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EXPONENTIAL ORLICZ SPACES AND INDEPENDENT RANDOM VARIABLES

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Abstract. In this paper some inequalities for sums of independent random variables belonging to exponential Orlicz spaces are obtained.

0. Introduction. Let $(\Omega, \mathscr{A}, P)$ be a non-atomic probability space and $p > 1$. The exponential Orlicz space $L_{(p)}(\Omega)$ consists of all random variables X defined on $(\Omega, \mathscr{A}, P)$ such that $\mathrm{E}\exp|\lambda^{-1}X|^p < \infty$ for some $\lambda > 0$. The norm is defined by the formula (see [4])

$$\|X\|_{(p)} = \inf\{\lambda > 0\colon \mathrm{E}\exp|\lambda^{-1}X|^p \leqslant 2\}.$$

Probability problems connected with the exponential Orlicz spaces were considered by many authors (see, e.g., [1], [2], [7]).

The following result is well known. For the proof see, e.g., [6].

PROPOSITION 1. *The following conditions are equivalent*:

(1) $\mathrm{E}\exp(tX) \leqslant \exp(B_1|t|^{p'})$ $(|t| \geqslant A_1)$ *for some* $A_1, B_1 > 0$, *where* $p' = p/(p-1)$;

(2) $\mathrm{E}\exp(A_2|X|^p) \leqslant B_2$ *for certain* $A_2, B_2 > 0$;

(3) $P[|X| \geqslant x] \leqslant A_3\exp(B_3x^p)$ *for some* $A_3, B_3 > 0$ *and all* $x > 0$.

Moreover, in each implication $(i) \Rightarrow (j)$ *the constants* A_j, B_j *depend only on* A_i, B_i.

1. Results.

THEOREM 1. *There exists a constant* $A = A(p)$ *such that for each set of independent random variables* $\{X_k\}_{k=1}^n \subset L_{(p)}(\Omega)$, $\mathrm{E}X_k = 0$,

$$\Big\|\sum_{k=1}^n X_k\Big\|_{(p)} \leqslant A\Big(\sum_{k=1}^n \|X_k\|_{(p)}^{p'}\Big)^{1/p'} \quad (p \geqslant 2), \tag{1}$$

$$\Big\|\sum_{k=1}^n X_k\Big\|_{(p)} \leqslant A\Big[\Big(\sum_{k=1}^n \|X_k\|^{p'}\Big)^{1/p'} + \Big(\sum_{k=1}^n \mathrm{E}X_k^2\Big)^{1/2}\Big] \quad (1 < p < 2). \tag{2}$$

This result is an analogue of the well-known inequalities of von Bahr and Esseen [2] and Rosenthal [8].

We denote by H_p the expressions of the right-hand side of (1) and (2). Using Proposition 1 we conclude that the inequalities (1) and (2) are equivalent for the estimate

$$P[|\sum_{k=1}^{n} X_k|/H_p \geqslant x] \leqslant B\exp(-Cx^p), \tag{3}$$

where B, $C > 0$ depend only on p.

Let $\{Y_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed symmetric random variables such that

$$P[|Y_k| \geqslant x] = \exp(-x^p) \tag{4}$$

for all $x > 0$. We write, as usual, for $a = \{a_k\}_{k=1}^{n}$

$$\|a\|_r = (\sum_{k=1}^{n} |a_k|^r)^{1/r}.$$

Let $r(p) = p'$ if $p \geqslant 2$ and $r(p) = 2$ if $1 < p < 2$.

THEOREM 2. *There exist positive constants $C_1(p)$ and $C_2(p)$ such that for each real vector $a = \{a_k\}_{k=1}^{n}$*

$$C_1(p)\|a\|_{r(p)} \leqslant \|\sum_{k=1}^{n} a_k Y_k\|_{(p)} \leqslant C_2(p)\|a\|_{r(p)}. \tag{5}$$

This result shows that the power p' in (1) is the best. The question about the best power in (2) is opened.

2. Some inequalities for characteristic functions. According to Proposition 1, if $X \in L_{(p)}(\Omega)$, then the corresponding characteristic function $f(t)$ is extended to the whole function. Put

$$Q_m(X, z) = \sum_{j=1}^{m} \frac{i^j \mathrm{E}X^j}{j!} z^j \quad (m = 1, 2, \ldots). \tag{6}$$

LEMMA 1. *Let $X \in L_{(p)}(\Omega)$, $\|X\|_{(p)} = 1$, $m = [p']$ and let $f(z)$ be the corresponding characteristic function. Then*

$$f(z) = 1 + Q_m(X, z) + R(z)|z|^{\max\{2, p'\}}$$

and $\sup\{|R(z)|\colon |z| \leqslant \alpha\} \leqslant \beta(p, \alpha) < \infty$ *for all $\alpha > 0$, where $\beta(p, \alpha)$ depends only on p and α.*

Proof. By Taylor's formula and the well-known equality $\mathrm{E}X^k = i^k f^{(k)}(0)$ we get

$$f(z) = 1 + Q_m(X, z) + T(z).$$

The remainder term is represented in the form $T(z) = f^{(m+1)}(u(z))z^{m+1}/(m+1)!$, where $u(z)$ belongs to the segment joining 0 and z. Using the formula

$$f^{(m+1)}(u) = \int_{-\infty}^{\infty} u^{m+1} e^{iux} dF(x),$$

where $F(x) = P[X < x]$, and by Proposition 1 we get the estimate

$$|f^{(m+1)}(u)| \leqslant \gamma(p, \alpha) < \infty,$$

where $|u| \leqslant \alpha$ and $\gamma(p, \alpha)$ depends only on p and α. Putting

$$R(z) = T(z)|z|^{-\max\{2, p'\}}$$

we obtain the required representation. Thus the lemma is proved.

Let $0 < r_1 < \ldots < r_n < \infty$. Then

$$\sum_{k=1}^{n} t^{r_k} \leqslant C(t^{r_1} + t^{r_n})$$

for all $t > 0$, where C depends only on $r_1, \ldots, r_n$. This implies the inequality

$$\sum_{k=1}^{n} \mathrm{E}|X|^{r_k} \leqslant C(\mathrm{E}|X|^{r_1} + \mathrm{E}|X|^{r_n}). \tag{7}$$

LEMMA 2. *Let* $X \in L_{(p)}(\Omega)$, $\mathrm{E}X = 0$, $1 < p < 2$. *Then for all complex* z

$$|f(z)| \leqslant \exp[C(p)(|z|^2 \mathrm{E}X^2 + |z|^{p'} \|X\|_{(p)}^{p'})].$$

If $p \geqslant 2$, *then*

$$|f(z)| \leqslant \exp[C(p) \min\{(|z| \, \|X\|_{(p)})^2, (|z| \, \|X\|_{(p)})^{p'}\}].$$

Proof. Assume $\|X\|_{(p)} = 1$. Then, by Proposition 1,

$$|f(z)| \leqslant \exp\big(B(p)|z|^{p'}\big) \tag{8}$$

for $|z| \geqslant A(p)$, where $A(p)$, $B(p) > 0$ are constants. Let $1 < p < 2$. Since $\mathrm{E}X = 0$, by (6) and (7) we get

$$|Q_m(X, z)| \leqslant \sum_{j=2}^{m} \mathrm{E}|zX|^j/j! + \mathrm{E}|zX|^{p'} \leqslant C(\mathrm{E}|zX|^2 + \mathrm{E}|zX|^{p'}).$$

There exists a constant $D = D(p)$ such that $\mathrm{E}|Y|^{p'} \leqslant D \|Y\|_{(p)}^{p'}$ for all $Y \in L_{(p)}(\Omega)$. Hence

$$|Q_m(X, z)| \leqslant C_1(p)(|z|^2 \mathrm{E}X^2 + |z|^{p'} \|X\|_{(p)}^{p'}).$$

Using the condition $\|X\|_{(p)} = 1$, Lemma 1 and the inequality $1 + x < \exp x$ we obtain

$$|f(z)| \leqslant \exp[C_2(p)(|z|^2 \mathrm{E}X^2 + |z|^{p'})] \qquad (|z| \leqslant A(p)).$$

From this and (8) the required estimate is deduced.

If $p \geqslant 2$, then $m = 1$. Since $\mathrm{E}X = 0$, we have $Q_m(X, z) = 0$. From Lemma 1 we obtain

$$|f(z)| \leqslant 1 + \beta(p)|z|^2 \leqslant \exp\big(\beta(p)(p)|z|^2\big)$$

for $|z| \leqslant A(p)$. Since $p' \leqslant 2$, from (8) we get

$$|f(z)| \leqslant \exp[C(p) \min\{|z|^2, |z|^{p'}\}].$$

Now we remove the assumption $\|X\|_{(p)} = 1$. Let $t = \|X\|_{(p)}$, $Y = t^{-1}X$ and let $g(z)$ be the characteristic function of Y. Then $g(z) = f(z/t)$. Using the estimates obtained for $g(z)$, we get the required estimate for $f(z)$. Thus the lemma is proved.

3. Proof of Theorem 1. Let $\{X_k\}_{k=1}^n \subset L_{(p)}(\Omega)$ be independent random variables, $\mathrm{E}X_k = 0$ and $f_k(z)$ be the corresponding characteristic functions. We denote by $f(z)$ the characteristic function of the sum $\sum_{k=1}^n X_k$. Then

$$f(z) = \prod_{k=1}^{n} f_k(z). \tag{9}$$

Let $1 < p < 2$ and let H_p be the expression of the right-hand side in (2). Then, by Lemma 2,

$$|f(z)| \leqslant \exp[C(p)(|zH_p|^2 + |zH_p|^{p'})]$$

for all complex z. Since $p' > 2$, we have

$$|f(z)| \leqslant \exp[2C(p)|zH_p|^{p'}]$$

for $|z| \geqslant H_p^{-1}$. Using Proposition 1 we obtain (3), which implies (2).

Let $p \geqslant 2$ and $t_k = \|X_k\|_{(p)}$. We can assume, without loss of generality, that

$$\sum_{k=1}^{n} t_k^{p'} = 1. \tag{10}$$

From (9) and Lemma 2 we obtain

$$|f(z)| \leqslant \exp\Big[C(p) \sum_{k=1}^{n} \min\{|t_k z|^2, |t_k z|^{p'}\}\Big].$$

Since $t_k \leqslant 1$ and $p' \leqslant 2$, we have $t_k^2 \leqslant t_k^{p'}$. Hence

$$\min\{|t_k z|^2, |t_k z|^{p'}\} \leqslant t_k^{p'} \min\{|z|^2, |z|^{p'}\}.$$

This inequality and (10) imply the estimate

$$|f(z)| \leqslant \exp[C(p)\min\{|z|^2, |z|^{p'}\}] = \exp[C(p)|z|^{p'}]$$

for $|z| \geqslant 1$. Using Proposition 1 we get (1). Thus Theorem 1 is proved.

4. Two lemmas. The results of this section will be used in the proof of Theorem 2. It is not difficult to show the next proposition.

LEMMA 3. *Assume that a symmetric random variable X has the whole characteristic function $f(z)$ and there exist constants $p > 1$ and $a, b > 0$ such that $P[|X| \geqslant x] \geqslant b\exp(-ax^p)$ for all $x > 0$. Then there exist constants $c, d > 0$, depending only on a, b, p, such that for $|t| \geqslant d$, $t \in \mathbf{R}$*

$$|f(-it)| \geqslant \exp(c|t|^{p'}).$$

LEMMA 4. *Let the conditions of Theorem 2 be fulfilled. Then for all $A, B > 0$ there exists a constant $D = D(A, B, p)$ such that if*

$$P\Big[\Big|\sum_{k=1}^{n} a_k Y_k\Big| \geqslant x\Big] \leqslant A\exp(-Bx^p) \tag{11}$$

for all $x > 0$, then $\sum_{k=1}^{n} |a_k|^{r(p)} \leqslant D$.

Proof. Let $p > 2$ and $f(z)$ be the characteristic function of Y_1. Since Y_1 is symmetric, $\mathrm{E}\,Y_1 = 0$. Hence $f(z) = 1-(\mathrm{E}\,Y_1^2/2)z^2+O(|z|^2)$ when $z \to 0$, $z \in \boldsymbol{C}$. Consequently, $f(-it) \geqslant \exp(ut^2)$ for sufficiently small $t \in \boldsymbol{R}$, where $u > 0$ is a constant. Applying (4), we get easily the strong inequality $f(-it) > 1$ for all $t \in \boldsymbol{R}$, $t \neq 0$. Using Lemma 3 we conclude that there exists a constant $C(p)$ such that for all $t \in \boldsymbol{R}$

$$f(-it) \geqslant \exp[C(p)\min\{t^2, |t|^{p'}\}].$$

Assume that (11) holds. The sum $\sum_{k=1}^{n} a_k Y_k$ has the characteristic function

$$g(z) = \prod_{k=1}^{n} f(a_k z).$$

From (11) and Proposition 1 we obtain $|g(z)| \leqslant \exp(B_1|z|^{p'})$ for $|z| \geqslant A_1$, where $A_1, B_1 > 0$ depend only on A, B, p. Using the last inequalities we obtain

$$C(p)\sum_{k=1}^{n} \min\{(a_k t)^2, |a_k t|^{p'}\} \leqslant B_1|t|^{p'} \quad \text{for } t \in \boldsymbol{R},\ |t| \geqslant A_1.$$

Since $p \geqslant 2$, we have $r(p) = p'$. Hence

$$\sum_{k=1}^{n} |a_k|^{r(p)} = \sum_{k=1}^{n} |a_k|^{p'} \leqslant B_1/C(p).$$

If $1 < p < 2$, then $r(p) = 2$. From (11) we get

$$\Big(\sum_{k=1}^{n} a_k^2\Big)\mathrm{E}\,Y_1^2 = \mathrm{E}\Big(\sum_{k=1}^{n} a_k Y_k\Big)^2 \leqslant C(A, B, p) < \infty.$$

This implies the required estimate and proves Lemma 4.

5. Proof of Theorem 2. The right-hand side inequality in (5) follows from Theorem 1. Suppose that the left-hand side inequality is not true. Then there exist some sets of real numbers $\{a_k^{(j)}\}_{k=1}^{n(j)}$ $(j = 1, 2, \ldots)$ such that

$$\sum_{k=1}^{n} |a_k^{(j)}|^{r(p)} = 1, \qquad \Big\|\sum_{k=1}^{n(j)} a_k^{(j)} Y_k\Big\|_{(p)} \leqslant 2^{-j}. \tag{12}$$

Put $m(0) = 0$, $m(j) = n(1)+\ldots+n(j)$ $(j \geqslant 1)$ and

$$S_l = \sum_{j=1}^{l}\sum_{k=1}^{n(j)} a_k^{(j)} Y_{m(j-1)+k} \quad (l = 1, 2, \ldots).$$

According to (12) we have $\|S_l\|_{(p)} \leqslant 1$. Using Proposition 1 we conclude that

$$P[|S_l| \geqslant x] \leqslant A\exp(-Bx^{p'}) \quad \text{for all } x > 0,$$

where $A, B > 0$ depend only on p. By Lemma 4 we have

$$\sum_{j=1}^{l} \sum_{k=1}^{n(j)} |a_k^{(j)}|^{r(p)} \leqslant D(p) < \infty.$$

But (12) implies that the sum in the left-hand side is equal to l. Hence the last estimate cannot be true for all l. This contradiction proves Theorem 2.

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