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THE DOMAIN OF ATTRACTION OF STABLE LAWS AND EXTREME ORDER STATISTICS

BY

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Abstract. The purpose of this paper is to study the asymptotic behaviour of trimmed sums of order statistics

$$Y_n = a_n^{-1}\Big(\sum_{i=k}^{k(n)} X_{i:n} + \sum_{i=n+1-r(n)}^{n-r} X_{i:n}\Big) - b_n,$$

where the order statistic $X_{i:n}$ arises from an i.i.d. sequence belonging to the domain of attraction of a stable law with index $0 < \alpha < 2$. If we use a special representation for $X_{i:n}$ related to $F^{-1}(U_{i:n})$ coming from uniformly distributed random variables $U_1, \ldots, U_n$, then we can prove the convergence in probability or even L^1-convergence for Y_n in various cases. Special attention is devoted to the convergence of Y_n to one-sided stable laws showing that we may choose $\min(k(n), r(n)) = 0$. As an example we obtain the limiting distribution of student's t type statistics.

1. Introduction. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of real valued random variables lying in the domain of attraction of a stable random variable Y with index α of stability, $0 < \alpha < 2$, i.e. there exist coefficients $a_n > 0$ and $b_n \in \boldsymbol{R}$ such that

$$a_n^{-1} \sum_{i=1}^{n} X_i - b_n \xrightarrow{D} Y \tag{1.1}$$

in distribution as $n \to \infty$. It was suggested by LePage, Woodroofe and Zinn [7] that only the extreme order statistics yield a contribution to the limit (1.1). Recently S. Csörgő, Horváth and Mason [4] have shown that for each sequence $k(n)$ such that

$$k(n) \to \infty \text{ and } k(n)/n \to 0 \text{ as } n \to \infty \tag{1.2}$$

the sum of the order statistics

$$a_n^{-1}\Big(\sum_{i=1}^{k(n)} X_{i:n} + \sum_{i=n-k(n)+1}^{n} X_{i:n}\Big) - b_n' \xrightarrow{D} Y \tag{1.3}$$

converges in distribution an $n \to \infty$. Here $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ denote the order statistics arising from $X_1, \ldots, X_n$. In addition, these authors studied the behaviour of the trimmed sums

$$a_n^{-1} = \sum_{i=k+1}^{n-k} X_{i:n} - b_n''. \tag{1.4}$$

Their methods heavily rely on a new Brownian bridge approximation. The authors remarked that the technic also applies to asymmetric trimmed sums but they decided to do not carry it out in view of the ensuing additional technicalities.

The purpose of the present paper is to present an elementary approach to the asymptotic behaviour of (1.3) and (1.4) based on well-known technics for order statistics. The method also applies to asymmetric trimmed sums and when (1.2) is violated. Note that for one-sided stable limit distributions Y the assumption $k(n) \to \infty$ is not necessary. Among various applications we give a new probabilistic proof for the sufficiency of conditions (2.1) and (2.2) below for the convergence of the normalized sums (1.1) to a stable random variable. The proof is very rapid. Only the investigation of the correct centering constants needs more effort. In addition we are able to find a probability space and random variables with the same distribution as in (1.1) such that convergence in probability or even L^1-convergence to the stable random variable Y holds.

In the sequel we will make use of some results for slowly varying functions, which can be found in [10].

2. The behaviour of the central part of the sum. Let X denote a real random variable with distribution function F. By Feller [5], p. 577, it is well-known that X belongs to the domain of attraction of a stable law with index α, $0 < \alpha < 2$, iff there exists a function L varying slowly at infinity such that

$$G(y) = P(\{|X| > y\}) = y^{-\alpha} L(y) \quad \text{as } y \uparrow \infty \tag{2.1}$$

and

$$(1 - F(y))/G(y) \to p \quad \text{and } F(-y)/G(y) \to q \quad \text{as } y \uparrow \infty \tag{2.2}$$

for some $p \in [0, 1]$, $p + q = 1$. Let $G^{-1}(s) = \inf\{t\colon G(t) \leqslant s\}$ denote the inverse of G and F^{-1} of F, respectively[1]. Subsequently let us always choose the normalizing constants a_n of (1.1) as

$$a_n = G^{-1}(1/n). \tag{2.3}$$

At this stage we recall a known representation for order statistics (see [2],

[1] More precisely: F^{-1} is the inverse distribution function.

section 13.6). Assume that $Y_1, Y_2, \ldots$ is an i.i.d. sequence of exponential distributed random variables. Set

$$(2.4) \qquad \Gamma_k = \sum_{i=1}^{k} Y_i.$$

Then it is well-known that the following random variables are equal in distribution

$$(2.5) \qquad (X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{D}}{=} (F^{-1}(\Gamma_1/\Gamma_{n+1}), \ldots, F^{-1}(\Gamma_n/\Gamma_{n+1})).$$

If we take (2.3) into account, we see that (2.2) yields

$$(2.6) \quad a_n^{-1} F^{-1}(\Gamma_k/\Gamma_{n+1}) = \frac{F^{-1}([n\Gamma_k/\Gamma_{n+1}]/n)}{G^{-1}(1/n)} \to -q^{1/\alpha}\Gamma_k^{-1/\alpha} \quad \text{as } n \to \infty,$$

which suggests that the extreme order statistics yield a major part of the stable distribution. The next lemma shows that the central part of the sum (1.1) vanishes asymptotically. Special attention is devoted to the L^1-convergence of the central part which has further applications.

(2.1) LEMMA. *Assume* (2.1) *and* (2.2). *Let* $k(n)$ *and* $r(n)$ *be integers such that* $0 \leqslant k(n) \leqslant n-r(n) \leqslant n$ *and*

$$(2.7) \qquad q/(k(n)+1) + p/(r(n)+1) \to 0 \quad \textit{as } n \to \infty.$$

(a) *There exists* $d_n \in \mathbf{R}$ *such that*

$$(2.8) \qquad a_n^{-1} \sum_{i=k(n)+1}^{n-r(n)} X_{i:n} - d_n \xrightarrow[\mathrm{P}]{} 0$$

in probability as $n \to \infty$.

(b) *If in addition*

$$(2.9) \qquad (k(n)+1) > 1/\alpha \quad \textit{and} \quad (r(n)+1) > 1/\alpha,$$

then the mean

$$(2.10) \qquad c_n = E\Big(a_n^{-1} \sum_{i=k(n)+1}^{n-r(n)} X_{i:n}\Big)$$

exists. Under (2.9) *we may choose* $d_n = c_n$ *in* (2.8) *and, in addition,* (2.8) *tends to zero in* L^1.

The proof of Lemma (2.1) (a) is elementary whereas part (b) requires further calculations. First recall the following well-known lemma for order statistics which is due to Bickel [1], Theorem (2.1):

(2.2) LEMMA. *Assume that* $E_1, \ldots, E_n$ *are i.i.d. random variables with compact support on* $\mathbf{R}$. *For each pair* $1 \leqslant j \leqslant k \leqslant n$ *the order statistics* $X_{j:n}$ *and* $X_{k:n}$ *are non-negative correlated.*

Proof. By [1] the order statistics are positive correlated whenever E_1 has a Lebesque density. A weak approximation of the distribution of E_1 by absolutely continuous distributions now yields the result.

Proof of Lemma (2.1) (a). Introduce, for $\varepsilon > 0$,

$$^{(\varepsilon)}X_i = X_i 1_{[-a_n\varepsilon, a_n\varepsilon]}(X_i) + a_n \varepsilon 1_{(a_n\varepsilon, \infty)}(X_i) - a_n\varepsilon 1_{(-\infty, a_n\varepsilon)}(X_i), \tag{2.11}$$

whee 1_A denotes the indicator function of a set A. We see that

$$\operatorname{Var}(a_n^{-1} \sum_{i=1}^{n} {}^{(\varepsilon)}X_i) \leqslant n a_n^{-2} [\int_{[-\varepsilon a_n, \varepsilon a_n]} x^2 dF(x) + \tag{2.12}$$

$$+ a_n^2 \varepsilon^2 P(\{|X_1| > a_n \varepsilon\})] = : f_n(\varepsilon).$$

An application of Lemma (2.2) for ${}^{(\varepsilon)}X_1, \ldots, {}^{(\varepsilon)}X_n$ yields

$$\operatorname{Var}(a_n^{-1} \sum_{i=k(n)+1}^{n-r(n)} {}^{(\varepsilon)}X_{i:n}) \leqslant f_n(\varepsilon). \tag{2.13}$$

From [5], p. 579, we recall that

$$\int_{[-z,z]} x^2 dF(x) \approx \alpha/(2-\alpha) z^2 G(z) \quad \text{as } z \to \infty. \tag{2.14}$$

Let now $\varepsilon_n \downarrow 0$ be any sequence such that $\varepsilon_n a_n \to \infty$. Then, from (2.14) and $G(a_n) = 1/n$, we obtain

$$f_n(\varepsilon_n) \sim [1+\alpha/(2-\alpha)] \varepsilon_n^2 G(a_n \varepsilon_n)/G(a_n) = \varepsilon_n^{2-\alpha} L(a_n\varepsilon_n)/L(a_n) \to 0 \text{ as } n \to \infty, \tag{2.15}$$

which is a well-known property of slowly varying functions. Combining (2.13) and (2.15) we see that

$$a_n^{-1} \sum_{i=k(n)+1}^{n-r(n)} ({}^{(\varepsilon_n)}X_{i:n} - E({}^{(\varepsilon_n)}X_{i:n})) \to 0 \quad \text{in } L^1. \tag{2.16}$$

From (2.6) we recall that, for fixed $k \in N$,

$$a_n^{-1} X_{k:n} \xrightarrow{\mathrm{D}} -q^{1/\alpha} \Gamma_k^{-1/\alpha} \quad \text{and} \quad a_n^{-1} X_{n+1-k:n} \xrightarrow{\mathrm{D}} p^{1/\alpha} \Gamma_k^{-1/\alpha} \tag{2.17}$$

converge in distribution as $n \to \infty$.

In view of (2.7) there exists a sequence $\varepsilon_n \to 0$ such that $a_n \varepsilon_n \to \infty$ and

$$P(A_n) \to 0 \quad \text{as } n \to \infty, \tag{2.18}$$

where $A_n = \{X_{k(n)+1:n} \leqslant -a_n \varepsilon_n\} \cup \{X_{n-r(n):n} \geqslant a_n \varepsilon_n\}$.

Since

$$\sum_{i=k(n)+1}^{n-r(n)} (X_{i:n} - {}^{(\varepsilon_n)}X_{i:n}) | A_n^c = 0 \tag{2.19}$$

restricted on the complement A_n^c of A_n, the assertions (2.16) and (2.18) yield the desired convergence (2.8) and the proof of Lemma (2.1) (a) is complete.

For the proof of Lemma (2.1) (b) it is not enough to consider (2.18) and (2.19). We need the following Lemma.

(2.3) LEMMA. *Assume* (2.1), (2.2) *and* $k > 1/\alpha$. *For each* $\varepsilon > 0$

$$E\Big(a_n^{-1} \sum_{i=k}^{n} X_{i:n} 1_{(-\infty, -\varepsilon a_n]}(X_{i:n})\Big) \to -q^{1/\alpha} \int_0^{q/\varepsilon^\alpha} y^{-1/\alpha}(1 - H_y(k-2))\, dy = h(k, \varepsilon), \quad as\ n \to \infty, \tag{2.20}$$

where H_y *denotes the distribution function of a Poisson random variable with mean* y.

Proof. Assumption (2.2) implies

$$G^{-1}(s) = s^{-1/\alpha} \tilde{L}(s), \tag{2.21}$$

where $\tilde{L}$ is a further function varying slowly at zero. The expectation (2.20) can be expressed by

$$a_n^{-1} \int F^{-1}(x) 1_{(-\infty, -\varepsilon a_n]}(F^{-1}(x)) \sum_{i=k}^{n} f_{i:n}(x)\, dx. \tag{2.22}$$

Here $f_{i:n}(x)$ denotes the density of $U_{i:n}$ arising from uniformly distributed i.i.d. random variables $U_1, \dots, U_n$, i.e.

$$f_{i:n}(x) = n \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}, \quad 0 < x < 1. \tag{2.23}$$

An application of the transformation $nx = y$ shows that (2.22) equals

$$\int F^{-1}(y/n)/G^{-1}(1/n)\, 1_{(-\infty, -\varepsilon a_n]}(F^{-1}(y/n)) \times \sum_{i=k-1}^{n-1} \binom{n-1}{i} (y/n)^i (1-(y/n))^{n-1-i}\, dy. \tag{2.24}$$

Thus by (2.2) and (2.21) the integrand of (2.24) tends for fixed $y > 0$ to

$$q^{1/\alpha} y^{-1/\alpha}(1 - H_y(k-2))\, 1_{[0, q/\varepsilon^\alpha]}(y). \tag{2.25}$$

We will prove that the dominated convergence theorem of Lebesgue can be applied. Note that, for $y > 0$,

$$\sum_{i=k-1}^{n-1} \binom{n-1}{i} (y/n)^i (1-(y/n))^{n-1-i} \leqslant y^{k-1}. \tag{2.26}$$

Let us now choose $\delta > 0$ such that

$$k - 1 - \delta - 1/\alpha > -1 \tag{2.27}$$

and $K > 0$ such that $F^{-1}(y/n) \leqslant -\varepsilon a_n$ implies $y \leqslant K$ uniformly in n.

By (2.21), (2.26) and $|F^{-1}(s)| \leqslant G^{-1}(s)$ the integrand of (2.24) is dominated by

$$(2.28)\quad G^{-1}(y/n)/G^{-1}(1/n)\,y^{k-1}\,1_{[0,K]}(y) = y^{k-1-\delta-1/\alpha}[y^{\delta}\tilde{L}(y/n)/\tilde{L}(1/n)]\,1_{[0,K]}(y).$$

Note that $[y^{\delta}\tilde{L}(y/n)/\tilde{L}(1/n)]\,1_{[0,K]}(y)$ is uniformly bounded. This is a consequence of the fact that $\tilde{L}(y/n)/\tilde{L}(1/n)\to 1$ uniformly in y on compact sets $C\subset(0,\infty)$ and $y_n^{\delta}\tilde{L}(y_n/n)/\tilde{L}(1/n)\to 0$ as $y_n\downarrow 0$ (cf. [10]). In view of (2.27) assertion (2.20) follows which proves Lemma (2.3).

Proof of Lemma (2.1) (b). Assume (2.7) and (2.9). By Lemma (2.3) there exists a sequence $\varepsilon_n\downarrow 0$ such that $a_n\varepsilon_n\to\infty$ and

$$(2.29)\quad a_n^{-1}\sum_{i=k(n)+1}^{n} X_{i:n}\,1_{(-\infty,-\varepsilon_n a_n)}(X_{i:n})\to 0,$$

$$(2.30)\quad a_n^{-1}\sum_{i=1}^{n-r(n)} X_{i:n}\,1_{(\varepsilon_n a_n,\infty)}(X_{i:n})\to 0,$$

both in L^1 as $n\to\infty$.

In accordance with (2.11) we may write

$$(2.31)\quad a_n^{-1}\sum_{i=k(n)+1}^{n-r(n)} X_{i:n} = a_n^{-1}\sum_{i=k(n)+1}^{n-r(n)} {}^{(\varepsilon_n)}X_{i:n} + a_n^{-1}\sum_{i=k(n)+1}^{n-r(n)}[X_{i:n}+\varepsilon_n a_n]\,1_{(-\infty,-\varepsilon_n a_n)}(X_{i:n}) + a_n^{-1}\sum_{i=k(n)+1}^{n-r(n)}[X_{i:n}-\varepsilon_n a_n]\,1_{(\varepsilon_n a_n,\infty)}(X_{i:n}).$$

If we center (2.31) at the mean, which obviously exists by Lemma (2.3), then we have convergence to zero in L^1 as claimed in Lemma (2.1) (b). Note that the first term of (2.31) can be treated as in (2.16), whereas the second and the third term tend in L^1 to zero by (2.29) and (2.30). Thus the proof is complete.

(2.4) Remarks. (a) If $k<1/\alpha$, then the mean of $X_{k:n}$ does not exist whenever $q\neq 0$ (use (2.21)).

(b) In view of the convergence (1.1) there exists a $b_n'\in R$ such that

$$(2.32)\quad a_n^{-1}\Big[\sum_{i=1}^{k(n)} X_{i:n}+\sum_{i=r(n)+1}^{n} X_{i:n}\Big]-b_n' \xrightarrow{D} Y,$$

provided the assumptions of Lemma (2.1) hold. We do not need the assumption $k(n)+r(n)=o(n)$ of [4]. In the special case, where Y is a one-sided stable distribution, i.e. $\min(p,q)=0$, we may choose $k(n)=0$ or $r(n)=0$, which also seems to be new.

In the next section we give a selfcontained proof of (2.32) without making use of (1.1), which also yields a stronger convergence result.

3. Convergence in probability and L^1 to stable type distributions. First we will introduce new random variables $Z_{i,n}$ which are equal to $X_{i:n}$ in distribution. Then we are able to prove convergence in probability or L^1 to Y for the new random variables.

Let $Y_1, Y_2, \ldots, \tilde{Y}_1, \tilde{Y}_2, \ldots$ be two sequences of jointly independent random variables with common exponential distribution with mean 1. As in (2.4) define

$$\tilde{\Gamma}_k = \sum_{i=1}^{k} \tilde{Y}_i \tag{3.1}$$

and for $\alpha \in (0, 2)$ set, for $k \in N$,

$$\gamma_k = \gamma_k(\alpha) = \begin{cases} 0 & \text{if } k \leqslant 1/\alpha, \\ E(\Gamma_k^{-1/\alpha}) & \text{otherwise.} \end{cases} \tag{3.2}$$

The law of iterated logarithm (LIL) shows that[2]

$$\Delta_1 := \sum_{k=1}^{\infty} (\Gamma_k^{-1/\alpha} - \gamma_k) \tag{3.3}$$

is almost surely convergent (cf. (2.4)). By definition,

$$\Delta_2 := \sum_{k=1}^{\infty} (\tilde{\Gamma}_k^{-1/\alpha} - \gamma_k) \tag{3.4}$$

is an independent copy of Δ_1. It turns out (by the arguments below or known results of the literature) that Δ_1 is a one-sided stable distribution whose Lévy spectral measure vanishes on $(-\infty, 0)$. For $\alpha > 1$ the random variable Δ_1 is centered at its mean, whereas the mean does not exist for $\alpha \leqslant 1$.

In the sequel we will introduce, more generally than in (2.5),

$$\Gamma_{k,n} = \begin{cases} \sum_{i=1}^{k} Y_i & \text{if } k \leqslant [n/2], \\ \Gamma_{[n/2],n} + \sum_{j=1}^{k-[n/2]} \tilde{Y}_{n+1-j-[n/2]} & \text{if } k > [n/2] \end{cases} \tag{3.5}$$

for $k = 1, \ldots, n$, where [ ] denotes the Gaussian bracket. As in (2.5) we see that for

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(2) Elementary computation yields

$$\sum_{k=1}^{\infty} |\gamma_k - k^{-1/\alpha}| < \infty.$$

The LIL shows $|\Gamma_k - k| = O(\sqrt{k}\sqrt{\log\log k})$, and by the mean value theorem we obtain $|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}| \leqslant K|\Gamma_k - k|\, k^{-1/\alpha - 1}$ for $k \geqslant k_0$. Thus 3.3 follows since $\alpha < 2$.

$$(3.6)\qquad Z_{i,n} := F^{-1}(\Gamma_{i,n+1}/\Gamma_{n+1,n+1})$$

equality in distribution

$$(3.7)\qquad (X_{1:n}, \ldots, X_{n:n}) \stackrel{\mathrm{D}}{=} (Z_{1,n}, \ldots, Z_{n,n})$$

holds.

(3.1) THEOREM. *Assume* (2.1)-(2.3) *and* (2.7). *Let k and r be non-negative integers.*

(a) *Then the following assertions hold:*

$$(3.8)\qquad a_n^{-1} \sum_{i=k+1}^{k(n)} (Z_{i,n} - d_{i,n}) \xrightarrow[\mathrm{P}]{} -q^{1/\alpha}\Big(\Delta_1 - \sum_{i=1}^{k} (\Gamma_i^{-1/\alpha} - \gamma_i)\Big)$$

and

$$(3.9)\qquad a_n^{-1} \sum_{i=n+1-r(n)}^{n-r} (Z_{i,n} - d_{i,n}) \xrightarrow[\mathrm{P}]{} p^{1/\alpha}\Big(\Delta_2 - \sum_{i=1}^{r} (\tilde{\Gamma}_i^{-1/\alpha} - \gamma_i)\Big)$$

in probability as $n \to \infty$.

The centering constants $d_{i,n}$ are equal to

$$(3.10)\qquad d_{i,n} = \begin{cases} E(Z_{i,n}) & \text{if } 1/\alpha < \min(i, n+1-i), \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Assume in addition that* $1/\alpha < k+1 \leqslant k(n) < n+1-1/\alpha$, *and* $1/\alpha < n+1-r(n) \leqslant n-r < n+1-1/\alpha$, *respectively, then the sequence of random variables* (3.8) *and* (3.9), *respectively, converges in L^1.*

The proof relies on the following well-known lemma showing that for sufficiently large k the expectations of the random variables of (2.6) are convergent to each other.

(3.2) LEMMA. *Under* (2.1)-(2.3) *we obtain, for $k > 1/\alpha$,*

$$(3.11)\qquad \begin{aligned} &a_n^{-1} E\big(X_{k:n} 1_{(-\infty,0)}(X_{k:n})\big) \to -q^{1/\alpha} E(\Gamma_k^{-1/\alpha}), \\ &a_n^{-1} E\big(X_{k:n} 1_{(0,\infty)}(X_{k:n})\big) \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Lemma (3.2) can be deduced from [8] and [9], section, 4.1. Note that (3.11) also follows from Lemma (2.3). The convergence of

$$(3.12)\qquad a_n^{-1} E\big(X_{k:n} 1_{(-\varepsilon a_n, 0)}(X_{k:n})\big) \to -q^{-1/\alpha} E\big(\Gamma_k^{-1/\alpha} 1_{(-\varepsilon,0)}(\Gamma_k^{-1/\alpha})\big)$$

is obvious. The assertion (2.20) yields that

$$(3.13)\qquad \limsup_{n\to\infty} a_n^{-1} E\big(X_{k:n} 1_{(-\infty,-\varepsilon a_n]}(X_{k:n})\big)$$

becomes arbitrary small for large ε. Similar arguments show that the expectation of the positive part of $X_{k:n}$ asymptotically vanishes.

Proof of Theorem (3.1). As in (2.6) we note that, for fixed i,

$$a_n^{-1} Z_{i,n} \to -q^{1/\alpha} \Gamma_i^{-1/\alpha} \tag{3.14}$$

and for $i < [n/2]$

$$a_n^{-1} Z_{n+1-i,n} = a_n^{-1} F^{-1}(1-\tilde{\Gamma}_i/\Gamma_{n+1,n+1}) \to p^{1/\alpha} \tilde{\Gamma}_i^{-1/\alpha} \tag{3.15}$$

almost surely as $n \to \infty$. Note that it is enough to prove (3.8). Assume first that the conditions of (b) are satisfied. Then it is well-known that the convergence of the L^1-norms and the almost sure convergence(3) show that (3.14) holds in L^1. Consequently,

$$a_n^{-1}(Z_{i,n}-d_{i,n}) + q^{1/\alpha}(\Gamma_i^{-1/\alpha} - \gamma_i) \to 0 \tag{3.16}$$

in L^1 as $n \to \infty$. Thus there exists a sequence $k'(n) \leqslant k(n)$ such that $q/(k'(n)+1) \to 0$ and

$$\sum_{i=k+1}^{k'(n)} [a_n^{-1}(Z_{i,n}-d_{i,n}) + q^{1/\alpha}(\Gamma_i^{-1/\alpha} - \gamma_i)] \to 0 \tag{3.17}$$

in L^1 as $n \to \infty$. On the other hand, it is easy to see by Lemma (2.1) (b) that (4)

$$a_n^{-1} \sum_{i=k+1}^{k'(n)} (Z_{i,n} - d_{i,n})$$

is a Cauchy sequence in L^1. Hence (3.17) and the almost sure convergence in (3.3) imply that

$$a_n^{-1} \sum_{i=k+1}^{k'(n)} (Z_{i,n}-d_{i,n}) \to -q^{1/\alpha}\left(\Delta_1 - \sum_{i=1}^{k} (\Gamma_i^{-1/\alpha} - \gamma_i)\right) \tag{3.18}$$

in L^1 as $n \to \infty$. If we now observe that, by Lemma (2.1) (b),

$$a_n^{-1} \sum_{i=k'(n)+1}^{k(n)} (Z_{i,n} - d_{i,n}) \to 0 \tag{3.19}$$

in L^1 as $n \to \infty$, then the desired L^1-convergence of (3.8) follows from (3.18).

In the situation of Theorem (3.1) (a) we remark that for $k \leqslant i \leqslant 1/\alpha$ assertion (3.14) remains true, which proves the convergence in probability of (3.8) in the general case. Hence the proof of Theorem (3.1) is finished.

(3.3) Discussion. Theorem (3.1) has various applications which are mentioned below. Let us keep in mind that for each index set $I \subset \{1, \ldots, n\}$ we obtain by (3.7) the equality in distribution of

$$\sum_{i \in I} X_{i:n} \stackrel{D}{=} \sum_{i \in I} Z_{i,n} \tag{3.20}$$

and recall the definition of the centering constants $d_{i,n}$ (cf. (3.10)).

(3) The assertion is known to be Vitali's theorem.

(4) Note that f_n is a Cauchy sequence in L^1 iff $f_n - f_{j(n)} \to 0$ for all sequences $j(n) \to \infty$, $j(n) \leqslant n$.

(a) If we choose $k(n) = n - r(n)$ with $\min(k(n), r(n)) \to \infty$, then

$$(3.21)\qquad a_n^{-1} \sum_{i=k+1}^{n-r} (Z_{i,n} - d_{i,n}) \to -q^{-1/\alpha} \sum_{i=k+1}^{\infty} (\Gamma_i^{-1/\alpha} - \gamma_i) + $$

$$+ p^{1/\alpha} \sum_{i=r+1}^{\infty} (\tilde{\Gamma}_i^{-1/\alpha} - \gamma_i) = : W_{k,r}$$

in probability as $n \to \infty$ and in L^1 whenever

$$(3.22)\qquad 1/\alpha < k+1 \quad \text{and} \quad 1/\alpha < r+1.$$

Note that $W_{0,0}$ is a stable distribution. Thus we have convergence to $W_{0,0}$ in L^1 for $\alpha > 1$.

(b) Assume that $k(n)$ and $r(n)$ satisfy (2.7). Then we conclude that

$$(3.23)\qquad a_n^{-1} \Big[\sum_{i=k+1}^{k(n)} (Z_{i,n} - d_{i,n}) + \sum_{i=n+1-r(n)}^{n-r} (Z_{i,n} - d_{i,n}) \Big] \to W_{k,r}$$

in probability as $n \to \infty$ and in L^1 if (3.22) holds. Note that for $p = 0$ the random variable $W_{0,0}$ is one-sided stable and we may choose $r(n) = 0$. In this case we can substitute $\Gamma_{k,n+1}$ in the definition of $Z_{k,n}$ by Γ_k (2.4).

(c) If we are not interested in the explicit form of the centering constants b_n in (1.1), then the proof above becomes quite simple and we obtain a short probabilistic proof for the sufficiency of conditions (2.1) and (2.2) for the convergence to a stable random variable (1.1). Let us sketch the proof. From (3.14) and (3.15) we conclude that there are $k'(n)$ and $r'(n)$ such that $\min(k'(n), r'(n)) \to \infty$ and

$$(3.24)\qquad \sum_{i=1}^{k'(n)} (a_n^{-1} Z_{i,n} + q^{1/\alpha} \Gamma_i^{-1/\alpha}) + \sum_{i=n+1-r'(n)}^{n} (a_n^{-1} Z_{i,n} - p^{1/\alpha} \tilde{\Gamma}_i^{-1/\alpha}) \underset{P}{\to} 0$$

in probability. If we now use Lemma (2.1) (a), and the almost sure convergence of (3.3), then there exists a b_n such that (1.1) holds.

Let $\operatorname{sign}(x)$ denote the sign of a real x.

Under the assumptions of Theorem (3.1) we obtain the subsequent result.

(3.4) Corollary. *Assume $\alpha < \beta$. (a) We have*

$$(3.25)\qquad a_n^{-\beta} \sum_{i=k+1}^{k(n)} \operatorname{sign}(Z_{i,n}) |Z_{i,n}|^{\beta} \underset{P}{\to} -q^{\beta/\alpha} \sum_{i=k+1}^{\infty} \Gamma_i^{-\beta/\alpha}$$

and

$$(3.26)\qquad a_n^{-\beta} \sum_{i=n+1-r(n)}^{n-r} \operatorname{sign}(Z_{i,n}) |Z_{i,n}|^{\beta} \underset{P}{\to} p^{\beta/\alpha} \sum_{i=k+1}^{\infty} \tilde{\Gamma}_i^{-\beta/\alpha}$$

in probability as $n \to \infty$.

(b) *Assume in addition* $\beta/\alpha < k+1 \leqslant k(n) \leqslant n+1-\beta/\alpha$, *and* $\beta/\alpha < n+1-r(n) \leqslant n-r < n+1-\beta/\alpha$, *respectively. Then the sequence of random variables* (3.25) *and* (3.26), *respectively, converges in* L^1.

Proof. Define

$$\tilde{X}_i = \operatorname{sign}(X_i)|X_i|^\beta \tag{3.27}$$

having the inverse distribution function

$$\tilde{F}^{-1}(s) = \operatorname{sign}(F^{-1}(s))|F^{-1}(s)|^\beta. \tag{3.28}$$

Thus Lemma (2.1) applies to $\operatorname{sign}(Z_{i,n})|Z_{i,n}|^\beta$. Assume first that the conditions of (b) are satisfied. As in (3.17) we obtain a sequence $k'(n) = \min(k(n), q(n))$, where $q(n) \to \infty$ as $n \to \infty$ and

$$\sum_{i=k+1}^{k'(n)} (a_n^{-\beta} \operatorname{sign}(Z_{i,n})|Z_{i,n}|^\beta + q^{\beta/\alpha} \Gamma_i^{-\beta/\alpha}) \to 0. \tag{3.29}$$

On the other hand, Lemma (2.1) yields

$$a_n^{-\beta} \sum_{i=k'(n)+1}^{k(n)} (\operatorname{sign}(Z_{i,n})|Z_{i,n}|^\beta - E(\operatorname{sign}(Z_{i,n})|Z_{i,n}|^\beta)) \to 0 \quad \text{in } L^1. \tag{3.30}$$

From [6], section 5, we recall that, for $\beta/\alpha + \varepsilon < 1$, $\varepsilon > 0$ and $i > \beta/\alpha$,

$$a_n^{-\beta} E(|Z_{i,n}|^\beta 1_{(-\infty,0)}(Z_{i,n})) \leqslant C i^{-\beta/\alpha+\varepsilon} \tag{3.31}$$

uniformly in n. Applying a similar formula for $|Z_{n+1-i,n}|^\beta 1_{(0,\infty)}(Z_{n+1-i,n})$, we see that, for $0 < q < 1$,

$$a_n^{-\beta} \sum_{i=k'(n)+1}^{k(n)} E(|Z_{i,n}|^\beta) \to 0 \quad \text{as } n \to \infty. \tag{3.32}$$

A monotonicity argument shows the same result for $q = 0$ or $q = 1$. Combining (3.29)–(3.32), we obtain as in the proof of Theorem (3.1) the result, which completes the proof of Corollary (3.4).

4. Asymptotic independence. As an application of Theorem (3.1) we will establish an asymptotic independence result for sums of truncated random variables. Observe that under the assumptions of Theorem (3.1) (a) the random variables

$$a_n^{-1} \sum_{i=1}^{k(n)} (X_{i:n} - d_{i,n}) \tag{4.1}$$

and

$$a_n^{-1} \sum_{i=n+1-r(n)}^{n} (X_{i:n} - d_{i,n}) \tag{4.2}$$

are asymptotically independent with a joint distribution

$$(4.3)\qquad (-q^{1/\alpha}\Delta_1,\, p^{1/\alpha}\Delta_2).$$

It turns out that instead of the random variables (4.1) and (4.2) the investigation of truncated random variables yields the same result. We generalize results of [11], [4] and [3], which proved, for certain centering constants $C_n^{(j)}$ $(j = 1, 2)$, the asymptotic independence of

$$(4.4)\qquad a_n^{-1}\Big(\sum_{i=1}^{n} X_i^- - C_n^{(1)}\Big),$$

and

$$(4.5)\qquad a_n^{-1}\Big(\sum_{i=1}^{n} X_i^+ - C_n^{(2)}\Big)$$

with the same joint distribution as in (4.3). Here we write $X_i^+ = \max(X_i, 0)$ and $X_i^- = \min(X_i, 0)$. Theorem (4.1) is an immediate consequence of Theorem (3.1).

(4.1) Theorem. *Assume that conditions* (2.1)–(2.3) *are satisfied. Let* $\varepsilon_n \to 0$ *and* $\varepsilon_n' \to 0$. *We introduce the centering constants*

$$(4.6)\qquad c_{i,n}^{(1)} = \begin{cases} E(X_{i:n} 1_{(-\infty,\varepsilon_n' a_n)}(X_{i:n})) & \text{if } 1/\alpha < i < n+1-1/\alpha,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$(4.7)\qquad c_{i,n}^{(2)} = \begin{cases} E(X_{i:n} 1_{(\varepsilon_n a_n,\infty)}(X_{i:n})) & \text{if } 1/\alpha < i < n+1-1/\alpha,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$(4.8)\qquad \Big(a_n^{-1}\sum_{i=1}^{n} [X_i(1_{(-\infty,\varepsilon_n' a_n)}(X_i) - c_{i,n}^{(1)}],\ a_n^{-1}\sum_{i=1}^{n} [X_i 1_{(\varepsilon_n a_n,\infty)}(X_i) - c_{i,n}^{(2)}]\Big)$$

converges in distribution to

$$(4.9)\qquad (-q^{1/\alpha}\Delta_1,\, p^{1/\alpha}\Delta_2) \quad \textit{as } n \to \infty.$$

Proof. Without restrictions we may assume that $\varepsilon_n \geq 0$ and $\varepsilon_n' \leq 0$ since the variance of the central portion vanishes, i.e.

$$(4.10)\quad \operatorname{Var}\Big(a_n^{-1}\sum_{i=1}^{n} X_i 1_{[-|\varepsilon_n|a_n,\,|\varepsilon_n|a_n]}(X_i)\Big) \leq n a_n^{-2}\Big[\int_{[-|\varepsilon_n|a_n,\,|\varepsilon_n|a_n]} x^2\, dF(x)\Big] \to 0$$

(cf. (2.14) and (2.15)).

Note that (4.10) also holds whenever $a_n|\varepsilon_n|$ remains bounded since $n a_n^{-2} \to 0$. Recall from section 3 that

$$(4.11)\qquad \sum_{i=1}^{n} X_i 1_{(\varepsilon_n a_n,\infty)}(X_i) \stackrel{\mathrm{D}}{=} \sum_{i=1}^{n} Z_{i,n} 1_{(\varepsilon_n a_n,\infty)}(Z_{i,n}).$$

Turning to the random variables $Z_{i,n}$, it suffices to prove the convergence in probability for each component of (4.8). Assume first that $\varepsilon_n = \varepsilon'_n = 0$ for each n. In this case we write

$$c_{i,n}^{(1)-} \quad \text{and} \quad c_{i,n}^{(2)+} \tag{4.12}$$

for the centering coefficients (4.6) and (4.7). By Discussion (3.3) (b) we obtain

$$a_n^{-1} \sum_{i=1}^{n} [Z_{i,n}^{+} + Z_{i,n}^{-} - c_{i,n}^{(1)-} - c_{i,n}^{(2)+}] \underset{P}{\to} -q^{1/\alpha} \Delta_1 + p^{1/\alpha} \Delta_2 \tag{4.13}$$

in probability as $n \to \infty$. Assume $p \neq 0$. Then X_i^+ lies in the domain of attraction of the one-sided stable law $p^{1/\alpha} \Delta_2$ and, similarly, we obtain

$$a_n^{-1} \sum_{i=1}^{n} [Z_{i,n}^{+} - c_{i,n}^{(2)+}] \underset{P}{\to} p^{1/\alpha} \Delta_2. \tag{4.14}$$

Combining (4.13) and (4.14) we conclude

$$a_n^{-1} \sum_{i=1}^{n} [Z_{i,n} - c_{i,n}^{(1)-}] \underset{P}{\to} -q^{1/\alpha} \Delta_1. \tag{4.15}$$

Note that for $p = 0$ assertion (4.15) follows from Discussion (3.3) (b) which then yields (4.14). Assume now that $\varepsilon_n \geqslant 0$ with $\varepsilon_n \to 0$. Then

$$\begin{aligned} (4.16) \quad & a_n^{-1} \sum_{i=1}^{n} [Z_{i,n} 1_{(\varepsilon_n a_n, \infty)}(Z_{i,n}) - c_{i,n}^{(2)} - Z_{i,n}^{+} + c_{i,n}^{(2)+}] \\ & = a_n^{-1} [\sum_{i=1}^{n} (Z_{i,n} 1_{(0, \varepsilon_n a_n]}(Z_{i,n}) - E(Z_{i,n} 1_{(0, \varepsilon_n a_n]}(Z_{i,n}))) + \\ & \qquad + \sum_{n+1-1/\alpha \leqslant i < n} E(Z_{i,n} 1_{(0, \varepsilon_n a_n]}(Z_{i,n}))]. \end{aligned}$$

By the same arguments as in (4.10) we see that (4.16) converges to zero in probability as $n \to \infty$. Hence the proof of Theorem (4.1) is complete.

Finally we will prove an asymptotic result for self-norming sums ([5]) or t-statistics (see [7] and references therein) of the type

$$(\sum_{j=1}^{n} X_j)/(\sum_{j=1}^{n} |X_j|^r)^{1/r}. \tag{4.17}$$

In [7] it is proved that (4.17) converges in distribution if X_i belongs to the domain of attraction of a stable law with index $\alpha < \min(2, r)$, where X_i is assumed to be symmetric or $\alpha < 1$.

Here is an example yielding convergence in probability for a more general class of distributions.

([5]) For a recent discussion of selt-normalized sums cf. M. Csörgő and L. Horváth, *Asymptotic representation of self-normalized sums*, Prob. Math. Statistics 9 (1988), p. 15–27 [added in proof].

(4.2) Example. Assume (2.1)–(2.3) and let $Z_{i,n}$ and $d_{i,n}$ be as in Theorem (3.1). For $\alpha < r$

$$(4.18)\qquad \Big[\sum_{i=1}^{n}(Z_{i,n}-d_{i,n})\Big]\Big/\Big(\sum_{i=1}^{n}|Z_{i,n}|^r\Big)^{1/r} \underset{P}{\rightarrow} W_{0,0}\Big/\Big(\sum_{j=1}^{\infty}(q^{r/\alpha}\Gamma_j^{-r/\alpha}+p^{r/\alpha}\tilde{\Gamma}_j^{-r/\alpha})\Big)^{1/r}$$

in probability as $n\to\infty$. Here $W_{0,0}$ denotes the stable law defined in (3.21).

Note that the convergence in probability of the denominator of (4.18) can be seen as follows. Write

$$(4.19)\qquad a_n^{-r}\sum_{i=1}^{n}|Z_{i,n}|^r = a_n^{-r}\sum_{i=1}^{n}\big[Z_{i,n}^{+r}+|Z_{i,n}^{-}|^r\big].$$

Then we can apply (3.26) to $Z_{i,n}^{+}$ instead of $Z_{i,n}$ showing that we can define $r(n) = n$ and $r = 0$. Thus, by Corollary (3.4),

$$(4.20)\qquad a_n^{-\beta}\sum_{i=1}^{n}(Z_{i,n}^{+})^r \underset{P}{\rightarrow} p^{\beta/\alpha}\sum_{i=1}^{\infty}\tilde{\Gamma}_i^{-\beta/\alpha}.$$

In the case $\alpha < 1$ the centering constants can be cancelled out. Note that then $W_{0,0}$ must be substituted by $W_{0,0}+d$, where d is a suitable shift.

5. Absolutely trimmed sums. In this section we will treat another trimmed partial sum where the k largest absolute values are neglected. For references concerning this problem cf. [4] and references therein. In the case $k = 0$ we obtain a representation for stable random variables of the type introduced by LePage, Woodroofe and Zinn [7]. It turns out that for the special random variables $Z_{i,n}$ the trimmed sums are convergent in probability. Subsequently we study the trimmed sums

$$(5.1)\qquad \sum_{i=1}^{n} Z_{i,n} - \sum_{i=1}^{k}\delta_{i,n}V_{i,n},$$

where $V_{i,n}$ denotes the order statistic with the index $n+1-i$ of the absolute values $|Z_{i,n}|$, $i = 1, \ldots, n$. For each $n\in N$ let $\sigma_n = (\sigma_{1n}, \ldots, \sigma_{nn})$ be a random permutation such that

$$(5.2)\qquad V_{i,n} = |Z_{\sigma_{in},n}|.$$

Then define

$$(5.3)\qquad \delta_{i,n} = \operatorname{sign}(Z_{\sigma_{in},n}).$$

Note that there may exist different δ_n satisfying (5.2). This leads to different random variables (5.1) and (5.3). In all cases we obtain the same asymptotic result. Next we similarly treat the limiting model. Consider the sequence

$$(5.4)\qquad (-q^{1/\alpha}\Gamma_1^{-1/\alpha},\ p^{1/\alpha}\tilde{\Gamma}_1^{-1/\alpha},\ -q^{1/\alpha}\Gamma_2^{-1/\alpha},\ p^{1/\alpha}\tilde{\Gamma}_2^{1/\alpha},\ \ldots).$$

Let V_i denote the i-th largest absolute value of the vector (5.4) and let δ_i, similarly as above, be the sign of the component of (5.4) which contributes to V_i. Note that in this case the random variables δ_i are uniquely defined with probability 1.

Before we can prove our main result we need some preparations. Let ε_x denote the one-point measure at x.

(5.1) LEMMA. (a) *$(\delta_i)_{i\in N}$ is an i.i.d. sequence with common distribution $p\varepsilon_1+q\varepsilon_{-1}$.*

(b) *The infinite vectors $(\delta_i)_{i\in N}$ and $(V_i)_{i\in N}$ are stochastically independent of each other.*

(c) *V_i and Γ_i are equal in distribution for each $i \geqslant 1$.*

(d) *The random variable*

$$\Gamma = \sum_{i=1}^{\infty} (\delta_i V_i - (p-q)\gamma_i) \tag{5.5}$$

is almost surely convergent.

(e) *There exists a constant γ such that*

$$-q^{1/\alpha}\Delta_1 + p^{1/\alpha}\Delta_2 = \Gamma + \gamma \tag{5.6}$$

almost surely.

Proof. (a) For fixed $i \in N$ and $n \geqslant 2i$ consider

$$a_n^{-1}(Z_{1,n}, \ldots, Z_{i,n}, Z_{n,n}, \ldots, Z_{n+1-i,n}) \tag{5.7}$$

which converges by (3.14) and (3.15) almost surely to

$$(-q^{1/\alpha}\Gamma_1^{-1/\alpha}, \ldots, -q^{1/\alpha}\Gamma_i^{-1/\alpha}, p^{1/\alpha}\tilde{\Gamma}_1^{-1/\alpha}, \ldots, p^{1/\alpha}\tilde{\Gamma}_i^{-1/\alpha}). \tag{5.8}$$

Thus we see that

$$V_{i,n} \to V_i \quad \text{and} \quad \delta_{i,n} \to \delta_i \tag{5.9}$$

almost surely as $n \to \infty$. Thus assertions (a) and (b) immediately follow from the asymptotic independence result of [7], Lemma 1. Note that also (c) is a consequence of (5.9).

In order to prove (d) and (e) note that

$$\gamma = \lim_{n\to\infty} \Big[(p-q)\sum_{i=1}^{n} \gamma_i + q^{1/\alpha}\sum_{i=1}^{[nq]} \gamma_i - p^{1/\alpha}\sum_{i=1}^{[np]} \gamma_i\Big] \tag{5.10}$$

exists. We will only sketch the proof of the existence of γ. It is well-known that for $\alpha < 1$

$$\sum_{i=1}^{\infty} \gamma_i < \infty. \tag{5.11}$$

Thus we may restrict ourselves to the case $\alpha \geqslant 1$. Then it is well-known that

$$j^{-1/\alpha} \leqslant \gamma_j \leqslant (j-1)^{-1/\alpha}. \tag{5.12}$$

We will show that γ is the limit of a Cauchy sequence (5.10). Consider $n < m$. By (5.12) we obtain

$$p \sum_{i=n+1}^{m} \gamma_i - p^{1/\alpha} \sum_{i=[np]+1}^{[mp]} \gamma_i \leqslant p\Big[\int_{n-1}^{m-1} x^{-1/\alpha}dx - \int_{([np]+1)/p}^{([mp]+1)/p} x^{-1/\alpha}dx\Big], \tag{5.13}$$

which becomes arbitrary small if $n, m \geqslant n_0$ are large enough. The rest of the proof of (5.10) follows the same line. Without restrictions assume that $0 < p < 1$. Define

$$M(n) = \sum_{i=1}^{n} 1_{\{1\}}(\delta_i) \quad \text{and} \quad N(n) = n - M(n). \tag{5.14}$$

Thus $M(n)$ is binomial distributed at sample size n with parameter p. Note that

$$\sum_{i=1}^{n} \delta_i V_i = -q^{1/\alpha} \sum_{i=1}^{N(n)} \Gamma_i^{-1/\alpha} + p^{1/\alpha} \sum_{i=1}^{M(n)} \tilde{\Gamma}_i^{-1/\alpha}. \tag{5.15}$$

Since

$$\sum_{i=1}^{N(n)} (\Gamma_i^{-1/\alpha} - \gamma_i) \to \Delta_1 \quad \text{and} \quad \sum_{i=1}^{M(n)} (\tilde{\Gamma}_i^{-1/\alpha} - \gamma_i) \to \Delta_2 \tag{5.16}$$

almost surely as $n \to \infty$, it suffices to prove that

$$q^{1/\alpha} \sum_{i=1}^{N(n)} \gamma_i - p^{1/\alpha} \sum_{i=1}^{N(n)} \gamma_i - (p-q) \sum_{i=1}^{n} \gamma_i \to \gamma \tag{5.17}$$

almost surely. Thus by (5.10) it remains to prove the almost sure convergence of

$$\sum_{i=1}^{[np]} \gamma_i - \sum_{i=1}^{M(n)} \gamma_i \to 0. \tag{5.18}$$

This assertion will be proved by a standard argument applying LIL to $M(n)$. Note that (5.18) is bounded above by

$$([np] - \min([np], M(n)))\,(M(n)-1)^{-1/\alpha} \to 0 \tag{5.19}$$

almost surely as $n \to \infty$ since

$$[np] - M(n) = O((n \log\log n)^{1/2}). \tag{5.20}$$

The other inequalities are treated similarly and Lemma (5.1) is proved.

(5.2) THEOREM. *Assume* (2.1)–(2.3) *and* (2.7) *and let* $k \geqslant 0$ *be an integer. For the random variables* $Z_{i,n}$ *and the centering constants* $d_{i,n}$, (3.6) *and* (3.10) *we have*

$$(5.21)\qquad a_n^{-1}\Big[\sum_{i=1}^{k(n)}(Z_{i,n}-d_{i,n})+\sum_{i=n+1-r(n)}^{n}(Z_{i,n}-d_{i,n})-\sum_{i=1}^{k}\delta_{i,n}V_{i,n}\Big]\underset{\mathrm{P}}{\to}\Gamma-\sum_{i=1}^{k}\delta_i V_i+\gamma$$

in probability as $n\to\infty$.

Proof. Assume first that $k(n)=n-r(n)$. Then

$$(5.22)\qquad a_n^{-1}\sum_{i=1}^{n}(Z_{i,n}-d_{i,n})\underset{\mathrm{P}}{\to}-q^{1/\alpha}\Delta_1+p^{1/\alpha}\Delta_2$$

in probability. By (5.9)

$$(5.23)\qquad \sum_{i=1}^{k}\delta_{i,n}V_{i,n}\underset{\mathrm{P}}{\to}\sum_{i=1}^{k}\delta_i V_i$$

in probability as $n\to\infty$. Combining (5.22) and (5.23) we obtain the result from Lemma (5.1) (e). If $k(n)<n-r(n)$, then we may apply Lemma (2.1) showing that the central part tends in probability to zero. Thus the proof of Theorem (5.2) is complete.

Concluding remarks. Assume above that $k=0$ and $k(n)=n-r(n)$. Then, by Theorem (5.2),

$$(5.24)\qquad a_n^{-1}\sum_{i=1}^{n}(\delta_{i,n}V_{i,n}-d_{i,n})=\sum_{i=1}^{n}(Z_{i,n}-d_{i,n})\underset{\mathrm{P}}{\to}\Gamma+\gamma$$

in probability. Note that

$$(5.25)\qquad \sum_{i=1}^{n}\delta_{i,n}V_{i,n}\overset{\mathrm{D}}{=}\sum_{i=1}^{n}X_i.$$

The random variable Γ is up to the centering constants of the same type as the random variable S^* in [7], Theorem 1.

For $\alpha>1$ the results of section 3 show that (5.24) converges in L^1.

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