalizations of several results proved in an interesting recent paper of Kuelbs [7] for the case of the exponent $p = 2$. We also prove a result on convergence of exponential moments in the central limit theorem in the framework of this section. Let us remark that, so far as we know, the results in Section 5 are new even for the real-valued case.

Notation. B will denote a separable Banach space, $B_r = \{x \in B\colon \|x\| \leqslant r\}$ $(r > 0)$. For a B-valued random vector (r.v.) X, we write

$$X_\tau = X I_{\{X\in B_\tau\}} \quad \text{and} \quad X^{(\tau)} = X - X_\tau.$$

By a *triangular array* we shall mean a doubly-indexed, row-wise independent family $\{X_{nj}\colon j = 1, \ldots, k_n;\ n \in N\}$ of B-valued r.v.'s. In all sections except Section 5 we write

$$S_n = \sum_{j=1}^{k_n} X_{nj};$$

in the case of series, we write similarly

$$S_n = \sum_{j=1}^{n} X_j.$$

Also,

$$S_{n,\tau} = \sum_j X_{nj\tau}, \quad S_n^{(\tau)} = \sum_j X_{nj}^{(\tau)},$$

$$M = \sup_n \|S_n\|, \quad M_\tau = \sup_n \|S_{n,\tau}\|, \quad M^{(\tau)} = \sup_n \|S_n^{(\tau)}\|.$$

2. An exponential inequality for the sum of independent a.s. bounded B-valued r.v.'s.

LEMMA 2.1 (Yurinskiĭ [8]). *Let $\{X_j: j = 1, \ldots, n\}$ be independent B-valued r.v.'s, let*

$$S_n = \sum_{j=1}^{n} X_j,$$

and assume $X_j \in L^1(B)$ $(j = 1, \ldots, n)$. Let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$ for $k = 1, \ldots, n$ and let $\mathcal{F}_0$ be the trivial σ-algebra. Then for $k = 1, \ldots, n$

$$\left|\mathrm{E}\{\|S_n\| \,|\mathcal{F}_k\} - \mathrm{E}\{\|S_n\| \,|\mathcal{F}_{k-1}\}\right| \leqslant \|X_k\| + \mathrm{E}\,\|X_k\| \quad a.s.$$

This is proved by an elementary argument with conditional expectations.

The next theorem extends a result of Bennett [4] for real-valued r.v.'s to the case of B-valued r.v.'s. For $c > 0$ and $\lambda > 0$, let $\varphi_c(\lambda) = c^{-2}(e^{\lambda c} - 1 - \lambda c)$.

THEOREM 2.1. *Let $\{X_j: j = 1, \ldots, n\}$ be independent B-valued r.v.'s, let*

$$S_n = \sum_{j=1}^{n} X_j,$$

and assume $\|X_j\| \leqslant c < \infty$ a.s. $(j = 1, \ldots, n)$. Let

$$a = \sum_{j=1}^{n} \mathrm{E}\,\|X_j\|^2.$$

Then for all $t > 0$

$$P\{\|S_n\| - \mathrm{E}\,\|S_n\| > t\} \leqslant \exp\left(\frac{t}{2c} - \left(\frac{t}{2c} + \frac{a}{c^2}\right)\log\left(1 + \frac{tc}{2a}\right)\right).$$

Proof. We first establish the following inequality:

$$\mathrm{E}\{\exp(\lambda(\|S_n\| - \mathrm{E}\,\|S_n\|))\} \leqslant \exp\left(\varphi_c(2\lambda) \sum_{j=1}^{n} \mathrm{E}\,\|X_j\|^2\right) \quad \text{for all } \lambda > 0. \tag{2.1}$$

Put $\eta_j = \mathrm{E}\{\|S_n\| \,|\, \mathcal{F}_j\} - \mathrm{E}\{\|S_n\| \,|\, \mathcal{F}_{j-1}\}$ $(j = 1, \ldots, n)$. Then

$$\|S_n\| - \mathrm{E}\,\|S_n\| = \sum_{j=1}^{n} \eta_j$$

and

$$\begin{aligned} \mathrm{E}\{\exp(\lambda(\|S_n\| - \mathrm{E}\,\|S_n\|))\} &= \mathrm{E}\Big(\mathrm{E}\Big\{\exp\big(\lambda \sum_{j=1}^{n} \eta_j\big) \,|\, \mathcal{F}_{n-1}\Big\}\Big) \\ &= \mathrm{E}\Big(\exp\big(\lambda \sum_{j=1}^{n-1} \eta_j\big)\, \mathrm{E}\{\exp(\lambda\eta_n) \,|\, \mathcal{F}_{n-1}\}\Big). \end{aligned} \tag{2.2}$$

Now

(2.3)

$$\mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\} = 1+\sum_{k=2}^{\infty}\frac{\lambda^k\,\mathrm{E}\{\eta_n^k|\mathscr{F}_{n-1}\}}{k!} \leqslant 1+\sum_{k=2}^{\infty}\frac{\lambda^k(2c)^{k-2}4\mathrm{E}\,\|X_n\|^2}{k!}$$

$$\leqslant 1+\varphi_c(2\lambda)\,\mathrm{E}\,\|X_n\|^2 \leqslant \exp\left(\varphi_c(2\lambda)\,\mathrm{E}\,\|X_n\|^2\right);$$

in the first step we have used $\mathrm{E}\{\eta_n|\mathscr{F}_{n-1}\} = 0$ and in the second

$$\mathrm{E}\{\eta_n^k|\mathscr{F}_{n-1}\} \leqslant (2c)^{k-2}\,\mathrm{E}\{\eta_n^2|\mathscr{F}_{n-1}\} \leqslant (2c)^{k-2}4\mathrm{E}\,\|X_n\|^2,$$

which follows from Lemma 2.1 and from the boundedness assumption.

By (2.2) and (2.3),

$$\mathrm{E}\{\exp(\lambda\sum_{j=1}^{n}\eta_j)\} \leqslant \exp\left(\varphi_c(2\lambda)\,\mathrm{E}\,\|X_n\|^2\right)\mathrm{E}\{\exp(\lambda\sum_{j=1}^{n-1}\eta_j)\}.$$

Iterating the same procedure yields (2.1).

By (2.1) and Markov's inequality, for all $\lambda > 0$ and $t > 0$ we have

$$P\{\|S_n\|-\mathrm{E}\,\|S_n\| > t\} \leqslant \exp\left(-\lambda t+a\varphi_c(2\lambda)\right).$$

For fixed $t > 0$, let $g_t(\lambda) = -\lambda t+a\varphi_c(2\lambda)$. By elementary calculus, g_t has a minimum at

$$\lambda_t = \frac{1}{2c}\log\left(1+\frac{tc}{2a}\right).$$

Since obviously $P\{\|S_n\|-\mathrm{E}\,\|S_n\| > t\} \leqslant \exp\left(g_t(\lambda_t)\right)$, one may complete the proof by elementary computations.

Remark. The inequality in Theorem 2.1 is slightly weaker than Bennett's [4] one-dimensional inequality in two respects: in the inequality in [4] the term $\mathrm{E}\,\|S_n\|$ on the left-hand side is absent, and the factor 1/2 multiplying $-t/c$ in the exponent on the right-hand side does not appear. However, Theorem 2.1 together with a somewhat delicate truncation argument will produce sharp integrability results in Theorems 3.2, 3.3 and 4.2-4.4.

3. Integrability of a.s. bounded *B*-valued series and triangular arrays. The first result refines in a particular case — namely, under the special assumption (c) — Theorems 3.1 and 3.2 of [1] and Theorem 2.1 of [3].

In Sections 3 and 4, we shall write

$$f_\alpha(x) = \exp\left(\alpha x\log(1+x)\right) \quad (\alpha\in R,\ x \geqslant 0).$$

The following obvious properties of the functions f_α will be useful:

(i) f_α is strictly increasing and convex,

(ii) if $\alpha\beta < \gamma$, then $f_\alpha(\beta t)/f_\gamma(t) \to 0$ as $t \to \infty$.

THEOREM 3.1. *Let $\{X_{nj}\}$ be a triangular array of B-valued r.v.'s. Assume*
(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,
(b) $\{S_n\}$ *is stochastically bounded,*
(c) $\sup_n \sum_j \mathrm{E}\,\|X_{nj} - \mathrm{E}\,X_{nj}\|^2 < \infty$.
Then

$$\sup_n \mathrm{E} f_\alpha(\|S_n\|) < \infty \quad \text{for every } \alpha < (4c)^{-1}.$$

Proof. Let

$$Y_{nj} = X_{nj} - \mathrm{E}X_{nj} \quad \text{and} \quad T_n = \sum_j Y_{nj} = S_n - \mathrm{E}S_n.$$

By well-known results (see, e.g., Theorem 3.1 of [1]), (a) and (b) imply

$$h = \sup_n \mathrm{E}\,\|S_n\| < \infty.$$

Now $\mathrm{E}\,\|T_n\| \leqslant 2h$; also, by (a), $\|Y_{nj}\| \leqslant 2c$ for all n, j. Let

$$a = \sup_n \sum_j \mathrm{E}\,\|Y_{nj}\|^2.$$

Since $\|S_n\| \leqslant \|T_n\| - \mathrm{E}\,\|T_n\| + 3h$, Theorem 2.1 gives

$$P\{\|S_n\| > t\} \leqslant P\{\|T_n\| - \mathrm{E}\,\|T_n\| > t - 3h\}$$
$$\leqslant \exp\left(\frac{t-3h}{4c} - \frac{t-3h}{4c}\log\left(1 + \frac{(t-3h)c}{a}\right)\right) \quad \text{for every } t > 3h.$$

The assertion follows at once from this inequality and from the formula

$$\mathrm{E} f_\alpha(\|S_n\|) = 1 + \int_0^\infty f'_\alpha(t) P\{\|S_n\| > t\}\,dt.$$

Remark. It is clear that the integrability statement holds for every $\alpha < (2c)^{-1}$ if one replaces (c) by the stronger assumption:

$$\sup_n \sum_j \mathrm{E}\,\|X_{nj}\|^2 < \infty.$$

LEMMA 3.1. *Let $\{X_j\colon j \in N\}$ be independent B-valued r.v.'s. Assume*
(a) $\|X_j\| \leqslant c < \infty$ *a.s. for all* $j \in N$,
(b) $\sum_{j=1}^{\infty} P\{\|X_j\| > \tau\} < \infty$ *for some* $\tau > 0$.
Then $\mathrm{E} f_\alpha(M^{(\tau)}) < \infty$ *for all* $\alpha < c^{-1}$.

Proof. We use an idea in [1], Theorem 3.2. Let

$$\varphi_j = I_{\{\|X_j\| > \tau\}}, \quad \varphi = \sum_{j=1}^{\infty} \varphi_j.$$

The φ_j's are independent; also $\|S_n^{(\tau)}\| \leqslant c\varphi$ for all n, which implies $M^{(\tau)} \leqslant c\varphi$.

For all $\lambda > 0$,

$$\begin{aligned}\mathrm{E}\{\exp(\lambda M^{(\tau)})\} \leqslant \mathrm{E}\{\exp(\lambda c\varphi)\} &= \prod_j \mathrm{E}\{\exp(\lambda c\varphi_j)\} \\ &= \prod_j (e^{\lambda c} P\{\varphi_j = 1\} + P\{\varphi_j = 0\}) \\ &= \prod_j (1 + (e^{\lambda c}-1) P\{\|X_j\| > \tau\}) \leqslant \exp((e^{\lambda c}-1)d),\end{aligned}$$

where

$$d = \sum_{j=1}^{\infty} P\{\|X_j\| > \tau\}.$$

By Markov's inequality,

$$P\{M^{(\tau)} > t\} \leqslant \exp(-\lambda t + d(e^{\lambda c}-1)) \quad \text{for all } \lambda > 0,\ t > 0.$$

Fix $t > 0$ and let $g_t(\lambda) = -\lambda t + d(e^{\lambda c}-1)$. Then g_t has a minimum at

$$\lambda_t = \frac{1}{c}\log\frac{t}{dc}.$$

It follows that

$$P\{M^{(\tau)} > t\} \leqslant \exp(g_t(\lambda_t)) = \exp\left(\frac{t}{c} - \frac{t}{c}\log\left(\frac{t}{dc}\right) - d\right).$$

The proof is completed by using the formula

$$\mathrm{E}f_\alpha(M^{(\tau)}) = 1 + \int_0^\infty f_\alpha'(t) P\{M^{(\tau)} > t\}\, dt.$$

The following result for series refines Theorems 2.3 and 2.5 of [1] in a particular case.

THEOREM 3.2. *Let $\{X_j\colon j \in N\}$ be independent B-valued r.v.'s. Assume*

(a) $\|X_j\| \leqslant c < \infty$ *a.s. for all* $j \in N$,

(b) $\sum_{j=1}^{\infty} \mathrm{E}\|X_j - \mathrm{E}X_j\|^2 < \infty$.

Then
(1) *if $\{S_n\}$ is stochastically bounded, then $\mathrm{E}f_\alpha(M) < \infty$ for all $\alpha < (8c)^{-1}$;*
(2) *if S_n converges a.s. in B, then $\mathrm{E}f_\alpha(M) < \infty$ for all $\alpha < c^{-1}$.*

Proof. (1) Let

$$M_n = \sup_{k \leqslant n} \|S_k\|.$$

Choose t_0 so that $\sup_k P\{\|S_k\| > t_0/2\} < 1/2$. By the Lévy-Ottaviani inequality,

$$P\{M_n > t\} \leqslant \left(1 - \max_{k \leqslant n} P\left\{\|S_k\| > \frac{t}{2}\right\}\right)^{-1} P\left\{\|S_n\| > \frac{t}{2}\right\}$$
$$\leqslant 2P\{\|2S_n\| > t\} \quad \text{for } t \geqslant t_0.$$

Now

$$\mathrm{E} f_\alpha(M_n) \leqslant f_\alpha(t_0) + \int_{t_0}^{\infty} f_\alpha'(t) P\{M_n > t\}\, dt$$
$$\leqslant f_\alpha(t_0) + 2\int_{t_0}^{\infty} f_\alpha'(t) P\{\|2S_n\| > t\}\, dt \leqslant f_\alpha(t_0) + 2\mathrm{E} f_\alpha(\|2S_n\|).$$

By monotone convergence and Theorem 3.1 we obtain $\mathrm{E}f_\alpha(M) < \infty$ for $\alpha < (8c)^{-1}$.

(2) First fix $\tau > 0$, and observe that

$$\sum_{j=1}^{\infty} P\{\|X_j\| > \tau\} < \infty \tag{3.1}$$

by the Borel-Cantelli lemma.

We claim next that

$$\sum_{j=1}^{\infty} \mathrm{E}\,\|X_{j\tau} - \mathrm{E}X_{j\tau}\|^2 < \infty. \tag{3.2}$$

In fact,

$$\sum_{j=1}^{\infty} |\mathrm{E}\,\|X_j - \mathrm{E}X_j\|^2 - \mathrm{E}\,\|X_{j\tau} - \mathrm{E}X_{j\tau}\|^2|$$
$$\leqslant \sum_{j=1}^{\infty} \mathrm{E}\,|(\|X_j - \mathrm{E}X_j\| + \|X_{j\tau} - \mathrm{E}X_{j\tau}\|)(\|X_j - \mathrm{E}X_j\| - \|X_{j\tau} - \mathrm{E}X_{j\tau}\|)|$$
$$\leqslant 4c \sum_{j=1}^{\infty} \mathrm{E}\,\|(X_j - X_{j\tau}) - \mathrm{E}(X_j - X_{j\tau})\|$$
$$\leqslant 8c^2 \sum_{j=1}^{\infty} P\{\|X_j\| > \tau\} < \infty \quad \text{by (3.1).}$$

Thus (3.2) follows by (b).

Assertion (3.1) and Lemma 3.1 imply that $\{S_n^{(\tau)}\}$ is stochastically bounded; since $S_{n,\tau} = S_n - S_n^{(\tau)}$ and $\{S_n\}$ is stochastically bounded, it follows that

(*) $\{S_{n,\tau}\}$ is stochastically bounded.

Now assume that $\alpha < c^{-1}$ is given. Choose $\beta \in (\alpha, c^{-1})$ and $\delta \in (\alpha/\beta, 1)$. Next select $\tau > 0$ so that $\tau < (1-\delta)(8\alpha)^{-1}$. By statement (1), taking into account (3.2) and (*), we have

(3.3) $$\mathrm{E}f_\gamma(M_\tau) < \infty, \quad \text{where } \alpha(1-\delta)^{-1} < \gamma < (8\tau)^{-1}.$$

On the other hand, Lemma 3.1 and (3.1) give

(3.4) $$\mathrm{E}f_\beta(M^{(\tau)}) < \infty.$$

Finally, since $M \leqslant M_\tau + M^{(\tau)}$, we have

$$f_\alpha(M) \leqslant f_\alpha(M_\tau + M^{(\tau)}) \leqslant (1-\delta)f_\alpha\big((1-\delta)^{-1}M_\tau\big) + \delta f_\alpha(\delta^{-1}M^{(\tau)})$$
$$\leqslant (1-\delta)c_0 f_\gamma(M_\tau) + \delta c_1 f_\beta(M^{(\tau)})$$

by properties (i) and (ii) of the functions f_α. Therefore

$$\mathrm{E}f_\alpha(M) \leqslant (1-\delta)c_0\,\mathrm{E}f_\gamma(M_\tau) + \delta c_1\,\mathrm{E}f_\beta(M^{(\tau)}) < \infty$$

by (3.3) and (3.4).

The following example is a slight modification of one presented in [1]. It shows that even on the real line Theorem 3.2 (2) is sharp in the following sense:

Example 3.1. *For every $c > 0$, there exists an independent sequence of real-valued r.v.'s $\{\xi_j\}$ such that*

(a) $|\xi_j| \leqslant c$ *for all* j,

(b) $S = \sum_{j=1}^{\infty} \xi_j$ *exists a.s.*,

(c) $\sum_{j=1}^{\infty} \mathrm{E}\xi_j^2 < \infty$,

but $\mathrm{E}f_\alpha(|S|) = \infty$ *for all* $\alpha > c^{-1}$.

Proof. It is clear that it is enough to prove the assertion for $c = 1$. Choose $\beta > 1$. Let $\{\xi_j\}$ be independent r.v.'s with $\mathscr{L}(\xi_j) = (1-p_j)\delta_0 + p_j\delta_1$, where $p_j = j^{-1}(\log j)^{-\beta}$. It is easily verified that (a)-(c) are satisfied.

Let

$$S_n = \sum_{j=1}^{n} \xi_j.$$

Then, as $n \to \infty$ we have

$$\mathrm{E}f_\alpha(S_n) \geqslant \exp(\alpha n \log n) P\{S_n = n\}$$
$$= \exp(\alpha n \log n)(n!)^{-1}\Big(\prod_{j=1}^{n} \log j\Big)^{-\beta} \geqslant \exp(\alpha n \log n) n^{-n} (\log n)^{-n\beta}$$
$$= \exp\big((\alpha-1)n\log n - n\beta \log(\log n)\big) \to \infty \quad \text{if } \alpha > 1.$$

The next proposition sharpens Theorem 3.1 for an important class of triangular arrays. We shall need the following lemma, which is similar to Lemma 3.1.

LEMMA 3.2. *Let* $\{X_{nj}\}$ *be a triangular array of B-valued r.v.'s. Assume*
(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,
(b) $\sup_n \sum_j P\{\|X_{nj}\| > \tau\} < \infty$ *for some* $\tau > 0$.
Then

$$\sup_n \mathrm{E} f_\alpha(\|S_n^{(\tau)}\|) < \infty \quad \textit{for every } \alpha < c^{-1}.$$

The proof is very similar to that of Lemma 3.1 and is therefore omitted.

THEOREM 3.3. *Let* $\{X_{nj}\}$ *be an infinitesimal triangular array of B-valued r.v.'s. Assume*
(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,
(b) $\{\mathscr{L}(S_n)\}$ *is relatively compact,*
(c) $\sup_n \sum_j \mathrm{E}\,\|X_{nj} - \mathrm{E}X_{nj}\|^2 < \infty$.
Then

$$\sup_n \mathrm{E} f_\alpha(\|S_n\|) < \infty \quad \textit{for all } \alpha < c^{-1}.$$

Proof. It is similar to that of Theorem 3.2. We will indicate the main steps.

Fix $\tau > 0$. By [2], Theorem 2.2,

$$\sup_n \sum_j P\{\|X_{nj}\| > \tau\} < \infty. \tag{3.5}$$

Arguing as in the proof of (3.2) in Theorem 3.2, we obtain

$$\sup_n \sum_j \mathrm{E}\,\|X_{nj\tau} - \mathrm{E}X_{nj\tau}\|^2 < \infty. \tag{3.6}$$

By Lemma 3.2, $\{S_n^{(\tau)}\}$ is stochastically bounded; since $S_{n,\tau} = S_n - S_n^{(\tau)}$, we may conclude that $\{S_{n,\tau}\}$ is stochastically bounded.

Given $\alpha < c^{-1}$, choose $\beta \in (\alpha, c^{-1})$ and $\delta \in (\alpha/\beta, 1)$. Next select $\tau > 0$ so that $\tau < (1-\delta)(4\alpha)^{-1}$. By Theorem 3.1, (3.6) and (*),

$$\sup_n \mathrm{E} f_\gamma(\|S_{n,\tau}\|) < \infty, \quad \text{where } \alpha(1-\delta)^{-1} < \gamma < (4\tau)^{-1}.$$

By Lemma 3.2 and (3.5),

$$\sup_n \mathrm{E} f_\beta(\|S_n^{(\tau)}\|) < \infty.$$

We may now complete the proof by writing

$$\mathrm{E} f_\alpha(\|S_n\|) \leqslant \mathrm{E} f_\alpha(\|S_{n,\tau}\| + \|S_n^{(\tau)}\|)$$

and proceeding as in Theorem 3.2.

In the following corollary we obtain a convergence result for a case not covered by the theorems on convergence of moments in the general central limit theorem in [3].

COROLLARY 3.1. *Let $\{X_{nj}\}$ be an infinitesimal triangular array of B-valued r.v.'s such that $\mathscr{L}(S_n) \underset{w}{\rightarrow} \nu$. Assume*

(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,

(b) $\sup_n \sum_j \mathrm{E}\,\|X_{nj} - \mathrm{E}X_{nj}\|^2 < \infty$.

Let $\varphi\colon B \to R^+$ be a continuous function such that $\varphi(x) \leqslant bf_\alpha(\|x\|)$ for all $x \in B$, for some $b > 0$ and $\alpha < c^{-1}$.

Then

$$\int \varphi d\nu < \infty \quad \text{and} \quad \lim_n \mathrm{E}\varphi(S_n) = \int \varphi d\nu.$$

Proof. The uniform integrability of $\{\varphi(S_n)\}$ follows easily from Theorem 3.3.

4. Triangular arrays, series and Poisson measures in spaces of cotype 2. The special assumption in Theorems 3.1-3.3 and in Corollary 3.1 may be dropped if B is a Banach space of cotype 2. Let us recall that if B is of cotype 2, then there exists $A > 0$ such that

$$\sum_{j=1}^{n} \mathrm{E}\,\|Y_j\|^2 \leqslant A\,\mathrm{E}\,\Big\|\sum_{j=1}^{n} Y_j\Big\|^2$$

for all finite independent sequences $\{Y_1, \ldots, Y_n\}$ of B-valued r.v.'s such that $\mathrm{E}\,\|Y_j\|^2 < \infty$ and $\mathrm{E}\,Y_j = 0$ $(j = 1, \ldots, n)$.

THEOREM 4.1. *Let B be a separable Banach space of cotype 2 and let $\{X_{nj}\}$ be a triangular array of B-valued r.v.'s. Assume*

(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,

(b) $\{S_n\}$ *is stochastically bounded.*

Then

$$\sup_n \mathrm{E}f_\alpha(\|S_n\|) < \infty \quad \textit{for every } \alpha < (4c)^{-1}.$$

Proof. Let

$$Y_{nj} = X_{nj} - \mathrm{E}X_{nj} \quad \text{and} \quad T_n = \sum_j Y_{nj} = S_n - \mathrm{E}S_n.$$

Then

$$\sum_j \mathrm{E}\,\|Y_{nj}\|^2 \leqslant A\mathrm{E}\,\|T_n\|^2 \leqslant 4A\mathrm{E}\,\|S_n\|^2.$$

Since

$$\sup_n \mathrm{E}\,\|S_n\|^2 < \infty$$

by well-known results (see, e.g., [1], Theorem 3.1), the assertion follows from Theorem 3.1.

THEOREM 4.2. *Let B be a separable Banach space of cotype 2 and let $\{X_j : j \in N\}$ be independent B-valued r.v.'s. Assume that $\|X_j\| \leqslant c < \infty$ a.s. for all $j \in N$. Then*

(1) *if $\{S_n\}$ is stochastically bounded, then* $\mathrm{E}f_\alpha(M) < \infty$ *for all* $\alpha < (8c)^{-1}$,

(2) *if S_n converges a.s. in B, then* $\mathrm{E}f_\alpha(M) < \infty$ *for all* $\alpha < c^{-1}$.

Proof is similar to that of the previous theorem, but using Theorem 3.2.

THEOREM 4.3. *Let B be a separable Banach space of cotype 2 and let $\{X_{nj}\}$ be an infinitesimal triangular array of B-valued r.v.'s. Assume*

(a) $\|X_{nj}\| \leqslant c < \infty$ *a.s. for all* n, j,

(b) $\{\mathscr{L}(S_n)\}$ *is relatively compact.*

Then

$$\sup_n \mathrm{E}f_\alpha(\|S_n\|) < \infty \quad \text{for all } \alpha < c^{-1}.$$

Proof. As in Theorems 4.1 and 4.2, but using Theorem 3.3.

Analogously, Corollary 3.1 is true without assumption (b) if B is a space of cotype 2.

The next result refines Corollary 3.3 of [1] for Poisson measures in spaces of cotype 2 (for definitions and basic facts on Poisson measures, see [2]). Theorem 4.4 is sharp in that we prove the finiteness of the integral $\int f_\alpha(\|x\|)(c_\tau \operatorname{Pois}\mu)(dx)$ for the maximum possible range of α's. (In fact, if $\mu = \delta_1$ on R^1 and $\nu = \operatorname{Pois}\mu = e^{-1}\exp(\delta_1)$, then $\int f_\alpha d\nu = \infty$ for $\alpha \geqslant 1$.) When applied to the case of Poisson measures on Hilbert space, Theorem 4.4 improves a result of Kruglov [6] in which the precise range of admissible α's is not specified.

THEOREM 4.4. *Let B be a separable Banach space of cotype 2. Let μ be a Lévy measure on B such that $\mu(B_r^c) = 0$ for some $r > 0$. Then for all $\alpha < r^{-1}$ and all $\tau > 0$*

$$\int f_\alpha(\|x\|)(c_\tau \operatorname{Pois}\mu)(dx) < \infty.$$

Proof. Just as in [1], Corollary 3.3, but using Theorem 4.3.

The results of Sections 3 and 4 lead to the following question: is it possible to eliminate assumption (c) in Theorems 3.1 and 3.3 (assumption (b) in Theorem 3.2)? Or, better, one may pose

PROBLEM I. Determine for what Banach spaces it is true that any triangular array satisfying assumptions (a) and (b) of Theorem 3.1 also satisfies its conclusion (a similar problem may be posed for the statements of Theorems 3.2 and 3.3).

Theorem 4.4 suggests the following very closely related

PROBLEM II. Determine for what Banach spaces it is true that any Poisson measure whose Lévy measure has bounded support satisfies the conclusion of Theorem 4.4. Also, is Corollary 3.3 of [1] the best possible result in a general Banach space?

5. Exponential moments of order $p \in (1, 2]$. Let $\{X_{nj}\}$ be a triangular array of B-valued r.v.'s, $\{b_{nj}\}$ a triangular array of real numbers, and let

$$S_n = \sum_j b_{nj} X_{nj}.$$

Let $p \in (1, 2]$. In this section we give conditions under which

$$\sup_n \mathrm{E}\{\exp(\alpha \|S_n\|^p)\} < \infty \quad \text{for some } \alpha > 0.$$

(Let us remark, although the results in this section are stated for Banach spaces, all of them except Theorem 5.4 carry over to the case of linear measurable spaces considered in [7].)

The key to the integrability results in this section is the exponential inequality given in Theorem 5.1; it is analogous to inequality (2.1).

By elementary calculus we get

LEMMA 5.1. *Let* $a > 0$ *and* $\alpha > 0$. *Then* $\beta > \max\{ae^{\alpha}, 1+\alpha\}$ *implies*

$$1 + aue^{\alpha u} < e^{\beta u} \quad \textit{for all } u > 0.$$

LEMMA 5.2. *Let* $\beta > 0$ *and* $p > 1$. *Then there exist* $\alpha > 0$ *and* $c > 0$ *such that*

$$\int_0^\infty te^{ut} \exp(-\beta t^p)\, dt \leqslant c \exp(\alpha u^q) \quad \textit{for all } u > 0,$$

where $p^{-1} + q^{-1} = 1$.

Proof. By convexity, for every $\lambda > 0, u > 0, t > 0$ we have

$$(5.1) \qquad -\beta t^p + ut = -\beta t^p + (\lambda t)\left(\frac{u}{\lambda}\right)$$
$$\leqslant -\beta t^p + \frac{\lambda^p t^p}{p} + \frac{u^q}{\lambda^q q} = -t^p\left(\beta - \frac{\lambda^p}{p}\right) + \frac{1}{\lambda^q q} u^q.$$

Let $\alpha > 1/(\beta p)^{q/p} q$ and take $\lambda = (\alpha q)^{-1/q}$. Then $\delta = \beta - \lambda^p/p > 0$ and from (5.1) we have for every $u > 0, t > 0$ the inequality

$$-\beta t^p + ut \leqslant -\delta t^p + \alpha u^q.$$

It follows that

$$\int_0^\infty t \exp(ut - \beta t^p)\, dt \leqslant \int_0^\infty t \exp(-\delta t^p + \alpha u^q)\, dt = c \exp(\alpha u^q),$$

where

$$c = \int_0^\infty t \exp(-\delta t^p)\, dt.$$

THEOREM 5.1. *For every $p > 1$ and every $\beta > 0$, $L > 0$, there exists $\gamma > 0$ such that, for every finite independent sequence $\{X_j : j = 1, \ldots, n\}$ of B-valued r.v.'s with $\mathrm{E}\{\exp(\beta \|X_j\|^p)\} \leqslant L$ $(j = 1, \ldots, n)$, the inequality*

$$\mathrm{E}\{\exp(\lambda(\|S_n\| - \mathrm{E}\|S_n\|))\} \leqslant \exp\Big(\gamma\big(\lambda^2 \sum_{j=1}^{n} b_j^2 + \lambda^q \sum_{j=1}^{n} |b_j|^q\big)\Big)$$

holds for every finite real sequence $\{b_j : j = 1, \ldots, n\}$ and for every $\lambda > 0$; here

$$S_n = \sum_{j=1}^{n} b_j X_j \quad \text{and} \quad p^{-1} + q^{-1} = 1.$$

Proof. As in the proof of Theorem 2.1, we may write

$$\|S_n\| - \mathrm{E}\|S_n\| = \sum_{j=1}^{n} \eta_j$$

and

$$\mathrm{E}\{\exp(\lambda(\|S_n\| - \mathrm{E}\|S_n\|))\} = \mathrm{E}\Big(\exp\big(\lambda \sum_{j=1}^{n-1} \eta_j\big)\, \mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\}\Big). \tag{5.2}$$

Obviously, we may assume $b_j \geqslant 0$, $j = 1, \ldots, n$. By Lemma 2.1, $|\eta_j| \leqslant b_j Y_j$, where $Y_j = \|X_j\| + \mathrm{E}\|X_j\|$ $(j = 1, \ldots, n)$. It is clear that we may choose $\delta > 0$ and $M > 0$, both depending only on p, β and L, such that $\mathrm{E}\{\exp(\delta Y_j^p)\} \leqslant M$ $(j = 1, \ldots, n)$.

Now

$$\mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\} = 1 + \sum_{k=2}^{\infty} \frac{\lambda^k \mathrm{E}\{\eta_n^k|\mathscr{F}_{n-1}\}}{k!} \leqslant 1 + \sum_{k=2}^{\infty} \frac{\lambda^k b_n^k \mathrm{E} Y_n^k}{k!}. \tag{5.3}$$

Since

$$\mathrm{E}Y_n^k = \int_0^{\infty} kt^{k-1} P\{Y_n > t\}\, dt \leqslant \int_0^{\infty} kt^{k-1} M \exp(-\delta t^p)\, dt,$$

formula (5.3) implies

$$\begin{aligned} \mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\} &\leqslant 1 + \sum_{k=2}^{\infty} \frac{\lambda^k b_n^k}{k!} \int_0^{\infty} kt^{k-1} M \exp(-\delta t^p)\, dt \\ &= 1 + M \int_0^{\infty} \Big(\sum_{k=2}^{\infty} \frac{\lambda^k b_n^k t^{k-1}}{(k-1)!}\Big) \exp(-\delta t^p)\, dt \\ &= 1 + M(\lambda b_n) \int_0^{\infty} [\exp(\lambda b_n t) - 1] \exp(-\delta t^p)\, dt \\ &\leqslant 1 + M(\lambda b_n)^2 \int_0^{\infty} t \exp\big((\lambda b_n) t\big) \exp(-\delta t^p)\, dt; \end{aligned} \tag{5.4}$$

in the last step we have used the obvious inequality $u(e^{ut}-1) \leqslant u^2 te^{ut}$ $(u \geqslant 0, t \geqslant 0)$. By Lemma 5.2, (5.4) yields

$$\mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\} \leqslant 1+MC(\lambda b_n)^2 \exp\left(\alpha(\lambda b_n)^q\right) \tag{5.5}$$

for certain constants $c > 0$ and $\alpha > 0$. By Lemma 5.1, putting $u = \lambda b_n$, we get

$$\begin{aligned}1+MCu^2 \exp(\alpha u^q) &\leqslant 1+MC \max\{u^2, u^q\} \exp(\alpha \max\{u^2, u^q\}) \\ &\leqslant \exp(\gamma \max\{u^2, u^q\}) \leqslant \exp\left(\gamma(u^2+u^q)\right)\end{aligned}$$

for a certain constant γ. Thus from (5.5) we obtain

$$\mathrm{E}\{\exp(\lambda\eta_n)|\mathscr{F}_{n-1}\} \leqslant \exp\left(\gamma(\lambda^2 b_n^2+\lambda^q b_n^q)\right)$$

and from (5.2) we get

$$\mathrm{E}\{\exp(\lambda \sum_{j=1}^{n} \eta_j)\} \leqslant \exp\left(\gamma(\lambda^2 b_n^2+\lambda^q b_n^q)\right) \mathrm{E}\{\exp(\lambda \sum_{j=1}^{n-1} \eta_j)\}.$$

The proof is completed by iterating the same procedure.

From Theorem 5.1 we obtain an integrability result for triangular arrays.

THEOREM 5.2. *Let $\{X_{nj}\}$ be a triangular array of B-valued r.v.'s, $\{b_{nj}\}$ a triangular array of real numbers, and*

$$S_n = \sum_j b_{nj} X_{nj}.$$

Let $1 < p \leqslant 2$. Assume

(a) $\sup_{n,j} \mathrm{E}\{\exp(\beta \|X_{nj}\|^p)\} < \infty$ *for some* $\beta > 0$,

(b) $\sup_n \sum_j b_{nj}^2 < \infty$,

(c) $\{S_n\}$ *is stochastically bounded.*

Then, for some $\alpha > 0$,

$$\sup_n \mathrm{E}\{\exp(\alpha \|S_n\|^p)\} < \infty.$$

Proof. By well-known arguments,

$$b = \sup_n \mathrm{E}\|S_n\| < \infty$$

(see, e.g., [7], Lemma 3.1). By Markov's inequality and Theorem 5.1, for all $t > 0$ and $\lambda > 0$ we obtain

$$\begin{aligned}P\{\|S_n\| > t+b\} &\leqslant P\{\|S_n\| - \mathrm{E}\|S_n\| > t\} \\ &\leqslant \exp(-\lambda t)\, \mathrm{E}\left\{\exp\left(\lambda(\|S_n\| - \mathrm{E}\|S_n\|)\right)\right\} \\ &\leqslant \exp(-\lambda t+\gamma c\lambda^2+\gamma d\lambda^q),\end{aligned} \tag{5.6}$$

where

$$c = \sup_n \sum_j b_{nj}^2 < \infty \quad \text{and} \quad d = \sup_n \sum_j b_{nj}^q;$$

observe that $d^{2/q} \leqslant c < \infty$ by (b) and the fact that $q \geqslant 2$. Fix $t > 0$ and let $g_t(\lambda) = -\lambda t + \gamma c\lambda^2 + \gamma d\lambda^q$. Let $\lambda = \delta t^{p-1}$ with δ to be determined in the sequel. We have

$$g_t(\delta t^{p-1}) = -(\delta - \gamma d\delta^q)t^p + (\gamma c\delta^2)t^{2p-2}.$$

If $p < 2$, choose $\delta > 0$ so that $\tau = \delta - \gamma d\delta^q > 0$; if $p = 2$, we further require that $\tau - \gamma c\delta^2 > 0$. Then from (5.6) we get

$$P\{\|S_n\| > t+b\} \leqslant \inf_{\lambda > 0} \exp(g_t(\lambda)) \leqslant \exp(-\tau t^p + (\gamma c\delta^2)t^{2p-2}) \quad \text{for all } t > 0.$$

The result follows at once from this inequality, as in Theorem 3.1.

We consider next the case of series of the form $\sum_j b_j X_j$. Theorem 5.3 generalizes a result of Kuelbs ([7], Theorem 3.2) for the exponent $p = 2$ to any exponent $p \in (1, 2]$ (our result for $p = 2$ improves slightly Theorem 3.2 of [7], where it is assumed that the X_j's have mean zero).

THEOREM 5.3. *Let $\{X_j : j \in N\}$ be independent B-valued r.v.'s, $\{b_j : j \in N\}$ a sequence of real numbers, and*

$$S_n = \sum_{j=1}^{n} b_j X_j, \qquad M = \sup_n \|S_n\|.$$

Let $1 < p \leqslant 2$. Assume

(a) $\sup_j \mathrm{E}\{\exp(\beta\|X_j\|^p)\} < \infty$ *for some* $\beta > 0$,

(b) $\sum_{j=1}^{\infty} b_j^2 < \infty$,

(c) $\{S_n\}$ *is stochastically bounded.*

Then

(1) $\mathrm{E}\{\exp(\alpha M^p)\} < \infty$ *for some* $\alpha > 0$;

(2) *if* (a) *holds for all* $\beta > 0$, *then* $\mathrm{E}\{\exp(\alpha M^p)\} < \infty$ *for all* $\alpha > 0$.

Proof. (1) follows from Theorem 5.2 by proceeding as in the proof of Theorem 3.2 (1).

(2) The proof is a variant of the argument in Theorem 5.2. We prove the statement for $p < 2$; a trivial modification of the argument gives a proof for $p = 2$. Given $\varrho > 0$, choose m so that

$$d_0 = \sum_{j=m}^{\infty} b_j^q < (\gamma \cdot 2^q \varrho^{q-1})^{-1}.$$

For fixed $t > 0$, let

$$g_t(\lambda) = -\lambda t + \gamma c_0\lambda^2 + \gamma d_0\lambda^q \quad \text{with } c_0 = \sum_{j=m}^{\infty} b_j^2.$$

Then

$$g_t(2\varrho t^{p-1}) = -(2\varrho-\gamma d_0(2\varrho)^q)t^p+\gamma c_0(2\varrho)^2 t^{2p-2} \leqslant -\varrho t^p+(4\gamma c_0\varrho^2)t^{2p-2}.$$

Arguing as in Theorem 5.2, for all $n \geqslant m$ and all $t > 0$ we obtain

$$P\{\|S_n-S_m\| > t+2b\} \leqslant \exp\left(-\varrho t^p+(4\gamma c_0\varrho^2)t^{2p-2}\right). \tag{5.7}$$

Now (5.7) and the assumption that (a) holds for all $\beta > 0$ imply

$$\sup_n \mathrm{E}\{\exp(\alpha\|S_n\|^p)\} < \infty \quad \text{for all } \alpha > 0.$$

Arguing as in (1) again yields (2).

It is possible to obtain, as corollaries to Theorem 5.3, generalizations of Corollaries 3.4 and 3.5 of [7] for series of the form $\sum_j Y_j x_j$, $\{Y_j: j\in N\}$ being an independent sequence of real-valued r.v.'s and $\{x_j: j\in N\}$ a sequence of points in B. We omit the statements, which are obvious modifications of those in [7] (the mean zero assumption should be deleted).

From Theorem 5.2 one may obtain results on the convergence of exponential moments in the central limit theorem covered neither by [3] nor by our Corollary 3.1. The single most interesting case is

THEOREM 5.4. *Let $\{X_j: j\in N\}$ be a sequence of independent identically distributed B-valued r.v.'s, and*

$$S_n = \sum_{j=1}^{n} X_j.$$

Let $1 < p \leqslant 2$. Assume

(a) $\mathrm{E}\{\exp(\beta\|X_1\|^p)\} < \infty$ for some $\beta > 0$,

(b) $\mathscr{L}(n^{-1/2}S_n) \underset{w}{\rightarrow} \gamma$.

Then

(1) if $p < 2$, then for every $\alpha > 0$ there exists $m\in N$ such that

$$\sup_{n\geqslant m} \mathrm{E}\{\exp(\alpha\|n^{-1/2}S_n\|^p)\} < \infty$$

and

$$\lim_n \mathrm{E}\{\exp(\alpha\|n^{-1/2}S_n\|^p)\} = \int \exp(\alpha\|x\|^p)\gamma(dx) < \infty;$$

(2) if $p = 2$, then there exists $\delta > 0$ such that for all $\alpha \leqslant \delta$

$$\lim_n \mathrm{E}\{\exp(\alpha\|n^{-1/2}S_n\|^2)\} = \int \exp(\alpha\|x\|^2)\gamma(dx) < \infty.$$

Proof. (2) follows at once from Theorem 5.2. In order to prove (1) we use again the method of proof of Theorem 5.2. Given $\varrho > 0$, choose m so that $m^{1-q/2} < (\gamma\cdot 2^q\varrho^{q-1})^{-1}$. For $n \geqslant m$ and a fixed $t > 0$, let

$$g_t(\lambda) = -\lambda t+\gamma\lambda^2+\gamma n^{1-q/2}\lambda^q.$$

Then

$$g_t(2\varrho t^{p-1}) \leqslant -\varrho t^p + (4\gamma\varrho^2) t^{2p-2}.$$

Arguing as in Theorem 5.2, for all $n \geqslant m$ and all $t > 0$ we get

$$P\{\|n^{-1/2} S_n\| > t+b\} \leqslant \exp\left(-\varrho t^p + (4\gamma\varrho^2) t^{2p-2}\right).$$

The statement follows easily by the standard formula used already in Theorem 3.1.

Remark. Since the limiting measure γ is necessarily Gaussian, it is always integrable in the stronger sense stated in (2) by Fernique's [5] theorem. It may be of interest to point out that Fernique's result can be obtained from Theorem 5.4 (2). This may be proved as follows. Let γ be a centered Gaussian measure on B, and Φ_γ its covariance. We claim

(**) there exist $c > 0$ and $\tau > 0$ such that

$$\Phi_\gamma(f,f) \leqslant c \int_{B_\tau} f^2 \, d\gamma \quad \text{for all } f \in B'.$$

In fact, choose τ so that $\gamma(B_\tau^c) < \varepsilon < 1/2$. Then

$$(5.8)\qquad 1-\exp\left(-\frac{1}{2}\Phi_\gamma(f,f)\right) = \int \left(1-\cos f(x)\right)\gamma(dx) \leqslant \frac{1}{2}\int_{B_\tau} f^2\, d\gamma + 2\varepsilon.$$

Let

$$\Psi(f, g) = \frac{1}{2}\int_{B_\tau} fg\,d\gamma \qquad (f, g \in B').$$

By (5.8), there exist $\delta > 0$ and $M > 0$ such that $\Psi(f,f) \leqslant \delta$ implies $\Phi_\gamma(f,f) \leqslant M$. Claim (**) follows by homogeneity.

Let $\{X_j : j \in N\}$ be independent B-valued r.v.'s with $\mathscr{L}(X_j) = \gamma$, and put

$$Y_j = c^{1/2} X_{j\tau}, \qquad T_n = \sum_{j=1}^{n} Y_j.$$

Then $\mathscr{L}(n^{-1/2} T_n) \underset{w}{\rightarrow} \mu$, a Gaussian measure on B (to see that $\{\mathscr{L}(n^{-1/2} T_n)\}$ is tight, use, e.g., Lemma 2.6 of [1]). By Theorem 5.4 (2) we have $\int \exp(\alpha \|x\|^2)\mu(dx) < \infty$ for sufficiently small $\alpha > 0$. But $\Phi_\gamma(f,f) \leqslant \Phi_\mu(f,f)$ for all $f \in B'$; by a well-known result, this implies that γ is a convolution factor of μ, and hence

$$\int \exp(\alpha \|x\|^2)\gamma(dx) \leqslant \int \exp(\alpha \|x\|^2)\mu(dx) < \infty$$

by convexity (of course, this proof could be simplified if one could exhibit at once a bounded r.v. belonging to the domain of normal attraction of γ).

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