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VERTICES OF GIVEN DEGREE IN A RANDOM GRAPH

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Abstract. The asymptotic distributions of the number of vertices of given degree in random graph $K_{n,p}$ are given. By using the method of Poisson convergence, Poisson and normal distributions are obtained.

In recent years many papers were devoted to the problem of degree of vertices in random graphs, mainly $K_{n,p}$. For a review we refer the reader to [4]. The aim of this note is to use the method of Poisson convergence (Barbour [1]) for simplification and generalization of the above-mentioned results.

Let $K_{n,p}$ denote a random graph on the set of n labelled vertices in which each of $\binom{n}{2}$ possible edges occurs with the same probability p $(0 < p < 1)$ independently of all other edges, $q = 1-p$. Let

$$X_{rn}^{(i)} = \begin{cases} 1 & \text{if the } i\text{-th vertex has degree } r, \\ 0 & \text{otherwise,} \end{cases}$$

$$X_{rn} = X_{rn}^{(1)} + \ldots + X_{rn}^{(n)}.$$

Then

$$EX_{rn}^{(i)} = \binom{n-1}{r} p^r q^{n-r-1}$$

and

$$\alpha_r(n) = EX_{rn} = n\binom{n-1}{r} p^r q^{n-r-1}.$$

Let

$$Y_{rn}^{(i)} = \begin{cases} 1 & \text{if the } i\text{-th vertex has degree } r \text{ and is joined} \\ & \text{with the } n\text{-th one,} \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{rn} = Y_{rn}^{(1)} + \ldots + Y_{rn}^{(n-1)}.$$

Then

$$EY_{rn}^{(i)} = \binom{n-2}{r-1} p^r q^{n-r-1} \quad \text{and} \quad EY_{rn} = (n-1)\binom{n-2}{r-1} p^r q^{n-r-1}.$$

By $K_{n,p}^{(s)}$ we denote a graph arisen from $K_{n-1,p}$ in the following manner. We add a new (say n-th) vertex and join it with exactly s other vertices by exactly s edges. In such a graph $K_{n,p}^{(s)}$ we define a random variable

$$Z_{rsn}^{(i)} = \begin{cases} 1 & \text{if the } i\text{-th vertex has degree } r \text{ and is joined with} \\ & \text{the } n\text{-th one,} \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{rsn} = Z_{rsn}^{(1)} + \ldots + Z_{rsn}^{(n-1)}.$$

Then

$$EZ_{rsn}^{(i)} = \frac{s}{n-1}\binom{n-2}{r-1} p^r q^{n-r-1}, \quad EZ_{rrn} = r\binom{n-2}{r-1} p^{r-1} q^{n-r-1}$$

and

$$EZ_{r+1,rn} = r\binom{n-2}{r} p^r q^{n-r-2}.$$

Let $Z^+ = \{0, 1, \ldots\}$ and

$$\mathrm{Po}(\lambda, A) = e^{-\lambda} \sum_{j \in A} \frac{\lambda^j}{j!} \quad \text{for } \lambda > 0.$$

If $Z_m = \{0, 1, \ldots, m\}$ and φ is such a function that

$$\lambda\varphi(m+1) = \frac{\mathrm{Po}(\lambda, A \cap Z_m) - \mathrm{Po}(\lambda, A)\,\mathrm{Po}(\lambda, Z_m)}{\mathrm{Po}(\lambda, \{m\})}$$

and $\varphi(0) = 0$, then (see [2])

$$\mathrm{Prob}(X \in A) - \mathrm{Po}(\lambda, A) = E\{\lambda\varphi(X+1) - X\varphi(X)\}$$

and

$$\Delta\varphi \equiv \sup_{m \in Z^+} |\varphi(m+1) - \varphi(m)| \leqslant \min\{1, \lambda^{-1}\}.$$

THEOREM. *For $0 < p < 1$ we have the following estimation*:

$$\sup_{A \subset Z^+} \left|\mathrm{Prob}(X_{rn} \in A) - \mathrm{Po}(\alpha_r(n), A)\right|$$
$$\leqslant \binom{n-2}{r} p^r q^{n-r-1} \left\{ \frac{(r+1)n - 2r - 1}{n-r-1} pq + \frac{r}{n-r-1} q + (n-1)p^2 \right\}.$$

Proof. From the obvious formula

$$X_{rn}^{(i)} = X_{rn} - X_{r,n-1} + Y_{r+1,n} - Y_{rn} \geqslant 0$$

we obtain

$$E(X_{rn}-X_{r,n-1}+Y_{r+1,n}-Y_{rn}) = EX_{rn}^{(i)} = \binom{n-1}{r} p^r q^{n-r-1}.$$

From the fact that

$$\mathrm{Prob}(X_{rn}=k \mid X_{rn}^{(n)}=1) = \mathrm{Prob}(X_{r,n-1}+Z_{rrn}-Z_{r,r+1,n}+1)$$

we have

$$\begin{aligned} E\{X_{rn}^{(i)}\varphi(X_{rn})\} &= \mathrm{Prob}(X_{rn}^{(i)}=1)\, E\{\varphi(X_{rn}) \mid X_{rn}^{(i)}=1\} \\ &= \alpha_r^{(n)}(n)\, E\{\varphi(X_{r,n-1}+1+Z_{rrn}-Z_{r,r+1,n})\}. \end{aligned}$$

Hence the following estimation holds:

$$\begin{aligned} &|E\{\alpha_r(n)\,\varphi(X_{rn}+1)-X_{r,n}\varphi(X_{rn})\}| \\ &\qquad = \alpha_r(n)\,|E\{\varphi(X_{rn}+1)-\varphi(X_{r,n-1}+Z_{rrn}-Z_{r,r+1,n}+1)\}| \\ &\qquad \leqslant \alpha_r(n)(\Delta\varphi)\, E|X_{rn}-X_{r,n-1}+Z_{rrn}-Z_{r,r+1,n}| \\ &\qquad \leqslant EX_{rn}^{(n)}+EZ_{r,r+1,n}+EZ_{rrn}+EY_{r+1,n}+EY_{rn}, \end{aligned}$$

which gives the conclusion. ∎

Assuming $r = \text{const}$ we immediately obtain the following result:

COROLLARY. *If $np = \omega(n)$, where $\omega(n)\to\infty$ but $\omega(n)/n^\alpha = o(1)$ for every α or $\omega(n) = o(1)$ and $r \geqslant 1$, then*

$$\sup_{A\subset \mathbf{Z}^+} \left|\mathrm{Prob}(X_{rn}\in A)-\mathrm{Po}(\alpha_r(n), A)\right| = o(1).$$

Proof. The theorem gives the following estimation:

$$\begin{aligned} &\sup_{A\subset \mathbf{Z}^+} \left|\mathrm{Prob}(X_{rn}\in A)-\mathrm{Po}(\alpha_r(n), A)\right| \\ &\qquad \leqslant \left(1-\frac{\omega(n)}{n}\right)^n \frac{\omega^{r-1}(n)}{(r-1)!}\left\{\frac{\omega(n)}{r}+r+\frac{r+1}{r}\omega(n)+\frac{\omega^2(n)}{r}+\omega(n)\right\} \\ &\qquad \leqslant \begin{cases} A\omega^{r-1}(n) & \text{if } \omega(n)=o(1), \\ B\omega^{r+1}(n)\left(1-\dfrac{\omega(n)}{n}\right)^n & \text{if } \omega(n)\to\infty, \end{cases} \end{aligned}$$

where A and B are some constants. Then we obtain the conclusion. ∎

From the above estimation many particular results can be obtained. For example, if

$$\omega(n) = \log n + r\log\log n + x + o(1),$$

then $\alpha_r(n)\to e^{-x}/r!$ and from the Corollary we infer that X_{rn} asymptotically has the Poisson distribution with mean $e^{-x}/r!$ (see [3]). Palka [5] showed that if

$$\omega(n) = \log n - \beta \log\log n + o(\log\log n) \quad \text{for } r = 0$$

and if

$$\log n - \beta \log\log n + o(\log\log n) \leqslant \omega(n) \leqslant \log n + (1-\gamma)\, r \log\log n + o(\log\log n) \quad \text{for } r \geqslant 1,$$

where $0 < \beta < \infty$ and $0 < \gamma < 1$, then $\big(X_{rn} - \alpha_r(n)\big)/\big(\alpha_r(n)\big)^{1/2}$ asymptotically has the $N(0, 1)$ distribution. It is easy to see that if $\alpha_r(n) \to \infty$, then the result follows from our Corollary.

A similar result can be obtained in the earlier moments of the evolution, e.g., when $\omega(n) \to 0$. In particular, if $\omega(n) = n^{\varepsilon - 1/r}$, where $r \geqslant 1$ is a constant and $0 < \varepsilon < 1/\big(r(r+1)\big)$, then for all $1 \leqslant s \leqslant r$ we have $\alpha_s(n) \to \infty$ and, consequently, $\big(X_{sn} - \alpha_s(n)\big)/\big(\alpha_s(n)\big)^{1/2}$ has approximately the $N(0, 1)$ distribution. This fact gives an answer for the question 1 from [4].

This method may be generalized for a more general case, where probabilities of occurrence for different edges are not obviously the same. Such a general case and a more detailed discussion for particular cases, e.g., for $K_{n,p}$, $K_{m,n,p}$ and, generally, k-partities random graphs and some other regular graphs, will be a subject of another paper.

Remark. After this work was completed, I have received the proof of the paper by M. Karoński and A. Ruciński, *Poisson convergence and semi-induced properties of random graphs*, Math. Proc. Cambridge Philos. Soc. 101 (1987), pp. 291–300. Theorem 3 from that paper gives our Corollary.

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