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# REMARK ON A MULTIPLICATIVE DECOMPOSITION OF PROBABILITY MEASURES

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*Abstract.* The aim of this note is to define a rather wide class of probability measures admitting a multiplicative decomposition.

Let $P$ be the set of all Borel probability measures on the real line. Given $\mu, \nu \in P$, by $\mu\nu$ we shall denote the probability distribution of the product $XY$ of two independent random variables $X$ and $Y$ with probability distributions $\mu$ and $\nu$, respectively. It is evident that the binary operation $\mu\nu$ is commutative, associative and distributive with respect to convex combinations of probability measures. In what follows $\mu^n$ will denote the $n$-th power under this operation. Further, by $\delta_c$ we denote the probability measure concentrated at the point $c$. It is easy to check that

$$(\mu\nu)(E) = \int_{x \neq 0} \mu(x^{-1} E)\, \nu(dx) + \nu(\{0\})\, \delta_0(E).$$

Put $I = (0, 1]$. By $P_I$ we denote the subset of $P$ consisting of all measures concentrated on $I$. We say that $\mu \in Q$ if $\mu \in P$ and, for every $x \in I$, there exists a positive number $c$ such that $\mu(x^{-1} E) \leqslant c\mu(E)$ for all Borel subsets $E$ of the real line.

Denote by $q(\mu, x)$ the infimum of all those numbers $c$. It is clear that $q(\mu, x) \geqslant 1$ whenever $x \in I$ and $q(\mu, 1) = 1$. Moreover, denoting by $\{E_n\}$ the sequence of all open intervals with rational endpoints, we have

$$\{x:\ q(\mu, x) \leqslant c\} = \bigcap_{n=1}^{\infty} \{x:\ \mu(x^{-1} E_n) \leqslant c\mu(E_n)\},$$

which shows that the function $q(\mu, \cdot)$ is Borel measurable on $I$.

A standard calculation leads to the following inequalities for $\mu \in Q$:

$$q(\mu, xy) \leqslant q(\mu, x) q(\mu, y), \quad x, y \in I,$$

(1) $$\int_0^1 q(\mu, x)(\lambda_1 \lambda_2 \ldots \lambda_n)(dx) \leqslant \prod_{j=1}^{n} \int_0^1 q(\mu, x) \lambda_j(dx)$$

for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in P_I$;

(2) $$(\lambda\mu)(E) \leqslant \mu(E) \int_0^1 q(\mu, x) \lambda(dx) \quad \text{for } \lambda \in P_I.$$

We note that the set $Q$ is closed under convolution and convex combinations. Moreover,

$$q(\mu * \nu, x) \leqslant q(\mu, x) q(\nu, x)$$

and

$$q\big(c\mu + (1-c)\nu, x\big) \leqslant \max\big(q(\mu, x), q(\nu, x)\big).$$

As a simple example of measures belonging to $Q$ we quote the Gaussian measure $\varrho$ with the mean $m$ and the variance $\sigma^2$. Then we have

$$q(\varrho, x) = x^{-1} \exp\left(\frac{m^2}{2\sigma^2} \frac{1-x}{1+x}\right), \quad x \in I.$$

Setting for any $b > 0$

$$\mu_b(E) = b \int_{E \cap I} x^{b-1} dx,$$

we have also $\mu_b \in Q$ and $q(\mu_b, x) = x^{-b} (x \in I)$. Furthermore, it is easy to check that all unimodal distributions with the mode at 0 belong to $Q$.

THEOREM. *Let $\mu \in Q$. For every $\lambda \in P_I$ satisfying the condition*

(3) $$\int_0^1 q(\mu, x) \lambda(dx) < 2\lambda(\{1\})$$

*there exists a measure $\nu \in P$, absolutely continuous with respect to $\mu$, such that $\lambda\nu = \mu$.*

Proof. The masure $\lambda$ can be written in the form

$$\lambda = p\delta_1 + (1-p)\eta,$$

where $p = \lambda(\{1\})$, $\eta \in P_I$ and $\eta(\{1\}) = 0$. In the case $p = 1$ we have $\lambda = \delta_1$

and our assertion is obvious with $\nu = \mu$. Suppose that $p < 1$. Since $q(\mu, x) \geqslant 1$, we have by (3) the inequality $p > 1/2$. Consequently,

$$0 < r = \frac{1-p}{p} < 1 \tag{4}$$

and

$$s = r \int_0^1 q(\mu, x)\eta(dx) < 1. \tag{5}$$

Further, inequalities (1) and (2) yield

$$(\eta^n \mu)(E) \leqslant \Big(\int_0^1 q(\mu, x)\eta(dx)\Big)^n \mu(E), \qquad n = 1, 2, \ldots \tag{6}$$

Setting

$$\beta = (1-r)^{-1}(\mu - r\eta\mu) \tag{7}$$

and taking into account (4) and (5) we infer that

$$\beta(E) = (1-r)^{-1}\Big(\mu(E) - r\int_0^1 \mu(x^{-1}E)\eta(dx)\Big)$$

$$\geqslant (1-r)^{-1}\mu(E)\Big(1 - s\int_0^1 q(\mu, x)\eta(dx)\Big) \geqslant 0.$$

Since $\beta$ is normed on the real line, we conclude that $\beta \in P$. Put

$$\nu = (1-r^2)\sum_{k=0}^{\infty} r^{2k}\eta^{2k}\beta,$$

where $\eta^0 = \delta_1$. Obviously, $\nu \in P$ and, by (7),

$$\nu = (1+r)\sum_{n=0}^{\infty} (-1)^n r^n \eta^n \mu. \tag{8}$$

Consequently, by (5) and (6),

$$\nu(E) \leqslant (1+r)\sum_{n=0}^{\infty} r^n (\eta^n \mu)(E) \leqslant \frac{1+r}{1-s}\mu(E),$$

which shows that $\nu$ is absolutely continuous with respect to $\mu$. Further, by (4) and (8),

$$\eta\nu = \frac{1+r}{r}\mu - \frac{1}{r}\nu = \frac{1}{1-p}\mu - \frac{p}{1-p}\nu.$$

Thus

$$\lambda\nu = p(\delta_1 \nu) + (1-p)(\eta\nu) = p\nu + (1-p)(\eta\nu) = \mu,$$

which completes the proof.

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