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DICHOTOMIES FOR CERTAIN PRODUCT MEASURES AND STABLE PROCESSES*

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Abstract. Necessary and sufficient conditions for equivalence or singularity of certain product measures are given and applied to the problem of distinguishing a sequence of random vectors from affine transformations of itself. In particular, sequences of independent stable random variables are considered and the singularity of sequences with different indexes of stability is proved. By using these results the dichotomy, "two processes are either equivalent or singular", is established for certain classes of stable processes, such as independently scattered measures and harmonizable processes. Also sufficient conditions for singularity and necessary conditions for absolute continuity are given for p^{th} order processes.

1. INTRODUCTION

This paper investigates the equivalence and singularity of measures induced by non-Gaussian stable processes.

For two Gaussian processes the following dichotomy prevails: they are either mutually absolutely continuous (equivalent) or else they are singular (see, e.g., [6]). For non-Gaussian stable processes some results are available in [13], [30], [29] and [24].

In Section 2 an idea of LeCam [23] is developed further to provide a necessary and sufficient condition for equivalence and for singularity of certain product measures (Proposition 2.1). As an application, the results on the discrimination between a sequence of random vectors and its perturbation by rigid motions in [27] are extended to more general classes of perturbations (Corollary 2.2). Also necessary and sufficient conditions are given for the equivalence and for the singularity of certain sequences of independent stable random variables (Corollaries 2.3 and 2.4); and the singularity of two sequences of independent symmetric stable variables with different indexes of stability is proved (Proposition 2.5).

In Section 3 an equivalence-singularity dichotomy is shown for certain symmetric stable processes (Proposition 3.2), including independently scattered measures (Proposition 3.1) and harmonizable processes (Corollary 3.3), and

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necessary and sufficient conditions for the two alternatives are given, identical to those in the Gaussian case. The singularity of an invertible symmetric stable process to its multiples is also proved (Corollary 3.4).

In Section 4, a necessary condition for equivalence of two Gaussian processes, namely the setwise equality of their reproducing kernel Hilbert spaces (RKHS's), is extended to symmetric stable processes with the function space of the process introduced in [24] replacing the RKHS (Proposition 4.2). Further, for p^{th} order processes with $1 < p < 2$, necessary conditions for absolute continuity and sufficient conditions for singularity are presented (Proposition 4.3) analogous to those in [12] for second order processes.

The following setting is considered: $X_i = \big(X_i(t) = X_i(t, \omega);\, t \in T\big)$, $i = 1, 2$, are stochastic processes on a probability space $(\Omega, \mathscr{F}, P)$ with parameter set T and real or complex values, i.e., values in $\boldsymbol{F} = \boldsymbol{R}$ or $\boldsymbol{C}$. When $X_i(t) \in L_p(\Omega, \mathscr{F}, P) = L_p(P)$ for all $t \in T$ and some $p > 0$, X_i is called a p^{th} *order process* and its linear space $\mathscr{L}(X_i)$ is the $L_p(P)$ *completion* of the set of finite linear combinations of its random variables

$$l(X_i) \overset{\Delta}{=} \operatorname{sp}\{X_i(t);\ t \in T\}.$$

$\boldsymbol{F}^T$ denotes the set of all extended $\boldsymbol{F}$-valued (i.e., real or complex valued) functions on T, $\mathscr{C} = \mathscr{C}(\bar{\boldsymbol{F}}^T)$ the σ-field generated by the cylinder sets of $\bar{\boldsymbol{F}}^T$, and μ_i (or μ_{X_i}) the distribution of the process X_i, i.e., the probability induced on $\mathscr{C}$ by X_i: $\mu_i(C) = P\big(\{\omega;\ X_i(\cdot, \omega) \in C\}\big)$, $C \in \mathscr{C}$. We are interested in the Lebesgue decomposition of μ_2 with respect to μ_1, and in particular in conditions for μ_1 and μ_2 to be singular ($\mu_1 \perp \mu_2$), for μ_2 to be absolutely continuous with respect to μ_1 ($\mu_2 \ll \mu_1$), and for μ_1 and μ_2 to be mutually absolutely continuous or equivalent ($\mu_1 \sim \mu_2$).

2. ON THE EQUIVALENCE AND SINGULARITY OF CERTAIN PRODUCT MEASURES

In this section we consider the case where $X_i = (X_{i,n};\, n \in N)$, $i = 1, 2$, are sequences of independent random variables or, equivalently, μ_1 and μ_2 are product measures on $\boldsymbol{F}^N$. The equivalence-singularity dichotomy of product measures was characterized in [18] in terms of the Hellinger distance of the marginal measures, which may be difficult to compute, e.g., for stable measures. The case of translates of product measures with identical marginals was solved in [25] under finite Fisher information. The sufficient condition for equivalence in [25] was extended in [23] to a more general scenario under LeCam's "l" condition. Proposition 2.1 derives a nearly complete extension of a result of Shepp in [25] under a condition closely related to LeCam's. As an application the equivalence-singularity dichotomy is established for a sequence of i.i.d. random vectors and an affine transformation of itself in Corollary 2.2 (extending the results in [27] about rigid motions), and for sequences of i.i.d. stable random variables in Corollaries 2.3 and 2.4.

Before stating the main results we need to introduce some concepts for which we refer to [28].

2.1. Preliminaries. The normalized Hellinger distance d of two probability measures P and Q on a measurable space $(\Omega, \mathscr{F})$ is defined by

$$d^2(P, Q) = \tfrac{1}{2} \int_\Omega |(dP/d\nu)^{1/2} - (dQ/d\nu)^{1/2}|^2 d\nu,$$

where ν is any σ-finite measure dominating $P+Q$, i.e., $P+Q \ll \nu$ (e.g., $\nu = P+Q$); and d does not depend on ν.

Kakutani's theorem [18] states that if $(\mu_n;\ n \in N)$ and $(\lambda_n,\ n \in N)$ are sequences of probability measures with $\mu_n \sim \lambda_n$ and $\mu = \mathsf{X}_{n=1}^\infty \mu_n$ and $\lambda = \mathsf{X}_{n=1}^\infty \lambda_n$ are their product measures, then

$$\text{(2.1)} \quad \mu \sim \lambda \Leftrightarrow \sum_{n=1}^\infty d^2(\mu_n, \lambda_n) < \infty \quad \text{and} \quad \mu \perp \lambda \Leftrightarrow \sum_{n=1}^\infty d^2(\mu_n, \lambda_n) = \infty.$$

We consider the following setting: $(\Omega, \mathscr{F}, \nu)$ is a σ-finite measure space, and $\{P_\theta, \theta \in \Theta\}$ a family of probability measures on $(\Omega, \mathscr{F})$ with $P_\theta \ll \nu$ and Θ an open subset of R^k. Then $F\colon \Theta \to L_2(\Omega, \mathscr{F}, \nu) = L_2(\nu)$ defined by $F(\theta) = 2[dP_\theta/d\nu]^{1/2}$ is said to be *differentiable at* θ if there exists a map $DF(\theta) := DF(\cdot, \theta)\colon \Omega \to R^k$ such that

$$\|DF(\theta)\|^2_{L_2(\Omega, \mathscr{F}, \nu; R^k)} = \int_\Omega \|DF(\omega, \theta)\|^2 \nu(d\omega) < \infty,$$

i.e., $DF(\theta) \in L_2(\Omega, \mathscr{F}, \nu; R^k)$, and

$$\int_\Omega |F(\theta+h) - F(\theta) - \langle DF(\theta), h\rangle|^2 d\nu = o(\|h\|^2) \quad \text{as } \|h\| \to 0.$$

As usual, F is said to be *differentiable* (on Θ) if it is differentiable at each $\theta \in \Theta$. The Fisher information matrix is defined by

$$\mathscr{I}(\theta) = \int_\Omega DF(\theta) DF(\theta)' d\nu$$

(where $DF(\theta)'$ is the transpose of the column vector $DF(\theta)$). It is non-negative definite as $a'\mathscr{I}(\theta)a = \int_\Omega (a'DF(\theta))^2 d\nu$, and is positive definite if and only if the components of $DF(\theta)$ are linearly independent functions in $L_2(\nu)$.

2.2. Main result. As in [23] our purpose is to consider product measures

$$\text{(2.2)} \qquad \mu = \mathop{\mathsf{X}}_{n=1}^\infty \mu_n, \quad \lambda = \mathop{\mathsf{X}}_{n=1}^\infty \lambda_n, \quad \text{where } \mu_n = P_\theta \text{ and } \lambda_n = P_{\theta+h_n},$$

$\theta \in \Theta$ is fixed and $\theta + h_n \in \Theta$, $n = 1, 2, \ldots$ Under LeCam's condition

$$\text{“}l\text{”:} \quad \limsup_{0 < \|h\| \to 0} d^2(P_{\theta+h}, P_\theta)/\|h\| < \infty,$$

Proposition 2 in [23] shows that $\sum_{n=1}^\infty \|h_n\|^2 < \infty$ implies $\mu \sim \lambda$. Here we obtain an equivalence-singularity dichotomy along with necessary and sufficient conditions for the two alternatives, when $\mathscr{I}(\theta)$ is positive definite at

θ and the following separation type condition (which is usually assumed in asymptotic statistical theory [16]) is satisfied:

(2.3) "for all sufficiently small $\delta > 0$, $\inf_{\|h\| > \delta} d^2(P_{\theta+h}, P_\theta) > 0$".

PROPOSITION 2.1. *Let μ and λ be as in* (2.2), *F be differentiable at θ and $\mathscr{I}(\theta)$ be positive definite.*

(i) *If $0 < \|h_n\| \to 0$ as $n \to \infty$, then*

$$\mu \sim \lambda \Leftrightarrow \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \quad \textit{and} \quad \mu \perp \lambda \Leftrightarrow \sum_{n=1}^{\infty} \|h_n\|^2 = \infty.$$

(ii) *If condition* (2.3) *is satisfied, then the conclusions of* (i) *hold for any sequence* $(h_n;\ n \in N)$.

The sufficiency for equivalence follows from [23, Proposition 2], since L_2-differentiability is clearly stronger than condition "l", but we include a simple complete proof here.

Proof. (i) Since F is differentiable at θ, as $0 < \|h\| \to 0$ we have

$$\big|\|F(\theta+h)-F(\theta)\|_{L_2(\nu)} - \|\langle DF(\theta), h\rangle\|_{L_2(\nu)}\big| = o(\|h\|).$$

Thus for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $0 < \|h\| < \delta$, then

$$\|h\|^{-1}\|\langle DF(\theta), h\rangle\|_{L_2(\nu)} - \varepsilon < \|h\|^{-1}\|F(\theta+h)-F(\theta)\|_{L_2(\nu)}$$
$$< \|h\|^{-1}\|\langle DF(\theta), h\rangle\|_{L_2(\nu)} + \varepsilon.$$

But $\|DF(\theta), h\rangle\|^2_{L_2(\nu)} = \int_\Omega |\langle DF(\theta), h\rangle|^2 d\nu = h'\mathscr{I}(\theta)h$ implies that, for all $h \neq 0$,

$$k(\theta) \leqslant \|h\|^{-1}\|\langle DF(\theta), h\rangle\|_{L_2(\nu)} \leqslant K(\theta),$$

where $k(\theta)$ and $K(\theta)$ are the smallest and the largest eigenvalues of $\mathscr{I}(\theta)$. Since $\mathscr{I}(\theta)$ is positive definite, $k(\theta) > 0$ and we can choose $0 < \varepsilon < k(\theta)$ so that, for all h, $0 < \|h\| < \delta$,

$$0 < L(\theta) < \|h\|^{-1}\|F(\theta+h)-F(\theta)\|_{L_2(\nu)} < U(\theta),$$

where $L(\theta) = k(\theta)-\varepsilon$ and $U(\theta) = K(\theta)+\varepsilon$. Thus since it follows that $d^2(P_\theta, P_{\theta'}) = \|F(\theta)-F(\theta')\|^2_{L_2(\nu)}/8$, we have for n large

$$0 < \tfrac{1}{8}L^2(\theta)\|h_n\|^2 < d^2(\mu_n, \lambda_n) < \tfrac{1}{8}U^2(\theta)\|h_n\|^2$$

and the result follows from (2.1).

(ii) If (2.3) is satisfied and $h_n \not\to 0$, then there exist $\delta > 0$ and a subsequence $(n_j;\ j \in N)$ with $\|h_{n_j}\| > \delta$. It follows that

$$\sum_{n=1}^{\infty} d^2(\mu_n, \lambda_n) \geqslant \sum_{j=1}^{\infty} d^2(\mu_{n_j}, \lambda_{n_j}) \geqslant \sum_{j=1}^{\infty} \inf_{\|h\| > \delta} d^2(P_{\theta+h}, P_\theta) = \infty,$$

and from (2.1) we obtain $\mu \perp \lambda$. This combined with (i) gives the result. ∎

It should be mentioned that the differentiability of $F(\theta)$ is generally difficult to verify, but is implied by the classical Cramér–Wald and Hájek regularity conditions, which play an important role in statistical estimation theory and are in principle easy to check (see, e.g., [28, Section 77]). However, L_2-differentiability is weaker than any of these classical conditions, and the definition of Fisher information presented here extends the classical one, namely $\mathscr{I}(\theta) = -\mathrm{E}\{\partial^2 \ln(dP_\theta/dv)/\partial\theta^2\}$ under the usual conditions on dP_θ/dv.

2.2. Examples.

Affine transformations in $\boldsymbol{R}^k$. Suppose $(X_n;\ n\in N)$ is a sequence of i.i.d. random vectors in $\boldsymbol{R}^k$, $(A_n;\ n\in N)$ a sequence of $(k\times k)$-matrices, and $(b_n;\ n\in N)$ a sequence of vectors in $\boldsymbol{R}^k$. In order to compare the sequence of random vectors $(X_n;\ n\in N)$ with $(A_nX_n+b_n;\ n\in N)$ we can take as a parameter space Θ any open subset of the set

$$\{\theta = (A, b);\ A = (a_{ij}):\ (k\times k)\text{-matrix},\ b = (b_i)\in \boldsymbol{R}^k\}$$
$$\equiv \{\theta;\ \theta = (a_{11}, \ldots, a_{1k}, \ldots, a_{kk}, b_1, \ldots, b_k)\} \equiv \boldsymbol{R}^{k^2+k} \equiv \boldsymbol{R}^{k^2}\times \boldsymbol{R}^k$$

containing the point $(I, 0)$, with norm

$$\|\theta\|^2_{\boldsymbol{R}^{(k\times k)+k}} = \|A\|^2_{\boldsymbol{R}^{k\times k}} + \|b\|^2_{\boldsymbol{R}^k} = \sum_{i,j=1}^{k} a_{ij}^2 + \sum_{i=1}^{k} b_i^2.$$

With P the common distribution of the i.i.d. random vectors X_n and $\theta = (A, b)$, we define

$$P_\theta(B) = P_{(A,b)}(B) = P(\{AX_n+b\in B\}) \tag{2.4}$$

and note that $P = P_{(I,0)}$. From Proposition 2.1 we have the following

COROLLARY 2.2. *Let the probability measures P_θ defined as in* (2.4) *be such that, for an open set $\Theta\subset \boldsymbol{R}^{k^2}\times\boldsymbol{R}^k$ with $(I, 0)\in\Theta$, the family $\{P_\theta;\ \theta\in\Theta\}$ is dominated by some σ-finite measure v on $\boldsymbol{R}^k$, $F(\theta)$ is differentiable at $(I, 0)$ and $\mathscr{I}(I, 0)$ is positive definite. If $A_n\to I$ and $b_n\to 0$ as $n\to\infty$, then*

$$(X_n)\sim(A_nX_n+b_n)\Leftrightarrow \sum_{n=1}^{\infty}\|b_n\|^2_{\boldsymbol{R}^k}<\infty \quad \textit{and} \quad \sum_{n=1}^{\infty}\|I-A_n\|^2_{\boldsymbol{R}^{k\times k}}<\infty,$$

and otherwise $(X_n)\perp(A_nX_n+b_n)$. Furthermore, if condition (2.3) *is satisfied, the above conclusions hold for all sequences (A_n, b_n) in Θ.*

Proof. Putting $\theta = (I, 0)$ and $(A_n, b_n) = \theta+h_n$ we have $h_n = (A_n-I, b_n)$ and

$$\|h_n\|^2_{\boldsymbol{R}^{(k\times k)+k}} = \|A_n-I\|^2_{\boldsymbol{R}^{(k\times k)}} + \|b_n\|^2_{\boldsymbol{R}^k}.$$

The conclusion then follows from Proposition 2.1. ∎

Remarks. (a) Since the space of $(k\times k)$-matrices is finite dimensional, any norm can be used in place of $\|\cdot\|_{\boldsymbol{R}^{k\times k}}$.

(b) When $A_n = I$ for all n, Corollary 2.2 extends the result on translates in [25] from random variables ($k = 1$) to random vectors ($k = 2$).

(c) Corollary 2.2 contains Theorems 1 and 2 in [27], which consider the case where A_n is a rotation, i.e., $A_n x + b_n$ is a rigid motion of $x \in \mathbf{R}^k$.

(d) When the X_n's are Gaussian random variables ($k = 1$) with mean zero and variance one, Corollary 2.2 can be checked directly by computing Hellinger distances. However, the computation of Hellinger distance is not simple in higher dimensions ($k \geqslant 2$) even for Gaussian random vectors.

Stable sequences. Here we denote by $f_{(\alpha,\beta,a,b)}$ the univariate stable density whose characteristic function $\int_{-\infty}^{\infty} \exp(iux) f_{(\alpha,\beta,a,b)}(x)dx$ is

$$\begin{aligned} &\exp\{-|au|^{\alpha}\exp[-i\pi\beta\,\mathrm{sgn}(u)/2]+ibu\} && \text{if } \alpha \neq 1, \\ &\exp\{-|au|-i(2\beta/\pi)au\ln(|au|)+ibu\} && \text{if } \alpha = 1, \end{aligned}$$

where $0 < \alpha \leqslant 2$, $|\beta| \leqslant \alpha \wedge (2-\alpha)$, $a > 0$ and $-\infty < b < \infty$ (see [9]). If $\beta = 0$ and $b = 0$, we have the symmetric α-stable case (SαS).

We establish the equivalence-singularity dichotomy for certain sequences of independent stable variables. Since results about L_2-differentiability and the validity of condition (2.3) at $\alpha = 1$ are not known, we consider only limiting values $\alpha \neq 1$.

COROLLARY 2.3. *Let $(X_{1n}, n \in N)$ be a sequence of i.i.d. stable variables with density $f_{(\alpha_0,\beta_0,a_0,b_0)}$ and let $(X_{2n};\ n \in N)$ be a sequence of independent stable variables where the density of each X_{2n} is $f_{(\alpha_n,\beta_n,a_n,b_n)}$ with $(\alpha_n, \beta_n, a_n, b_n) \to (\alpha_0, \beta_0, a_0, b_0)$ and $\alpha_0 \neq 1$. Then*

$$(X_{1n}) \sim (X_{2n}) \Leftrightarrow \begin{cases} \sum_{n=1}^{\infty} (\alpha_n-\alpha_0)^2 < \infty, & \sum_{n=1}^{\infty} (\beta_n-\beta_0)^2 < \infty, \\ \sum_{n=1}^{\infty} (a_n-a_0)^2 < \infty, & \sum_{n=1}^{\infty} (b_n-b_0)^2 < \infty, \end{cases}$$

and otherwise $(X_{1n}) \perp (X_{2n})$.

Proof. Let Θ be any open subset of the set

$$\{\theta = (\alpha, \beta, a, b);\ \alpha \in (0, 1) \cup (1, 2),\ |\beta| < \alpha \wedge (2-\alpha),\ a > 0,\ -\infty < b < \infty\}$$

containing the point $\theta_0 = (\alpha_0 \wedge \beta_0, a_0, b_0)$. It is known that the densities $\{f_\theta, \theta \in \Theta\}$ satisfy the usual Cramér–Wald regularity conditions ([9], p. 952); hence $f_\theta^{1/2}$ is L_2(Leb)-differentiable at each $\theta \in \Theta$ (see, e.g., [28, Section 77]). Moreover, the Fisher information matrix $\mathscr{I}(\theta_0)$ is positive definite [9, p. 954]. Therefore, the assumptions of Proposition 2.1 (i) hold at θ_0. Since for $h_n = (\alpha_n - \alpha_0, \beta_n - \beta_0, a_n - a_0, b_n - b_0)$ we have

$$\|h_n\|_{\mathbf{R}^4}^2 = (\alpha_n-\alpha_0)^2 + (\beta_n-\beta_0)^2 + (a_n-a_0)^2 + (b_n-b_0)^2,$$

the result follows. ∎

When all parameters except shift b are kept fixed, the separation condition (2.3) follows from the inequality in [16, Example 3, p. 57], and when $\beta = 0$ and $\alpha \in (0, 2]$ is fixed, it has been proved in [19]. Hence we have the following

COROLLARY 2.4. *Let $(X_n;\ n \in N)$ be a sequence of i.i.d. standard* SαS *variables with density $f_{(\alpha,0,1,0)}$ and $\alpha \in (0, 2]$, and let (a_n, b_n) and (a'_n, b'_n) be two sequences of pairs of real numbers with $a_n \neq 0$. Then*

$$(a_n X_n + b_n) \sim (a'_n X_n + b'_n) \Leftrightarrow (X_n) \sim \big((a_n/a'_n) X_n + (b_n - b'_n)/a'_n\big)$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \{1 - |a_n/a'_n|\}^2 < \infty \ \textit{and} \ \sum_{n=1}^{\infty} \{(b_n - b'_n)/a'_n\}^2 < \infty,$$

and otherwise $(a_n X_n + b_n) \perp (a'_n X_n + b'_n)$.

Proof. The first equivalence follows since the map $(x_n) \to ((x_n - b'_n)/a'_n)$ is invertible and the second follows from Corollary 2.2 since $(a_n/a'_n) X_n + (b_n - b'_n)/a_n$ has density $f_{(\alpha,0,a_n/a'_n,(b_n-b'_n)/a_n)}$. ■

We next explore the tail behaviour of a stable distribution to show that two infinite sequences of independent symmetric stable variables with two different indexes of stability are singular.

PROPOSITION 2.5. *Let $X_i = (X_{in};\ n \in N)$, $i = 1, 2$, be two sequences of (non-degenerate) independent symmetric stable variables with index of stability α_i in $(0, 2]$ and scale parameters (a_{in}). If $\alpha_1 \neq \alpha_2$, then $\mu_1 \perp \mu_2$.*

Proof. Assume $\alpha_1 < \alpha_2 < 2$. For each $\gamma \in (0, 2)$ let Z_γ denote an SγS r.v. with scale parameter 1. Thus

$$\mu_{in}(B) \triangleq P(X_{in} \in B) = P(a_{in} Z_{\alpha_i} \in B).$$

Since $c^\gamma P(|Z_\gamma| > c^\gamma) \to C_\gamma$ as $c \to \infty$, where C_γ is a positive constant (see, e.g., [11]), given any $\varepsilon > 0$, there exist $M_{\gamma,\varepsilon}$ such that, for $c > M_{\gamma,\varepsilon}$,

$$(C_\gamma - \varepsilon) c^{-\gamma} < P(|Z_\gamma| > c) < (C_\gamma + \varepsilon) c^{-\gamma}.$$

From now on fix ε such that $0 < \varepsilon < \min(C_{\alpha_1}, C_{\alpha_2})$.

Case 1. Assume

$$\sigma_n \triangleq a_{1n}/a_{2n} \to 0 \quad \text{as } n \to \infty.$$

Define Ψ: $F^N \to F^N$ by $\Psi(x) = \big(\Psi_n(x) = x_n/a_{2n};\ n \in N\big)$. It follows that Ψ is an i.i.d. sequence of standard Sα_2S r.v.'s under μ_2, and under μ_1 an independent sequence of Sα_1S r.v.'s with scale parameter $a_{1n}/a_{2n} = \sigma_n$.

As before let d_v denote the total variation distance between probability measures. For $c > M(1 + \sup_n \sigma_n)$, where $M = \max(M_{\alpha_1}, M_{\alpha_2})$, we have

$$d_v(\mu_{1n} \Psi_n^{-1}, \mu_{2n} \Psi_n^{-1}) \geqslant P(|Z_{\alpha_2}| > c) - P(|\sigma_n Z_{\alpha_1}| > c)$$

$$> (C_{\alpha_2} - \varepsilon) c^{\alpha_2} - \sigma_n^{\alpha_1} (C_{\alpha_1} + \varepsilon) c^{\alpha_1},$$

and thus

$$\liminf_{n\to\infty} d_v(\mu_{1n}\Psi_n^{-1}, \mu_{2n}\Psi_n^{-1}) \geqslant (C_{\alpha_2}-\varepsilon)/c^{\alpha_2} > 0.$$

Since $d_v \leqslant 2d$, where d denotes the Hellinger distance (see, e.g., [28]), we have

$$\sum_{n=1}^{\infty} d(\mu_{1n}\Psi_n^{-1}, \mu_{2n}\Psi_n^{-1}) = \infty,$$

and therefore, by Kakutani's Theorem, $\mu_1\Psi^{-1} \perp \mu_2\Psi^{-1}$, which implies $\mu_1 \perp \mu_2$.

Case 2. Assume $\sigma_n \nrightarrow 0$. Thus there exist $\delta > 0$ and a sequence $(n_k;\ k\in N)$ such that $\sigma_{n_k} \geqslant \delta$, i.e., $\sigma_{n_k}^{-1} \leqslant \delta^{-1}$. Define Φ: $F^N \to F^N$ by

$$\Phi(x) = \left(\Phi_k(x) = x_k/a_{1n_k};\ k\in N\right).$$

Then Φ is an i.i.d. sequence of standard $S\alpha_1 S$ r.v.'s under μ_1, and under μ_2 an independent sequence of $S\alpha_2 S$ r.v.'s with scale parameter $\alpha_{2n_k}/a_{1n_k} = \sigma_{n_k}^{-1}$. For $c > M(1+\delta^{-1})$ we have

$$\begin{aligned} d_v(\mu_{1n_k}\Phi_{n_k}^{-1}, \mu_{2n_k}\Phi_{n_k}^{-1}) &\geqslant P(|Z_{\alpha_1}| > c) - P(|\sigma_{n_k}^{-1} Z_{\alpha_2}| > c) \\ &> (C_{n_k}-\varepsilon)c^{\alpha_1} - \sigma_{n_k}^{-1}(C_{\alpha_2}+\varepsilon)c^{\alpha_2} > (C_{n_k}-\varepsilon)c^{\alpha_1} - \delta^{-1}(C_{\alpha_2}+\varepsilon)c^{\alpha_2} \triangleq \delta'(c). \end{aligned}$$

Since $\alpha_1 < \alpha_2$, we have $\delta'(c) > 0$ if and only if $c^{\alpha_2-\alpha_1} > \delta^{-1}(C_{\alpha_2}+\varepsilon)/(C_{\alpha_1}-\varepsilon)$. Thus, fixing

$$c > M\left(1+\delta^{-1}+(c_{\alpha_2}+\varepsilon)/(C_{\alpha_1}-\varepsilon)^{1/(\alpha_2-\alpha_1)}\right),$$

we obtain

$$\limsup_{n\to\infty} d_v(\mu_{1n}\Phi_n^{-1}, \mu_{2n}\Phi_n^{-1}) > \delta'(c) > 0,$$

and the conclusion follows as in case 1.

If $\alpha_2 = 2$, the result can be shown with minor modifications in the proof. ∎

3. DICHOTOMIES FOR CERTAIN SαS PROCESSES

For stochastic processes the equivalence-singularity dichotomy has been proved for product measures [18], for Gaussian processes ([10] and [14]), and for certain ergodic measures [20]. In [24], it was shown that this dichotomy prevails for translates of certain SαS processes. Such a dichotomy for general SαS measures has been conjectured in [7] but the problem remains open. In this section we show that an equivalence-singularity dichotomy holds for certain SαS processes, e.g., independently scattered SαS measures and harmonizable SαS processes, and we give necessary and sufficient conditions for the two alternatives for all $\alpha\in(0, 2]$.

Recall that a random variable X is SαS with scale parameter $\|X\|_\alpha \in (0, \infty)$ if

$$\mathrm{E}\{\exp(iuX)\} = \exp(-\|X\|_\alpha^\alpha |u|^\alpha),$$

and a stochastic process $X = (X(t);\ t\in T)$ is SαS if all linear combinations $\sum_{k=1}^{n} a_k X(t_k)$ are SαS variables. When $\alpha = 2$, we have zero mean Gaussian variables and processes, respectively. When $0 < \alpha < 2$, the tails of the distributions are heavier and only moments of order $p\in(0, \alpha)$ are finite.

We first prove a dichotomy for independently scattered SαS measures. Let I be an arbitrary set and $\mathscr{I}$ a δ-ring of subsets of I with the property that there exists an increasing sequence $(I_n;\ n\in N)$ in $\mathscr{I}$ with $\bigcup_n I_n = I$. A real stochastic process $Z = (Z(B);\ B\in\mathscr{I})$ is called an *independently scattered* SαS *measure* if, for every sequence $(B_n;\ n\in N)$ of disjoint sets in $\mathscr{I}$, the random variables $\{Z(B_n);\ n\in N\}$ are independent, and whenever $\bigcup_n B_n \in \mathscr{I}$ we obtain $Z(\bigcup_n B_n) = \sum_n Z(B_n)$ a.s., and for every $B\in\mathscr{I}$ the random variable $Z(B)$ is SαS, i.e.,

$$\mathrm{E}\{\exp(iuZ(B))\} = \exp\{-m(B)|u|^{\alpha}\}, \quad \text{where } m(B) = \|Z(B)\|_{\alpha}^{\alpha}.$$

Then m is a measure on $\mathscr{I}$ which extends uniquely to a σ-finite measure on $\sigma(\mathscr{I})$, and is called the *control measure* of Z. The existence of an independently scattered SαS measure with a given control measure is a consequence of Kolmogorov's consistency theorem. If I is an interval of the real line and the conrol measure m is Lebesgue measure, then X has stationary independent increments,

$$\mathrm{E}\{\exp(iu[X(t)-X(t')])\} = \exp\{-|t-t'|\,|u|^{\alpha}\},$$

and is called an SαS *motion* on I.

The following notation will be used in Proposition 3.1. Recall that if a σ-finite measure space $(I, \sigma(\mathscr{I}), m)$ is such that $\sigma(\mathscr{I})$ contains all single point sets (e.g., I is a Polish space, $\sigma(\mathscr{I})$ its Borel sets, and $\mathscr{I}$ the δ-ring of Borel sets with finite m-measure), then m can be decomposed into $m = m_a + m_d$, where m_a is purely atomic and m_d is diffuse (non-atomic) [21], and the set of atoms is at most countable, say $A = \{a_n\}$. Thus if $Z = (Z(B);\ B\in\mathscr{I})$ is an independently scattered SαS measure with control measure m, it can be decomposed into $Z = Z_a + Z_d$, where Z_a and Z_d are independent SαS independently scattered measures defined for all $B\in\mathscr{I}$ by $Z_a(B) = Z(A\cap B)$ and $Z_d(B) = Z(A^c\cap B)$, and have control measures m_a and m_d, respectively. The atomic component has a series expansion $Z_a(B) = \sum_n 1_B(a_n) Z(\{a_n\})$ which can be normalized by using the i.i.d. standard SαS variables $Z_n \triangleq Z(\{a_n\}) m^{-1/\alpha}(\{a_n\})$ with $\mathrm{E}\{\exp(iuZ_n)\} = \exp(-|u|^{\alpha})$ as follows:

$$Z_a(B) = \sum_n 1_B(a_n) m^{1/\alpha}(\{a_n\}) Z_n.$$

PROPOSITION 3.1. *For $i = 1, 2$, let $Z_i = (Z_i(B);\ B\in\mathscr{I})$ be an independently scattered* Sα_iS *measure with $\alpha_i \in (0, 2]$ and control measure m_i which is not purely discrete with a finite number of atoms. Then $\mu_1 \sim \mu_2$ if any only if the following conditions are satisfied*:

(i) $\alpha_1 = \alpha_2$,

(ii) $m_{1d} = m_{2d}$,

(iii) *m_1 and m_2 have the same set of atoms $A = \{a_n\}$ and*

$$\sum_n [1 - m_1(\{a_n\})/m_2(\{a_n\})]^2 < \infty.$$

Furthermore, if any of these conditions fails, then $\mu_1 \perp \mu_2$.

Proof. First suppose that m_1 and m_2 are not equivalent, e.g., $m_2 \not\ll m_1$. Then there exists $B \in \sigma(\mathscr{I})$ such that

$$\|Z_1(B)\|_{\alpha_1}^{\alpha_1} = m_1(B) = 0 \quad \text{and} \quad \|Z_2(B)\|_{\alpha_2}^{\alpha_2} = m_2(B) > 0.$$

Define Γ_B: $F^{\mathscr{I}} \to F$ by $\Gamma_B(x) = x(B)$. It follows that $\mu_1 \Gamma_B^{-1} \perp \mu_2 \Gamma_B^{-1}$, and thus $\mu_1 \perp \mu_2$. From now on we assume $m_1 \sim m_2$.

Suppose $\alpha_1 \neq \alpha_2$. Since m_1 and m_2 are not purely atomic with a finite number of atoms, we can choose an infinite sequence $(B_n; n \in N)$ of disjoint sets in $\mathscr{I}$ such that $m_i(B_n) > 0$, $i = 1, 2$. Define Ψ: $F^{\mathscr{I}} \to F^N$ by $\Psi(x) = (\Psi_n(x) = x(B_n); n \in N)$. Thus, for $i = 1, 2$, under μ_i, Ψ is a sequence of independent Sα_iS r.v.'s with $\|\Psi_n\|_{\alpha_i}^{\alpha_i} = m_i(B_n)$. It follows from Proposition 2.5 that if $\alpha_1 \neq \alpha_2$, then $\mu_1 \Psi^{-1} \perp \mu_2 \Psi^{-1}$, so that $\mu_1 \perp \mu_2$. From now on we assume $\alpha_1 = \alpha_2 = \alpha$.

Since $m_1 \sim m_2$, we have $m_{1d} \sim m_{2d}$. Suppose $m_{1d} \neq m_{2d}$, so that

$$m_{id}(\{dm_{2d}/dm_{1d} \neq 1\}) > 0, \quad i = 1, 2;$$

hence

$$m_{id}(\{0 < dm_{2d}/dm_{1d} < 1\}) > 0 \quad \text{or} \quad m_{id}(\{dm_{2d}/dm_{1d} > 1\}) > 0.$$

Assume $m_{id}(\{dm_{2d}/dm_{1d} > 1\}) > 0$. Then there exists $\delta > 1$ such that

$$m_{id}(\{dm_{2d}/dm_{1d} > \delta\}) > 0.$$

Since m_{id} is non-atomic, we can find a sequence $(B_n; n \in N)$ of disjoint subsets of $\{dm_{id}/dm_{1d} > \delta\}$ such that $m_{1d}(B_n) > 0$. Let Φ: $F^{\mathscr{I}} \to F^N$ be the map defined by

$$\Phi(x) = \{\Phi_n(x) = x(A^c \cap B_n)/m_{1d}(B_n)^{1/\alpha};\ n \in N\}.$$

Under μ_1, Φ is an i.i.d. sequence of standard SαS r.v.'s, and under μ_2, Φ is an independent sequence of SαS r.v.'s with $\|\Phi_n\|_\alpha^\alpha = m_{2d}(B_n)/m_{1d}(B_n)$. It follows from Corollary 2.4 that $\mu_1 \Phi^{-1}$ and $\mu_2 \Phi^{-1}$ are either equivalent or singular, and they are singular if and only if

$$\sum_n \{1 - [m_{2d}(B_n)/m_{1d}(B_n)]^{1/\alpha}\}^2 = \infty. \tag{3.1}$$

Now, by construction,

$$m_{2d}(B_n) = \int_{B_n} \frac{dm_{2d}}{dm_{1d}} dm_{1d} > \delta m_{1d}(B_n).$$

Hence $1 < \delta < m_{2d}(B_n)/m_{1d}(B_n)$, so that (3.1) holds. Thus $\mu_1 \Phi^{-1} \perp \mu_2 \Phi^{-1}$, which implies $\mu_1 \perp \mu_2$.

If $m_{id}(\{dm_{2d}/dm_{1d} > 1\}) = 0$, we have $m_{id}(\{dm_{1d}/dm_{2d} > 1\}) > 0$ and an identical argument applies. Therefore $m_1 \sim m_2$ and $m_{1d} \neq m_{2d}$ implies $\mu_1 \perp \mu_2$.

Now assume $m_{1d} = m_{2d}$. Since $m_1 \sim m_2$, they have the same set of atoms $A = \{a_n\}$. Suppose $\mu_2 \ll \mu_1$ and let Ξ: $F^{\mathscr{I}} \to F^N$ be defined by

$$\Xi(x) = \{\Xi_n(x) = x(\{a_n\})/m_1(\{a_n\})^{1/\alpha};\ n \in N\}.$$

Thus $\mu_2 \Xi^{-1} \ll \mu_1 \Xi^{-1}$, and Ξ is an i.i.d. sequence of standard SαS r.v.'s under μ_1, and under μ_2 an independent sequence of SαS r.v.'s with $\|\Xi\|_\alpha^\alpha = m_2(\{a_n\})/m_1(\{a_n\})$. Hence, by Corollary 2.4,

$$\sum_n \{1 - [m_2(\{a_n\})/m_1(\{a_n\})]^{1/\alpha}\}^2 < \infty. \tag{3.2}$$

Also, if (3.2) does not hold, again Corollary 2.4 implies $\mu_1 \Xi^{-1} \perp \mu_2 \Xi^{-1}$ so that $\mu_1 \perp \mu_2$.

Note that (3.1) and (3.2) are symmetric in m_1 and m_2 and independent of α as for $q \neq 0$, $\sum_n (1-u_n)^2 < \infty$ if and only if $\sum_n (1-u_n^q)^2 < \infty$. Hence (3.2) can be replaced by (iii).

Conversely, suppose that (i)–(iii) hold. Since $m_{1d} = m_{2d}$, we have

$$Z_i \stackrel{L}{=} Z_{ia} + Z_d, \quad i = 1, 2,$$

where Z_{ia} and Z_d are independent, independently scattered SαS measures with control measures m_{ia} and $m_d = m_{1d} = m_{2d}$, respectively, and $\stackrel{L}{=}$ denotes equality in law. Let Φ: $F^N \to F^{\mathscr{I}}$ be defined by

$$[\Phi(y)](B) = \Phi(y, B) = \sum_{n=1}^{\infty} 1_B(a_n) m_1(\{a_n\})^{1/\alpha} y_n, \quad y = (y_n) \in Z^N.$$

Thus $(\Phi \circ \Xi)(Z_i) \stackrel{L}{=} Z_{ia}$, so that $\mu_{ia} = (\mu_i \Xi^{-1})\Phi^{-1}$, $i = 1, 2$. Now, by Corollary 2.4, condition (iii) implies $\mu_1 \Xi^{-1} \sim \mu_2 \Xi^{-1}$; hence $\mu_{1a} \sim \mu_{2a}$. Therefore, since $\mu_i = \mu_{ia} * \mu_d$, $i = 1, 2$, it follows that $\mu_1 \sim \mu_2$. ∎

The results in Proposition 3.1 can be extended to certain symmetric (dependent) stable processes. Let Z be an independently scattered SαS measure with control measure m. For any function $f \in L_\alpha(I, \sigma(\mathscr{I}), m) = L_\alpha(m)$ the stochastic integral $\int_I f\,dZ$ can be defined in the usual way and is an SαS variable with $\|\int_I f\,dZ\|_\alpha = \|f\|_{L_\alpha(m)}$. The map $f \to \int_I f\,dZ$ from $L_\alpha(m)$ into $\mathscr{L}(Z)$ is an isometry and

$$\mathscr{L}(Z) = \{\int_I f\,dZ;\ f \in L_\alpha(m)\}. \tag{3.3}$$

The stochastic integral allows for the construction of SαS processes with generally dependent values by means of the spectral representation

$$X(t) = \int_T f(t, u) Z(du), \quad t \in T, \tag{3.4}$$

where $\{f(t, \cdot);\ t \in T\} \subset L_\alpha(m)$. In fact, every S$\alpha$S process X has such a spectral representation in law, in the sense that, for some family $\{f(t, \cdot),\ t \in T\}$ in some $L_\alpha(m)$,

$$(X(t);\ t \in T) \stackrel{L}{=} (\int_I f(t, u) Z(du);\ t \in T) \tag{3.5}$$

(see, e.g., [22] and [15]). Some examples of SαS processes will be considered at the end of the section.

Let $X = (X(t);\ t \in T)$ be an SαS process with spectral representation as in (3.4). It follows from the continuity of the stochastic integral map $f \to \int f dZ$ that the representing functions $\{f(t, \cdot);\ t \in T\}$ are linearly dense in $L_\alpha(m)$, $\overline{\mathrm{sp}}\{f(t, \cdot);\ t \in T\} = L_\alpha(m)$, if and only if $\mathscr{L}(X) = \mathscr{L}(Z)$. Processes satisfying this condition will be said to have an *invertible spectral representation* or, more simply, to be *invertible*. Gaussian processes are invertible [1]. For non-Gaussian SαS processes this is not generally true [5]. Conditions for invertible representation are given in [3] and [5]. SαS processes with invertible representation in $L_2([0, 1])$ are considered in [30].

Let $X_i = (X_i(t);\ t \in T)$, $i = 1, 2$, be two invertible Sα_iS processes with spectral representations $X_i(t) = \int_I f(t, u) Z_i(du)$, where Z_i are independently scattered Sα_iS measures with control measures m_i and $f(\cdot, t) \in L_{\alpha_1}(m_1) \cap L_{\alpha_2}(m_2)$, $t \in T$. X_1 and X_2 will be called *simultaneously invertible* if for each $B \in \mathscr{I}$ there exist $N_n(B)$, $a_{n1}(B), \ldots, a_{nN_n(B)}(B)$, $t_{n1}(B), \ldots, t_{nN_n(B)}(B)$ such that

$$\sum_{k=1}^{N_n(B)} a_{nk} f(t_{nk}(B), \cdot) \to 1_B(\cdot) \quad \text{as } n \to \infty$$

in $L_{\alpha_i}(m_i)$ for both $i = 1, 2$. For example X_1 and X_2 are simultaneously invertible if they are invertible, and either $\alpha_1 = \alpha_2$ and dm_1/dm_2 is bounded above or below, or their associated random measures Z_1 and Z_2 are equivalent (cf. Proposition 3.1). The simultaneous invertibility of X_1 and X_2 allows for the study of the equivalence and singularity of μ_1, μ_2 in terms of that of Z_1, Z_2. Indeed, $X_i(t) = \int f(t, u) Z_i(du)$ is, roughly speaking, $X_i = L(Z_i)$, where L is a linear map from $\mathscr{L}(Z_i)$ into $\mathscr{L}(X_i)$. Simultaneous invertibility is like having $Z_i = L^{-1}(X_i)$, so the singularity of Z_1, Z_2 should imply the singularity of X_1 and X_2, and vice-versa for equivalence. The next proposition makes this precise.

PROPOSITION 3.2. *Let $X_i = (X_i(t);\ t \in T)$ be two simultaneously invertible* Sα_iS *processes with $\alpha_i \in (0, 2]$ and spectral representations $X_i(t) = \int_I f(t, u) Z_i(du)$, where Z_i are independently scattered* Sα_iS *measures with control measures m_i which are not purely discrete with a finite number of atoms. Then μ_{X_1} and μ_{X_2} are either equivalent or singular, and*

$$\mu_{X_1} \sim \mu_{X_2} \Leftrightarrow \mu_{Z_1} \sim \mu_{Z_2}, \qquad \mu_{X_1} \perp \mu_{X_2} \Leftrightarrow \mu_{Z_1} \perp \mu_{Z_2},$$

i.e., $\mu_{X_1} \sim \mu_{X_2}$ if and only if conditions (i)–(iii) *of Proposition* 3.1 *are satisfied, and otherwise $\mu_{X_1} \perp \mu_{X_2}$.*

Proof. For $B \in \mathscr{I}$ we can define

$$\Phi_n(B, x) = \sum_{k=1}^{N_n(B)} a_{nk}(B) x(t_{nk}(B)), \qquad x \in F^T,$$

so that $\Phi_n(B, X_i(\cdot, \omega)) \to Z_i(B, \omega)$ in probability as $n \to \infty$, $i = 1, 2$. Let $(\Phi_{n_k}(B, \cdot);\ k \in N)$ be a subsequence converging a.s. (μ_i), $i = 1, 2$, and put

$$\tilde{Z}(B) = \tilde{Z}(B, \cdot) = \liminf_{k\to\infty} \Phi_{n_k}(B, \cdot) 1_{\{x;\Phi_{n_k}(x)\ \text{converges}\}}(\cdot).$$

Hence $\tilde{Z}(B, X_i(\cdot, \omega)) = Z_i(B, \omega)$ a.s., $i = 1, 2$. The stochastic process $\tilde{Z} = (\tilde{Z}(B), B \in \mathscr{I})$ defined on $(F^T, \mathscr{C})$ is an independently scattered $S\alpha_i S$ measure with control measure m_i under μ_{X_i}. If we also denote by $\tilde{Z}$ the map $x \to \tilde{Z}(\cdot, x)$, then

$$\mu_{X_1} \sim \mu_{X_2} \Rightarrow \mu_{X_1}\tilde{Z}^{-1} \sim \mu_{X_2}\tilde{Z}^{-1} \text{ (i.e. } \mu_{Z_1} \sim \mu_{Z_2})$$

and

$$\mu_{Z_1} \perp \mu_{Z_2} \text{ (i.e. } \mu_{X_1}\tilde{Z}^{-1} \perp \mu_{X_2}\tilde{Z}^{-1}) \Rightarrow \mu_{X_1} \perp \mu_{X_2}.$$

On the other hand, if $\mu_{Z_1} \sim \mu_{Z_2}$, i.e., $\mu_{X_1}\tilde{Z}^{-1} \sim \mu_{X_2}\tilde{Z}^{-1}$, it follows that (i)–(iii) of Proposition 3.1 hold. Thus, we can construct independent processes $\tilde{X}_d$ and $\tilde{X}_{ia}$ on $(F^{\mathscr{I}}, \mathscr{C}(F^{\mathscr{I}}), \mu_{Z_i})$ such that

$$X_i \stackrel{L}{=} \tilde{X}_d + \tilde{X}_{ia}, \quad i = 1, 2,$$

with $\mu_{\tilde{X}_{1a}} \sim \mu_{\tilde{X}_{2a}}$. Since $\mu_{X_1} = \mu_{\tilde{X}_d} * \mu_{\tilde{X}_{ia}}$, we have $\mu_{X_1} \sim \mu_{X_2}$.

Now, if μ_{X_1} and μ_{X_2} are not equivalent, it follows that $\mu_{Z_1} \perp \mu_{Z_2}$ (since otherwise $\mu_{Z_1} \sim \mu_{Z_2}$, which implies $\mu_{X_1} \sim \mu_{X_2}$, i.e., a contradiction), and this was shown to imply $\mu_{X_1} \perp \mu_{X_2}$. ■

It follows from Proposition 3.2 that simultaneously invertible processes are singular whenever their indexes of stability are different. This is not generally true for symmetric stable processes with different indexes of stability. Indeed, let $G = (G(t);\ t \in T)$ be a Gaussian process, and for $i = 1, 2$ let A_i be a standard positive $(\alpha_i/2)$-stable random variable where $\alpha_1 \neq \alpha_2$, and consider the sub-Gaussian $S\alpha_i S$ processes

$$X_i = (X_i(t) = A_i^{1/2} G(t);\ t \in T).$$

We have $\mu_{X_i}(B) = \int_{R^+} \mu_{xG}(B)\mu_{A_i}(dx)$. Since the distribution μ_{A_i} of A_i has positive density in R^+, we get $\mu_{A_1} \sim \mu_{A_2}$, so that, by the Corollary to Theorem 18.1 in [26], $\mu_{X_1} \sim \mu_{X_2}$. Since the linear space of a sub-Gaussian process does not contain (non-degenerate) independent random variables (see [5]), sub-Gaussian processes are not invertible (nor simultaneously invertible). Further examples of symmetric stable processes with different indexes of stability which are equivalent are

$$X_i = \left(X_i(t) = \sum_{n=1}^{N} A_{in}^{1/2} G_n(t);\ t \in T\right),$$

where for each $i = 1, 2$ the vector $(A_{i1}, \ldots, A_{iN})$ is positive $(\alpha_i/2)$-stable, independent of the mutually independent Gaussian processes $G_n = (G_n(t);\ t \in T)$, $n = 1, \ldots, N$.

As a consequence of Proposition 3.2, harmonizable processes are either equivalent or singular and necessary and sufficient conditions for the two alternatives are provided.

COROLLARY 3.3. *Let* $X_k = (X_k(t); t \in T)$, $k = 1, 2$, *be two harmonizable* $S\alpha_k S$ *processes, with* $\alpha_k \in (0, 2]$, *i.e.*,

$$X_k(t) = \int_I \exp(\langle i\langle t, u\rangle) Z_k(du), \qquad t \in T,$$

where $I = R^d$, *respectively* $[-\pi, \pi]^d$, *for* $T = R^d$, *respectively* Z^d, *and* Z_k *are independently scattered* $S\alpha_k S$ *measures with finite spectral measures* m_k *which are not purely discrete with a finite number of atoms. Then* μ_{X_1} *and* μ_{X_2} *are equivalent if and only if* (i)–(iii) *of Proposition* 3.1 *are satisfied, and they are singular otherwise.*

Proof. Clearly, X_1 and X_2 are simultaneously invertible, since indicator functions can be approximated uniformly, and hence in $L^{\alpha_k}(m_k)$, by linear combinations of the functions $f(t, u) = \exp(i\langle t, u\rangle)$. Hence the result follows from Proposition 3.2. ∎

As a special case, let S and N be harmonizable $S\alpha S$ signal and noise processes as in Corollary 3.3, that are independent of each other. Then μ_{S+N} and μ_N are equivalent if any only if $m_{S,d} = 0$, the atoms of m_S are atoms of m_N, and

$$\sum_n \left[\frac{m_S(\{a_n\})}{m_S(\{a_n\}) + m_N(\{a_n\})}\right]^2 < \infty.$$

Otherwise, μ_{S+N} and μ_N are singular, and the presence of the random signal S in the additive noise N can be detected with probability one (at least in principle). In particular, μ_{S+N} and μ_N are singular when the signal has continuous spectrum or the noise has no atomic spectrum. (Similar results hold when the signal and noise processes have simultaneously invertible representations as in Proposition 3.2.)

The results in Propositions 3.1 and 3.2 and Corollary 3.3 are identical in the non-Gaussian stable case and in the Gaussian case [6]. However, in the case of Corollary 3.3 much more is known for Gaussian processes. Namely, for stationary Gaussian processes ($d = 1$) restricted over a *finite* interval, the equivalence-singularity dichotomy prevails and necessary and sufficient conditions for the two alternatives are known (see, e.g., [17]). Both of these important questions remain open in the non-Gaussian stable case.

Another consequence of Proposition 3.2 is the singularity of multiples of invertible processes.

COROLLARY 3.4. *Let* $X = (X(t);\ t \in T)$ *be an invertible* $S\alpha S$ *process with* $\alpha \in (0, 2]$ *and control measure m which is not purely atomic with a finite number of atoms. Then* X *and* bX *are singular wherever* $|b| \neq 1$.

Proof. If $X(t) = \int f(t, u)Z(du)$, where Z has control measure m, then $bX(t) = \int f(t, u)Z_b(du)$, where $Z_b = bZ$ has control measure $|b|^\alpha m$. Clearly, X and bX are simultaneously invertible and the result follows from Proposition 3.2. ∎

The result in Corollary 3.4 is known to hold for every Gaussian process with infinite dimensional linear space. Here again the class of SαS sub-Gaussian processes provides an example to show that the result is not true for all infinite dimensional SαS processes. In fact, if $X = (A^{1/2}G(t);\, t \in T)$, as before, for each $b > 0$ we have

$$\mu_{bX}(B) = \int_{R^+} \mu_{xG}(B)\mu_{bA}(dx).$$

The distributions μ_A and μ_{bA} are equivalent for all $b > 0$ so that $\mu_X \sim \mu_{bX}$.

In the Gaussian case the multiple b in Corollary 3.4 is allowed to be a function $b(t)$, but this problem remains open in the non-Gaussian stable case. Corollary 3.4 is relevant to the detection of a constant signal in multiplicative noise (see [2]).

4. REMARKS ON SINGULARITY AND ABSOLUTE CONTINUITY OF p^{th} ORDER AND SαS PROCESSES

For two Gaussian processes, the setwise equality of their RKHS's is a necessary condition for equivalence. For two second order processes a necessary condition for absolute continuity and a sufficient condition for singularity in terms of their RKHS's are proved in [12]. We show that these results remain true for SαS processes and for p^{th} order processes with $1 < p < 2$, respectively, with the RKHS replaced by an appropriate function space $\mathscr{F}$ specified in the sequel.

The function space of a p^{th} order process $X = (X(t);\, t \in T)$ is defined in [24] by

$$\mathscr{F} = \left\{ s\colon\ T \to F;\ \|s\|_{\mathscr{F}} \triangleq \sup_{\substack{a_1,\ldots,a_N \\ t_1,\ldots,t_N}} \frac{\left|\sum_{n=1}^{N} a_n s(t_n)\right|}{\left\|\sum_{n=1}^{N} a_n X_i(t_n)\right\|_{L_p(P)}^{1/p}} < \infty \right\}.$$

Note that when $p = 2$, $\mathscr{F} = \text{RKHS}$. If $X_i = (X_i(t);\, t \in T)$, $i = 1, 2$, are two p^{th} order processes, we say that X_1 *dominates* X_2 if there exists $0 < K < \infty$ such that for all $N \in N$, $a_1, \ldots, a_N \in R^1$ and $t_1, \ldots, t_N \in T$,

$$\left\| \sum_{n=1}^{N} a_n X_2(t_n) \right\|_{L_p(P)} \leqslant K \left\| \sum_{n=1}^{N} a_n X_1(t_n) \right\|_{L_p(P)}.$$

The relationship between domination and the function spaces is clarified in the following

PROPOSITION 4.1. *Let $X_i = (X_i(t);\, t \in T)$ be p^{th} order processes with function space $\mathscr{F}_i$, $i = 1, 2$.*

(i) *If* X_1 *dominates* X_2, *then* $\mathscr{F}_2 \subset \mathscr{F}_1$.

(ii) X_1 *dominates* X_2 *if and only if there exists a bounded linear transformation* $V\colon \mathscr{L}(X_1) \to \mathscr{L}(X_2)$ *satisfying* $V(X_1(t)) = X_2(t)$, $t \in T$. *Consequently, if* X_1 *dominates* X_2 *and vice versa, then* $\mathscr{F}_1 = \mathscr{F}_2$ *(setwise),* $\|\cdot\|_{\mathscr{F}_1}$ *and* $\|\cdot\|_{\mathscr{F}_2}$ *are equivalent, and the transformation* V *has bounded inverse.*

Proof. (i) If X_1 dominates X_2, it follows that, for all functions s, $\|s\|_{\mathscr{F}_1} \leqslant K\|s\|_{\mathscr{F}_2}$, and thus $\mathscr{F}_2 \subset \mathscr{F}_1$.

(ii) Let $V\colon \mathscr{L}(X_1) \to \mathscr{L}(X_2)$ be defined by

$$V\Big(\sum_{n=1}^{N} a_n X_1(t_n)\Big) = \sum_{n=1}^{N} a_n X_2(t_n).$$

It is clear that V is a well-defined bounded linear transformation and as such it can be extended to $\mathscr{L}(X_1)$ if and only if X_1 dominates X_2. ∎

For SαS processes, the next proposition shows that mutual domination is a necessary condition for absolute continuity, i.e., non-domination is a sufficient condition for singularity. This proposition is a stochastic process version of Proposition 7 in [30].

PROPOSITION 4.2. *Let* $X_i = (X_i(t);\, t \in T)$, $i = 1, 2$, *be two* Sα_iS *processes. If* μ_1 *and* μ_2 *are not singular, then* X_1 *dominates* X_2, X_2 *dominates* X_1, *and* $\mathscr{F}_1 = \mathscr{F}_2$. *Equivalently, if* $\mathscr{F}_1 \neq \mathscr{F}_2$, *then either* X_1 *does not dominate* X_2 *or* X_2 *does not dominate* X_1 *and* $\mu_1 \perp \mu_2$.

Proof. Since for $Y \in \mathscr{L}(X_i)$, $\|Y\|_{L_p(P)} = C_{p,\alpha_i}\|Y\|_{\alpha_i}$ (see [4]), X_1 dominates X_2 if and only if

$$\Big\|\sum_{n=1}^{N} a_n X_2(t_n)\Big\|_{\alpha_2} \leqslant K \Big\|\sum_{n=1}^{N} a_n X_1(t_n)\Big\|_{\alpha_1}.$$

Assume X_1 does not dominate X_2. Then for any positive sequence $K_n \to \infty$, as $n \to \infty$, there exist

$$Y_n^{(i)} = \sum_{k=1}^{N_n} a_{n,k} X_i(t_{n,k}), \quad i = 1, 2,$$

such that $\|Y_n^{(2)}\|_{\alpha_2} \geqslant K_n \|Y_n^{(1)}\|_{\alpha_1}$, $n = 1, 2, \ldots$ Without loss of generality we can assume $\|Y_n^{(1)}\|_{\alpha_2} = 1$ for all n. Thus

$$\|Y_n^{(1)}\|_{\alpha_2} \leqslant 1/K_n \to 0 \quad \text{as } n \to \infty.$$

Now consider the sequence of random variables $(Y_n;\, n \in N)$ defined on $(F^T, \mathscr{C})$ by

$$Y_n(x) = \sum_{n=1}^{N} a_{n,k} x(t_{n,k}), \quad x \in F^T.$$

It follows that

$$\int_{F^T} \exp(iuY_n)\,d\mu_1 = \exp(-\|Y_n^{(1)}\|_{\alpha_1}^{\alpha_1}|u|^{\alpha_1}) \to 1 \quad \text{as } n \to \infty.$$

Hence a subsequence $(Y_{n_k};\ k \in N)$ can be chosen such that if $C_0 = \{x;\ Y_{n_k}(x) \to 0$ as $k \to \infty\}$, then $\mu_{X_1}(C_0) = 1$. Clearly, C_0 is a measurable linear subspace of F^T and, since μ_2 is an $S\alpha_2S$ measure of $(F^T, \mathscr{C})$, it follows by the zero-one law for stable measures [8] that $\mu_2(C_0) = 0$ or 1. On the other hand,

$$\int_{F^T} \exp(iuY_{n_k})d\mu_2 = \exp(-\|Y_{n_k}\|_{\alpha_2}^{\alpha_2}|u|^{\alpha_2}) = \exp(-|u|^{\alpha_2}),$$

which implies that $\mu_2(C_0) = 0$, and thus $\mu_1 \perp \mu_2$. ∎

The crucial result used in the proof of Proposition 4.2 is the zero-one law, which is not available for general p^{th} order processes. However, the proposition has some partial analogs for certain p^{th} order processes.

As in [12] we call a p^{th} order process $X = (X(t);\ t \in T)$ *non-reduced* if there exists some $\varepsilon \in (0, 1]$ such that, for all countable subsets T_0 of T, $P(\{\omega;\ X(t, \omega) = 0,\ t \in T_0\}) \geq \varepsilon$; otherwise, X is called *reduced*. Non-trivial $S\alpha S$ processes are reduced. When X is separable and T an interval of the real line, it is shown in [12] that X is reduced if and only if $P(\{X(t) = 0,\ t \in T\}) = 0$, and non-reduced if and only if $P(\{X(t) = 0,\ t \in T\}) \geq \varepsilon$ for some $\varepsilon \in (0, 1]$.

Next we generalize to p^{th} order processes with $1 < p < 2$ the results in [12], Théorèmes (3.2) and (3.3.2). The proof is essentially identical to Fortet's and is presented in a shorter form.

PROPOSITION 4.3. *Let $X_i = (X_i(t);\ t \in T)$ be a p^{th} order process with $1 < p < 2$ and function space $\mathscr{F}_i$, $i = 1, 2$.*

(i) *If $\mu_2 \ll \mu_1$, then $\mathscr{F}_1 \cap \mathscr{F}_2$ is dense in $\mathscr{F}_2$.*

(ii) *If either X_1 or X_2 is reduced and $\mathscr{F}_1 \cap \mathscr{F}_2 = \{0\}$, then $\mu_1 \perp \mu_2$.*

Proof. (i) Fix $s \in \mathscr{F}_2$. By Proposition 1 in [24] we have

$$s(t) = \mathrm{E}(X_2(t) Y^{\langle p-1 \rangle}) = \int_{F^T} x(t) a(x)^{\langle p-1 \rangle} \mu_{X_2}(dx),$$

where $z^{\langle q \rangle} = |z|^{q-1}\bar{z}$, $Y \in \mathscr{L}(X_2)$ and $a(x)$ is a representation of Y in $L_p(\mu_1) - \overline{\mathrm{sp}}\{x(t);\ t \in T\} \subset F^T$, $Y(\omega) = a(X(\cdot, \omega))$. Let

$$\mu_2(E) = \int_E g d\mu_1 + \mu_2(E \cap N)$$

be the Lebesgue decomposition of μ_2 with respect to μ_1. Define

$$E_n = \{x;\ 0 < g(x) \leq n\} \cap N^c$$

and

$$s_n(t) = \int_{F^T} x(t) a(x)^{\langle p-1 \rangle} 1_{E_n}(x) \mu_2(dx) = \int_{F^T} x(t) a(x)^{\langle p-1 \rangle} g(x) 1_{E_n}(x) \mu_1(dx).$$

Since $a^{\langle p-1 \rangle} 1_{E_n} \in L_{p^*}(\mu_2)$ and $a^{\langle p-1 \rangle} g 1_{E_n} \in L_{p^*}(\mu_1)$, we have $s_n \in \mathscr{F}_1 \cap \mathscr{F}_2$. Also

$$\Big|\sum_{k=1}^{K} c_k (s - s_n)(t_k)\Big| \leq \Big[\int_{F^T} \Big|\sum_{k=1}^{K} c_k x(t_k)\Big|^p d\mu_2\Big]^{1/p} \Big[\int_{F^T} |a^{\langle p-1 \rangle}|^{p^*} 1_{E_n^c} d\mu_2\Big]^{1/p^*}.$$

Thus

$$\|s-s_n\|_{\mathscr{F}_2}^{p*} \leqslant \int_{E_n^c} |a^{\langle p-1\rangle}|^{p*} d\mu_2 = \int_{\{g\geqslant n\}} |a|^p g d\mu_1 \to 0 \quad \text{as } n\to\infty,$$

i.e., $\mathscr{F}_1 \cap \mathscr{F}_2$ is dense in $\mathscr{F}_2$.

(ii) For a fixed $t_0 \in T$, let $a_0(x) = x(t_0)$ and define

$$s_0(t) = \int_{F^T} x(t) a_0(x)^{\langle p-1\rangle} \mu_2(dx).$$

By Proposition 1 in [24], $s_0 \in \mathscr{F}_2$, since $a_0(x) \in L_{p*}(\mu_2)$. Let

$$s_{0n}(t) = \int_{F^T} x(t) a_0(x)^{\langle p-1\rangle} 1_{E_n}(x) \mu_2(dx) = \int_{F^T} x(t) a_0(x)^{\langle p-1\rangle} g(x) 1_{E_n}(x) \mu_1(dx),$$

so that $s_{0n} \in \mathscr{F}_1 \cap \mathscr{F}_2$. Since $\mathscr{F}_1 \cap \mathscr{F}_2 = \{0\}$, $s_{0n} \equiv 0$, i.e., $s_{0n}(t) = 0$ for all $t\in T$. In particular,

$$s_{0n}(t_0) = \int_{\{0<g<n\}} |x(t_0)|^p g(x) \mu_1(dx) = 0 \quad \text{for } n = 1, 2, \ldots,$$

and hence

$$\int_{\{0<g<\infty\}} |x(t_0)|^p g(x) \mu_1(dx) = 0.$$

Consequently, since $t_0 \in T$ is arbitrary, we have $x(t) = 0$ a.e. (μ_1) on $\{0<g<\infty\}$ for each $t\in T$. But this implies that X_1 is non-reduced if

$$\mu_1(\{x;\ x(t)=0,\ t\in T\}) \geqslant \mu_1(\{x;\ 0<g(x)<\infty\}).$$

On the other hand, if $\mu_1(\{x;\ 0<g(x)<\infty\}) > 0$, then $x(t) = 0$ a.e. $(g\mu_1)$ for each t and $\int_{\{0<g<\infty\}} g d\mu_1 > 0$. Hence

$$\mu_2(\{x;\ x(t)=0,\ t\in T_0\}) \geqslant \int_{\{0<g<\infty\}} g d\mu_1 > 0,$$

i.e., X_2 is non-reduced. Since either X_1 or X_2 is reduced, we must have

$$\mu_1(\{x;\ 0<g(x)<\infty\}) = 0, \quad \text{i.e.,} \quad \mu_1 \perp \mu_2. \ \blacksquare$$

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