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LIMIT LAWS FOR GENERALIZED CONVOLUTIONS

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Abstract. The paper deals with stochastically compact sequences of scalar modifications of powers of probability measures taken in the sense of a generalized convolution. Our aim is to give a characterization of all possible limit laws for these sequences.

1. Notation and preliminaries. In this paper we adopt definitions and notation given in [3] and [4]. In particular, $\boldsymbol{P}$ will denote the space of all Borel probability measures defined on the positive half-line $[0, \infty)$. The space $\boldsymbol{P}$ is endowed with the topology of weak convergence. For any $a \in (0, \infty)$, T_a will denote the scale change $(T_a \mu)(E) = \mu(a^{-1} E)$ $(\mu \in \boldsymbol{P})$. Further, δ_c will denote the probability measure concentrated at the point c. A continuous commutative and associative $\boldsymbol{P}$-valued binary operation $\circ$ on $\boldsymbol{P}$ is called a *generalized convolution* if it is distributive with respect to the convex combinations of measures and the operations T_a $(a > 0)$, δ_0 is its unit element and an analogue of the law of large numbers is fulfilled:

$$T_{c_n} \delta_1^{\circ n} \to \gamma \neq \delta_0 \tag{1.1}$$

for a choice of a norming sequence c_n of positive numbers. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$. The limit measure γ is uniquely defined up to a scale change and is called the *characteristic measure* of the generalized convolution in question. Generalized convolution algebras admitting a non-constant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations are called *regular*. All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular. Moreover, we shall always assume that the characteristic measure γ has finite q-th moment, where q denotes the characteristic exponent of the generalized convolution in question. It has been proved in [3], Theorem 6, that the convolution $\circ$ admits the characteristic function, i.e., the map $\mu \to \hat{\mu}$ from $\boldsymbol{P}$ into the set of continuous bounded real-valued functions commuting with convex combinations and scale change and fulfilling the condition $(\mu \circ \nu)^\wedge = \hat{\mu}\hat{\nu}$. Moreover, the characteristic function is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx) \mu(dx)$$

with a continuous kernel Ω satisfying the conditions $|\Omega(t)| \leqslant 1$ and

$$\Omega(t) = 1-t^q L(t),$$

where the function L is continuous at the origin and $L(0) > 0$.

A measure μ from $\boldsymbol{P}$ is said to be *infinitely divisible* if for every positive integer n there exists a measure $\mu_n \in \boldsymbol{P}$ such that $\mu = \mu_n^{\circ n}$. By the condition $L(0) > 0$ and Theorem 13 in [3] the family of the characteristic functions of infinitely divisible measures μ from $\boldsymbol{P}$ coincides with the family of functions

$$\hat{\mu}(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{m(x)} M(dx), \tag{1.2}$$

where $m(x) = \min(1, x^q)$ and M runs over all finite Borel measures on the positive half-line $[0, \infty)$. The integrand is defined as its limiting value $-L(0)$ when $x = 0$. Changing the scale if necessary we may assume without loss of generality that $L(0) = 1$. In what follows the measure μ with the characteristic function given by formula (1.2) will be denoted by $e(M)$.

The paper [5] has been devoted to the study of limit sets consisting of cluster points of normalized powers under a generalized convolution of probability measures. The family $\boldsymbol{F}$ consists of all cluster points of sequences $T_{a_n}\lambda^{\circ n}$, where $a_n > 0$, $\lambda \in \boldsymbol{P}$, the sequence $T_{a_n}\lambda^{\circ n}$ is conditionally compact and all its cluster points are non-degenerate laws. It is clear by Theorem 12 in [3] that the family $\boldsymbol{F}$ is contained in the family of infinitely divisible probability measures. Put $\boldsymbol{H} = \{e(M)\colon M \in \boldsymbol{F}\}$. For ordinary convolution a nice analytic characterization of the family $\boldsymbol{H}$ has been given by Pruitt in [2]. Our aim is to extend this result to the case of generalized convolutions. Namely, we shall prove the following statement:

THEOREM. *A measure M belongs to $\boldsymbol{H}$ if and only if it does not vanish identically and there exists a positive number c such that*

$$x^q \int_0^\infty \frac{M(dy)}{m(y)} \leqslant c \int_0^x \frac{y^q}{m(y)} M(dy) \tag{1.3}$$

for all $x \in (0, \infty)$.

The proof of the necessity of the condition is in the next section. The final section contains the proof of the sufficiency.

2. The necessity of the condition. Suppose that $M \in \boldsymbol{H}$. There exist then a measure $\lambda \in \boldsymbol{P}$ and a sequence a_n of positive numbers such that the sequence of measures $T_{a_n}\lambda^{\circ n}$ is conditionally compact and its set of cluster points consists of non-degenerate measures and contains the measure $e(M)$. By Theorem 4.1 in [5] we infer that

$$x^q \lambda((x, \infty)) \leqslant c \int_0^x y^q \lambda(dy) \tag{2.1}$$

for some $c > 0$, $x_0 > 0$ and all $x \geqslant x_0$. Put

$$M_n(E) = n \int_E m(y) T_{a_n} \lambda(dy) \quad (n = 1, 2, \ldots). \tag{2.2}$$

By the Jurek Theorem on accompanying laws in [1] we conclude that the measure M is a cluster point of the sequence M_n. Suppose that $M_{n_k} \to M$. Further, suppose that x is not an atom of the limiting measure M. Then

$$\int_{x+}^{\infty} \frac{M_{n_k}(dy)}{m(y)} \to \int_{x+}^{\infty} \frac{M(dy)}{m(y)}$$

and

$$\int_0^x \frac{y^q}{m(y)} M_{n_k}(dy) \to \int_0^x \frac{y^q}{m(y)} M(dy) \quad \text{as } k \to \infty.$$

Taking into account the definition (2.2) we have

$$\int_{x+}^{\infty} \frac{M_{n_k}(dy)}{m(y)} = n_k \lambda\big((x a_{n_k}^{-1}, \infty)\big)$$

and

$$\int_0^x \frac{y^q}{m(y)} M_{n_k}(dy) = n_k a_{n_k}^q \int_0^{x a_{n_k}^{-1}} y^q \lambda(dy),$$

which, by (2.1), yields inequality (1.3) for all positive x which are not atoms of the measure M. The general case follows from the continuity on the right of both sides of inequality (1.3). This completes the proof of the necessity of our condition.

3. The sufficiency of the condition. We may restrict ourselves to the case of the measures M which are not concentrated at the origin. In fact, if M is concentrated at 0, then $e(M)$ is of the form $T_b \gamma$, where γ is the characteristic measure of the convolution in question. In this case the relation $M \in \boldsymbol{H}$ is a direct consequence of formula (1.1). In what follows we assume that inequality (1.3) is fulfilled and

$$M\big((0, \infty)\big) > 0. \tag{3.1}$$

We introduce auxiliary functions for $x \in (0, \infty)$

$$f(x) = x^q \int_{x+}^{\infty} \frac{M(dy)}{m(y)}, \qquad g(x) = \int_0^x \frac{y^q}{m(y)} M(dy)$$

and

$$h(x) = f(x) + g(x) = \int_0^{\infty} \frac{\min(x^q, y^q)}{m(y)} M(dy).$$

It is easy to verify the following formulae:

$$\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x)x^{-q} = 0, \tag{3.2}$$

$$\lim_{x\to 0} f(x)x^{-q} = \int_{0+}^{\infty} \frac{M(dy)}{m(y)} > 0, \tag{3.3}$$

$$\lim_{x\to 0} g(x) = M(\{0\}), \tag{3.4}$$

$$\lim_{x\to\infty} g(x) = \int_0^\infty \max(1, y^q)M(dy) > M(\{0\}), \tag{3.5}$$

$$\lim_{x\to\infty} g(x)x^{-q} = 0. \tag{3.6}$$

Of course, inequality (1.3) can be written in the form

$$f(x) \leqslant cg(x) \quad \text{for } x\in(0, \infty). \tag{3.7}$$

Observe that, by (3.3) and (3.6),

$$\lim_{\substack{x\to 0\\ y\to\infty}} x^q y^{-q} f^{-1}(x)g(y) = 0.$$

Consequently, we can choose a pair r_1, s_1 of positive numbers fulfilling the conditions $r_1 < 1 < s_1$ and

$$d = \sup\{x^q y^{-q} f^{-1}(x)g(y)\colon x \leqslant r_1, y \geqslant s_1\} < 1. \tag{3.8}$$

We start the construction by choosing sequences $r_1 > r_2 > \ldots \to 0$ and $s_1 < s_2 < \ldots \to \infty$ and setting $A_0 = 1$,

$$A_n = \prod_{j=1}^{n} \frac{g(s_j)}{h(r_j)} \quad (n = 1, 2, \ldots).$$

Applying (3.2), (3.4) and (3.5) we get the formula

$$\lim_{n\to\infty} A_n = \infty. \tag{3.9}$$

Moreover, introducing the notation $B = M(\{0\})/\int_0^\infty \max(1, y^q)M(dy)$ we have, by (3.5),

$$\lim_{n\to\infty} A_n^{-1} \sum_{j=1}^{n-1} A_j = B/(1-B). \tag{3.10}$$

Next, we let $p_1 = r_1^{-1}$, $t_1 = p_1^{-q}h^{-1}(r_1)$ and, for $n > 1$,

$$p_n = \frac{s_1 s_2 \ldots s_{n-1}}{r_1 r_2 \ldots r_n}, \qquad t_n = p_n^{-q}h^{-1}(r_n)A_{n-1}.$$

Put $J_n = (r_n, s_n]$ and $I_n = p_n J_n$ $(n = 1, 2, \ldots)$. Since $r_n p_n = s_{n-1} p_{n-1}$, the intervals $I_1, I_2, \ldots$ are disjoint and

$$\bigcup_{n=1}^{\infty} I_n = (1, \infty).$$

Define the sequence $Q_1, Q_2, \ldots$ of measures on the half-line $[0, \infty)$ by setting

$$Q_n(E) = t_n \int_{p_n^{-1}E} 1_{J_n}(y) \frac{M(dy)}{m(y)}, \tag{3.11}$$

where 1_Z denotes the indicator of the set Z. It is clear that the measure Q_n is concentrated at the set I_n. We list a few facts for later reference. Given $u, v \in J_n$ fulfilling the condition $u < v$ we have the formula

$$Q_n((up_n, vp_n]) = t_n\big(u^{-q} f(u) - v^{-q} f(v)\big). \tag{3.12}$$

In particular, for $u = r_n$, $v = s_n$ we have

$$Q_n(I_n) = t_n\big(r_n^{-q} f(r_n) - s_n^{-q} f(s_n)\big), \tag{3.13}$$

which, by (3.2) and (3.3), yields

$$Q_n(I_n) > 0 \quad \text{for sufficiently large } n, \tag{3.14}$$

Further, for the same pair u, v we have the formula

$$\int_{up_n}^{vp_n} y^q Q_n(dy) = t_n p_n^q\big(g(v) - g(u)\big), \tag{3.15}$$

which yields

$$\int_{I_n} y^q Q_n(dy) = A_n - A_{n-1} \frac{g(r_n)}{h(r_n)}. \tag{3.16}$$

Put $w_n = t_n r_n^{-q} h(r_n)$ $(n = 1, 2, \ldots)$. Taking into account notation (3.8) we obtain the inequality

$$\frac{w_{n+1}}{w_n} = r_n^q s_n^{-q} h^{-1}(r_n) g(s_n) \leqslant d \quad (n = 1, 2, \ldots),$$

which yields

$$\sum_{n=k}^{\infty} t_n r_n^{-q} h(r_n) \leqslant (1-d)^{-1} t_k r_k^{-q} h(r_k) \quad (k = 1, 2, \ldots). \tag{3.17}$$

Observe that, by (3.13), $Q_n(I_n) \leqslant t_n r_n^{-q} h(r_n)$, which, by (3.17), yields $\sum_{n=1}^{\infty} Q_n(I_n) < \infty$. Moreover, by (3.14), $\sum_{n=1}^{\infty} Q_n(I_n) > 0$. Now we may define a probability measure λ by setting

$$\lambda = b \sum_{n=1}^{\infty} Q_n, \quad \text{where } b^{-1} = \sum_{n=1}^{\infty} Q_n(I_n).$$

To prove the sufficiency of our condition it is enough to show that the measure λ fulfils condition (2.1) for a certain constant c and belongs to the domain of attraction of the measure $e(M)$.

First we shall prove the inequality

$$\lambda((x, \infty)) \leqslant b(2-d)(1-d)^{-1} t_k x^{-q} p_k^q h(xp_k^{-1}) \tag{3.18}$$

for $x \in I_k$ $(k = 1, 2, \ldots)$. From the formula

$$\lambda((x, \infty)) = b \sum_{n=k+1}^{\infty} Q_n(I_n) + bQ_k((x, s_k p_k])$$

by virtue of (3.12), (3.13) and (3.17) we get the inequality

$$\begin{aligned}\lambda((x, \infty)) &\leqslant b \sum_{n=k+1}^{\infty} t_n r_n^{-q} h(r_n) + bt_k x^{-q} p_k^q h(xp_k^{-1}) \\ &\leqslant b(1-d)^{-1} t_{k+1} r_{k+1}^{-q} h(r_{k+1}) + bt_k x^{-q} p_k^q h(xp_k^{-1}).\end{aligned}$$

Since

$$t_{k+1} \leqslant t_k r_{k+1}^q s_k^{-q} h(s_k) h^{-1}(r_{k+1})$$

and the function $x^{-q}h(x)$ is monotone non-increasing, we have the inequality

$$t_{k+1} \leqslant t_k r_{k+1}^q x^{-q} p_k^q h(xp_k^{-1}) h^{-1}(r_{k+1}),$$

which yields (3.18).

Further, for any $x \in I_k$ $(k = 1, 2, \ldots)$ the formula

$$\int_0^x y^q \lambda(dy) = b \sum_{n=1}^{k-1} \int_{I_n} y^q Q_n(dy) + b \int_{r_k p_k}^{x} y^q Q_k(dy)$$

together with (3.15) and (3.16) yields

$$\begin{aligned}\int_0^x y^q \lambda(dy) &\geqslant b \sum_{n=1}^{k-1} (A_n - A_{n-1}) + bt_k p_k^q g(xp_k^{-1}) - bA_{k-1} \\ &= bt_k p_k^q g(xp_k^{-1}) - b.\end{aligned}$$

Taking into account (3.7) and the inequality

$$t_k p_k^q h(xp_k^{-1}) \geqslant t_k p_k^q h(r_k) = A_{k-1}$$

we conclude, by (3.9), that for sufficiently large k the inequality

$$\int_0^x y^q \lambda(dy) \geqslant \frac{b}{2(1+c)} t_k p_k^q h(xp_k^{-1}) \tag{3.19}$$

holds. Comparing this with (3.18) we obtain the inequality

$$x^q \lambda((x, \infty)) \leqslant 2(1+c)(2-d)(1-d)^{-1} \int_0^x y^q \lambda(dy)$$

for sufficiently large x. This shows, by Theorem 4.1 in [5], that there exists a sequence of norming constants a_n such that the sequence $T_{a_n}\lambda^{\circ n}$ is conditionally compact and all its cluster points are non-degenerate.

Put $n_k = [b^{-1}t_k^{-1}]$ and $b_k = p_k^{-1}$ $(k = 1, 2, \ldots)$, where the square brackets denote the integer part. Since, by (3.8),

$$t_k \leqslant d^{k-1} r_k^q h^{-1}(r_k)$$

and, by (3.3) and (3.6),

$$\lim_{x\to 0} \frac{h(x)}{x^q} = \int_{0+}^{\infty} \frac{M(dy)}{m(y)} > 0,$$

we infer that $t_k \to 0$ and, consequently, $n_k \to \infty$. Using Lemma 1.1 from [4] to prove that $M \in H$ it suffices to show that $T_{b_k}\lambda^{\circ n_k} \to e(M)$ as $k \to \infty$. By the Jurek Theorem on accompanying laws in [1] the last statement is equivalent to $M_k \to M$, where

$$M_k(E) = n_k \int_E m(y) T_{b_k}\lambda(dy) \quad (k = 1, 2, \ldots).$$

It is evident that the inequality $n < k$ yields $s_n p_n < r_k p_k$. Consequently, $p_k^{-1} I_n \subset (0, r_k] \subset (0, 1]$. Hence and from (3.16) we get the formula

$$\int_{I_n} m(yp_k^{-1}) Q_n(dy) = p_k^{-q}(A_n - A_{n-1} g(r_n) h^{-1}(r_n)).$$

Using this formula and applying (3.2), (3.4) and (3.10) we get

$$(3.20)\qquad t_k^{-1} \sum_{n=1}^{k-1} \int_{I_n} m(yp_k^{-1}) Q_n(dy)$$
$$= h(r_k) A_{k-1}^{-1} \sum_{n=1}^{k-1} (A_n - A_{n-1} g(r_n) h^{-1}(r_n)) \to M(\{0\}) \quad \text{as } k \to \infty.$$

Observe that the inequality $n > k$ yields $r_n p_n > s_k p_k$. Hence $p_k^{-1} I_n \subset (s_k, \infty) \subset (1, \infty)$. Consequently,

$$\int_{I_n} m(yp_k^{-1}) Q_n(dy) = Q_n(I_n).$$

Now, applying (3.13) and (3.17) we get the inequality

$$t_k^{-1} \sum_{n=k+1}^{\infty} \int_{I_n} m(yp_k^{-1}) Q_n(dy) \leqslant t_k^{-1}(1-d)^{-1} t_{k+1} r_{k+1}^{-q} h(r_{k+1})$$
$$= (1-d)^{-1} s_k^{-q} g(s_n),$$

which, by (3.6), yields

$$(3.21)\qquad t_k^{-1} \sum_{n=k+1}^{\infty} \int_{I_n} m(yp_k^{-1}) Q_n(dy) \to 0 \quad \text{as } k \to \infty.$$

Finally, we note that by (3.11)

$$t_k^{-1} \int_{I_k} m(yp_k^{-1}) Q_k(dy) = M(J_k) \to M((0, \infty)) \quad \text{as } k \to \infty. \tag{3.22}$$

Moreover, for any $x \in J_k$ being a continuity point of M

$$t_k^{-1} \int_{r_k p_k +}^{x p_k} m(yp_k^{-1}) Q_k(dy) = M((r_k, x]) \to M((0, x]) \quad \text{as } k \to \infty. \tag{3.23}$$

Formulae (3.20)–(3.22) yield the relation

$$M_k([0, \infty)) \to M([0, \infty)).$$

Furthermore, (3.20), (3.21) and (3.23) imply

$$M_k([0, x)) \to M([0, x))$$

for any $x \in (0, \infty)$ being a continuity point of the limiting measure M. This shows that $M_k \to M$, which completes the proof of the sufficiency of the condition in question. The Theorem is thus proved.

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