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ON AN INVARIANCE PRINCIPLE FOR UNIFORMLY STRONG MIXING STATIONARY SEQUENCES WHEN $\mathscr{E}X^2 = \infty$

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Abstract. We prove that for uniformly strong mixing strictly stationary sequences a weak invariance principle holds for random variables with the second moment divergent. This is an extension of the result of Peligrad [8] for random variables with finite variance.

1. Introduction and notation. Let $\{X_k\}_{k\in Z}$ be a strictly stationary random sequence on probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and let $\mathscr{F}_k^m$ denote the σ-field generated by $\{X_i;\ m \leqslant i \leqslant k\}$. Define:

$$\varphi_n = \varphi_n(\{X_k\}) = \sup\{|\mathscr{P}(B/A) - \mathscr{P}(B)|;\ A \in \mathscr{F}_{-\infty}^0,\ B \in \mathscr{F}_n^\infty,\ \mathscr{P}(A) > 0\},$$

$$\varrho_n = \varrho_n(\{X_k\}) = \sup\{|\mathrm{Corr}(f, g)|;\ f, g - \text{real}, f \in L^2(\mathscr{F}_{-\infty}^0),\ g \in L^2(\mathscr{F}_n^\infty)\}.$$

The sequence $\{X_k\}_k$ is said to be *uniformly strong mixing* or *φ-mixing* if $\lim_{n\to\infty} \varphi_n = 0$. It is well known that $\varrho_n \leqslant 2\varphi_n^{1/2}$.

In this note, unless otherwise stated, we shall deal with strictly stationary φ-mixing sequences only.

Let $S_n = \sum_{k=1}^n X_i$ and define the random element in $\mathscr{D}((0, 1])$:

$$\mathscr{X}_n(t) = \sigma_n^{-1} S_{[nt]}, \qquad t \in (0, 1],$$

where $\sigma_n^2 = \mathrm{Var}\, S_n$ and [ ] denotes the greatest integer function. $\mathscr{X}_n$ satisfies the *weak invariance principle* (WIP) if $\mathscr{X}_n$ converges weakly ($\Rightarrow_w$) to the standard Wiener measure $\mathscr{W}$.

Peligrad [8] proved that in the case $\mathscr{E}X_1^2 < \infty$ WIP is equivalent to the Lindenberg condition. On the other hand, in the iid case the Central Limit Theorem holds for random variables with the second moment barely divergent [2].

The purpose of this note is to formulate and prove a WIP when $\mathscr{E}X_1^2 = \infty$. We use the following notation: let $b_n \to_n +\infty$ for every $n \in N$ and denote by

$\{\hat{X}_k\}_k$ an independent copy of $\{X_k\}_k$;

$$X_i^n = X_i I(|X_i| < b_n) - \mathscr{E} X_i I(|X_i| < b_n);$$

$$\hat{X}_i^n = \hat{X}_i I(|\hat{X}_i| < b_n) - \mathscr{E} \hat{X}_i I(|\hat{X}_i| < b_n);$$

$$U_i^n = X_i^n - \hat{X}_i^n; \quad T_k^n = \sum_{i=1}^{k} X_i^n; \quad Z_k^n = \sum_{i=1}^{k} U_i^n; \quad T_n = T_n^n; \quad Z_n = Z_n^n;$$

$$Y_i^n = X_i I(|X_i| \geqslant b_n); \quad R_k^n = \sum_{i=1}^{k} Y_i^n; \quad R_n = R_n^n;$$

$$\hat{S}_n = \sum_{i=1}^{n} \hat{X}_i; \quad (\tau_k^n)^2 = \operatorname{Var} T_k^n; \quad (z_k^n)^2 = \operatorname{Var} Z_k^n; \quad \tau_n = \tau_n^n;$$

$$z_n = z_n^n; \quad \mathscr{W}_n'(t) = \tau_n^{-1} T_{[nt]}^n; \quad \mathscr{W}_n''(t) = \tau_n^{-1} S_{[nt]};$$

$$\mathscr{W}_n(t) = \tau_n^{-1}\big(S_{[nt]} - [nt]\mathscr{E} X_1 I(|X_1| < b_n)\big).$$

The Theorem we shall prove, in the case $b_n = +\infty$ for all $n \in N$, is Corollary 2.2 in [8]. As an application two corollaries will be proved, the second one is a recent result of Peligrad [9].

2. Auxiliary results and definitions. In this section we group some facts adapted for this note from more general theorems.

(2.1) $\{\max_{1 \leqslant i \leqslant n} \tau_n^{-2} (X_i^n)^2\}_n$ is uniformly integrable if and only if so is

$$\{\max_{1 \leqslant i \leqslant n} \tau_n^{-2} (T_i^n)^2\}_n$$

(see the proof of Proposition 2.1 in [8]).

(2.2) Let $\{X_k\}_k$ be a centered L^2-stationary random sequence; then

$$(1 - \varrho_p)^{1/2} \max_{1 \leqslant i \leqslant n} \sigma_i \leqslant \sigma_n + 2p\sigma_1$$

(see Lemma 4.2 in [7]).

(2.3) For any $\{X_k\}_k$ such that

$$\varphi_1 + \max_{1 \leqslant i \leqslant n} \mathscr{P}(|S_n - S_i| > x_0) \leqslant \eta < 1,$$

for $x \geqslant x_0$ we have

$$\mathscr{P}(\max_{1 \leqslant i \leqslant n} |S_i| > 2x) \leqslant (1-\eta)^{-1} \mathscr{P}(|S_n| > x)$$

(see Lemma 1.1.6 in [4]).

(2.4) Let $\{X_k^*\}_k$ denote an iid sequence with $\mathscr{L}(X_1^*) = \mathscr{L}(X_1)$; then for $x > 0$:

$$(1-\varphi_1)\mathscr{P}(\max_{1 \leqslant i \leqslant n} |X_i^*| > x) \leqslant \mathscr{P}(\max_{1 \leqslant i \leqslant n} |X_i| > x) \leqslant (1+\varphi_1)\mathscr{P}(\max_{1 \leqslant i \leqslant n} |X_i^*| > x)$$

(see Proposition 3.1 in [9]).

(2.5) $\mathscr{L}(X_1)$ is said to be *in the domain of attraction of the normal law* ($\mathscr{L}(X_1) \in \mathscr{D}\mathscr{A}(2)$) if there exist sequences $\{A_n\}_n$ and $\{b_n\}_n$ such that

$$\mathscr{L}(b_n^{-1} \sum_{k=1}^{n} X_i^* - A_n) \xrightarrow{w} \mathscr{N}(0, 1), \quad n \to +\infty.$$

This is equivalent [2] to the slow variation of $\mathscr{E}X_1^2 I(|X_1| < x)$, and then

$$b_n := \inf\{x;\ x^{-2}\mathscr{E}X_1^2 I(|X_1| < x) \leqslant 1/n\}.$$

(2.6) If $\mathscr{E}X_1^2 I(|X_1| < x)$ is slowly varying, then for $\{b_n\}_n$ from (2.5) we obtain

$$\frac{n}{b_n}\mathscr{E}|X_1| I(|X_1| > b_n) \xrightarrow{n} 0, \quad n \to +\infty$$

(this follows easily from Theorem 2, VIII, §9, in [2]).

(2.7) If $x^2 \mathscr{P}(|X_1| > x)$ is a slowly varying function, then so is $\mathscr{E}X_1^2 I(|X_1| < x)$ (see the same Theorem as in (2.6)); however, according to Exercise 32, VII, §10, in [2], the converse is not true.

(2.8) If $x^2 \mathscr{P}(|X_1| > x)$ is a slowly varying function, then

$$n\mathscr{P}(|X_1| > a_n) \xrightarrow{n} 1, \quad a_n = \inf\{x;\ \mathscr{P}(|X_1| > x) \leqslant 1/n\}$$

(see Lemma 1.8 in [10]).

(2.9) If $x^2 \mathscr{P}(|X_1| > x)$ is a slowly varying function, then

$$\mathscr{E}|X_1| I(|X_1| > x) \sim 2x\mathscr{P}(|X_1| > x), \quad x \to +\infty$$

(see Theorem 8.1.4 in [1]).

(2.10) Assume $n\mathscr{P}(|X_1| > b_n) \xrightarrow{n} 0$, and $\tau_n \to +\infty$, $n \to +\infty$, and $\{\tau_n^{-2} T_n^2\}_n$ is uniformly integrable. Then

$$(\mathscr{W}_n'(1)) \xrightarrow{w} \mathscr{N}(0, 1), \quad n \to +\infty$$

(see Theorem 3 in [6]).

3. Results and proofs.

THEOREM. *Assume that*

(3.1) $$\lim_{n\to\infty} n\mathscr{P}(|X_1| > b_n) = 0,$$

(3.2) $$\lim_{n\to\infty} \tau_n = +\infty,$$

(3.3) $$\lim_{n\to\infty} \tau_n^{-2}\mathscr{E}(\max_{1\leqslant i\leqslant n} (X_i^n)^2) = 0.$$

Then

(3.4) $$\mathscr{W}_n \stackrel{w}{\Longrightarrow} \mathscr{W}, \quad n \to +\infty.$$

Conversely, if $\varphi_1 < 1$ *and* (3.4) *holds, then* (3.3) *is satisfied.*

COROLLARY 1. *Let* $\mathscr{L}(X_1)\in\mathscr{D}\mathscr{A}(2)$, $\mathscr{E}X_1 = 0$ *and*

$$\liminf_{n\to\infty} \tau_n b_n^{-1} > 0, \tag{3.5}$$

where b_n *is defined in* (2.5). *Then*

$$\mathscr{W}_n'' \xrightarrow{w} \mathscr{W}, \quad n \to +\infty. \tag{3.6}$$

COROLLARY 2. *Assume* $x^2\mathscr{P}(|X_1| > x)$ *is slowly varying,* $\mathscr{E}X_1 = 0$, $\varphi_1 < 1$. *Then* (3.6) *holds, and*

$$\sqrt{\pi/2}\,\mathscr{E}|S_n| \sim \tau_n, \quad n \to +\infty, \tag{3.7}$$

for some $\{b_n\}_n$.

Proof of the Theorem. We shall consider only the case $\mathscr{E}X_1^2 = \infty$, i.e., $b_n \xrightarrow{n} +\infty$, since the other case can be proved analogously. From (3.1) we see that

$$\max_{1\leqslant k\leqslant n} \tau_n^{-1}|R_k^n| \xrightarrow{\mathscr{P}} 0, \quad n \to +\infty.$$

Thus in the proof we can restrict ourselves to $\mathscr{W}_n'$ random elements.

The direct half. An examination of the proof of Theorems 1 and 2 in [5] shows that it is enough to prove that

$$\max_{1\leqslant i\leqslant [n\delta_n]} \frac{(\tau_i^n)^2}{(\tau_n)^2} \xrightarrow{n} 0, \quad n \to +\infty,$$

for any $\{\delta_n\}_n$ such that $\lim_n \delta_n = 0$. By (2.2), for any $\varepsilon > 0$ and $n\in N$ such that $\delta_n \leqslant \varepsilon$, we have

$$\max_{1\leqslant i\leqslant [n\delta_n]} \frac{\tau_i^n}{\tau_n} \leqslant (1-\varrho_p)^{-1/2}\left(\frac{\tau_{[n\varepsilon]}^n}{\tau_n} + 2p\frac{\tau_1^n}{\tau_n}\right),$$

so the required condition is satisfied if $(\tau_n)^2$ is a regularly varying sequence with index 1 (see [1], p. 52), and

$$\frac{(\tau_{[nt]}^n)^2}{(\tau_{[nt]})^2} \xrightarrow{n} 1, \quad t\in(0, 1],\ n \to +\infty. \tag{3.8}$$

From (2.1) we infer that $\{\tau_n^{-2}T_n^2\}_n$ is uniformly integrable, so by (2.10) and (3.1) we obtain

$$\mathscr{L}(z_{[nt]}^{-1} Z_{[nt]}^n) \xrightarrow{w} \mathscr{N}(0, 1), \quad n \to +\infty. \tag{3.9}$$

On the other hand, by (2.2) we have

$$(\tau_n)^2 = \mathscr{E}\Big(\sum_{j=1}^{[n/[nt]]}\sum_{i=1}^{[nt]} X^n_{[nt](j-1)+i} + \sum_{i=[n/[nt]][nt]+1}^{n} X^n_i\Big)^2$$
$$\leqslant 2^{n/[nt]}(\tau^n_{[nt]})^2 + 2\max_{1\leqslant k\leqslant [nt]}(\tau^n_k)^2$$
$$\leqslant (\tau^n_{[nt]})^2\big(2^{2/t}+4(1-\varrho_p)^{-1}\big)+8p^2(1-\varrho_p)^{-1}(\tau^n_1)^2,$$

so there exists a constant $C = C(\varrho_p, t)$ such that

(3.10) $$\liminf_{n\to\infty} \tau_n^{-1}\tau^n_{[nt]} \geqslant C > 0,$$

since $\lim_{n\to\infty}\tau_n^{-1}\tau^n_1 = 0$ by (3.3). From (3.10) and (2.1) we infer that $\{(\tau^n_{[nt]})^{-2}(T^n_{[nt]})^2\}_n$ is uniformly integrable for $t\in(0, 1]$, so by (3.1) and (2.10) we get

(3.11) $$\mathscr{L}\big((z^n_{[nt]})^{-1}Z^n_{[nt]}\big) \xrightarrow{w} \mathscr{N}(0, 1), \quad n\to +\infty.$$

From (3.11), (3.9) and the Theorem of Convergence of Types we get (3.8).

Now observe that by assumption and (2.10) we have

$$\mathscr{L}\big(z_n^{-1}(S_n-\hat{S}_n)\big) \xrightarrow{w} \mathscr{N}(0, 1), \quad n\to +\infty.$$

Thus, by Theorem 18.1.1 in [3] we have

(3.12) $$\frac{(\tau_{kn})^2}{(\tau_n)^2} \xrightarrow{n} k, \quad k\in N,\ n\to +\infty.$$

Since

$$\mathscr{P}(|X_1-\hat{X}_1| > \varepsilon z_n) \leqslant \mathscr{P}(|X^n_1-\hat{X}^n_1| > 2^{-1}\varepsilon z_n)+2\mathscr{P}(|X_1| \geqslant b_n)$$
$$\leqslant 4\varepsilon^{-2}(z^n_1)^2 z_n^{-2}+n\mathscr{P}(|X_1| \geqslant b_n),$$

so by (3.3) and (3.1) we obtain

$$\mathscr{L}\big(z_n^{-1}(S_{n+1}-\hat{S}_{n+1})\big) \xrightarrow{w} \mathscr{N}(0, 1), \quad n\to +\infty.$$

Thus $\lim_{n\to\infty} z_n z_{n+1}^{-1} = 1$, so

(3.13) $$\tau_{n+1}\tau_n^{-1} \xrightarrow{n} 1, \quad n\to +\infty.$$

Let $q\in N$; then

$$\frac{(\tau_{q[nq^{-1}]})^2}{(\tau_{[nq^{-1}]})^2} \xrightarrow{n} q, \quad n\to +\infty.$$

But $q[nq^{-1}] = n, n-1, \ldots, n-q-1$ and, by (3.13),

(3.14) $$\frac{(\tau_n)^2}{(\tau_{[nq^{-1}]})^2} \xrightarrow{n} q, \quad n\to +\infty,$$

so by (3.12) we have

$$\frac{(\tau_{[\omega n]})^2}{(\tau_n)^2} \xrightarrow{n} \omega, \quad n \to +\infty, \tag{3.15}$$

for every ω rational. Let r be irrational and $r \in (0, 1]$, $c = r - \omega > 0$. We show, following Peligrad [7], that

$$\frac{(\tau_{[rn]})^2}{(\tau_n)^2} \xrightarrow{n} r, \quad n \to +\infty. \tag{3.16}$$

From (2.2) we have

$$|\tau^n_{[\omega n]} - \tau^n_{[rn]}| \leqslant \tau^n_{[rn]-[\omega n]} \leqslant (1-\varrho_p)^{-1/2}(\tau^n_{[n(r-\omega)]+2} + 2\tau^n_1),$$

so taking lim sup over both sides we have, by (3.3),

$$\limsup_{n\to\infty} \tau_n^{-1} |\tau^n_{[\omega n]} - \tau^n_{[rn]}| \leqslant (1-\varrho_p)^{-1/2} \limsup_{n\to\infty} \tau_n^{-1} \tau^n_{[n(r-\omega)]}.$$

Now, it remains to show that the right-hand side disappears when $\omega \nearrow r$. We have

$$\frac{\tau^n_{[nc]}}{\tau_n} = \frac{\tau^n_{[n/2]}}{\tau_n} \times \frac{\tau^n_{[n/2^2]}}{\tau^n_{[n/2]}} \times \frac{\tau^n_{[n/2^3]}}{\tau^n_{[n/2^2]}} \times \frac{\tau^n_{[n/2^4]}}{\tau^n_{[n/2^3]}} \times \ldots \times \frac{\tau^n_{[nc]}}{\tau^n_{[n/2^{[-\log c/\log 2]}]}}.$$

Note that lim sup of the last multiplier is bounded by $(1-\varrho_p)^{-1/2}$, so

$$\limsup_{n\to\infty} \frac{\tau^n_{[n(r-\omega)]}}{\tau_n} \leqslant (1-\varrho_p)^{-1/2} 2^{-(1/2)([-\log c/\log 2]-1)} \leqslant K(r-\omega),$$

where K is a constant depending on ϱ_p only, i.e., (3.16) holds. By (3.8) and (3.16), for every $r \in (0, 1]$ we have

$$\frac{(\tau_{[rn]})^2}{(\tau_n)^2} \xrightarrow{n} r, \quad n \to +\infty,$$

so by Theorem 1.3 in [10] the above holds for every $r > 0$, i.e., $\{(\tau_n)^2\}_n$ forms a regularly varying sequence with index 1.

The converse half. We have

$$\varphi_1 + \max_{1\leqslant j\leqslant n} \mathscr{P}(|Z_n - Z^n_j| > z_n x_0) \leqslant \varphi_1 + \max_{1\leqslant j\leqslant n} \mathscr{P}(|Z_n - Z_j| > 2^{-1} z_n x_0)$$
$$+ \max_{1\leqslant j\leqslant N_\delta} \mathscr{P}(|Z_j - Z^n_j| > 2^{-1} z_n x_0) + \max_{N_\delta < j\leqslant n} \mathscr{P}(|Z_j - Z^n_j| > 2^{-1} z_n x_0),$$

where N_δ is such that $\mathscr{P}(\tau_n^{-1}|R_n| > 2^{-1} x_0) \leqslant n\mathscr{P}(|X_1| > b_n) \leqslant \delta$ for $n > N_\delta$. The right-hand side of the above inequality can be estimated by

$$\varphi_1 + \frac{8}{x_0^2}\left(1 + \max_{1\leqslant j\leqslant n} \frac{(\tau_j)^2}{(\tau_n)^2}\right) + o(1) + \delta,$$

i.e., there exists $N_0 = N(\delta, \varphi_1)$ such that for $n \geqslant N_0$ and sufficiently large x_0

$$\varphi_1 + \max_{1 \leqslant j \leqslant n} \mathscr{P}(|Z_n - Z_j^n| > z_n x_0) \leqslant \eta < 1,$$

since $\max_{1 \leqslant j \leqslant n} \tau_j \tau_n^{-1}$ is bounded, by (3.4). Using (2.3), for $n \geqslant N_0$, $x \geqslant x_0$ we obtain

$$\mathscr{P}(\max_{1 \leqslant i \leqslant n} |Z_i^n| > 2xz_n) \leqslant (1-\eta)^{-1} \mathscr{P}(|Z_n| > xz_n), \tag{3.17}$$

and since

$$\mathscr{P}(\max_{1 \leqslant i \leqslant n} |U_i^n| > x) \leqslant 2\mathscr{P}(\max_{1 \leqslant i \leqslant n} |Z_i^n| > 2^{-1} x),$$

so, by (3.17), $\{\max_{1 \leqslant i \leqslant n} z_n^{-2} (U_i^n)^2\}_n$ is uniformly integrable. By the proof of Theorem 1 in [5] we have

$$\max_{1 \leqslant i \leqslant n} \tau_n^{-1} |X_i^n| \xrightarrow{\mathscr{P}} 0, \quad n \to +\infty, \tag{3.18}$$

so for $\mu_n = \operatorname{med}(\max_{1 \leqslant i \leqslant n} \tau_n^{-1} |X_i^n|)$ we obtain

$$\mu_n \xrightarrow{n} 0, \quad n \to +\infty. \tag{3.19}$$

Thus

$$\mathscr{P}(\max_{1 \leqslant i \leqslant n} z_n^{-1} |X_i^n| \geqslant x) \leqslant \mathscr{P}(\big|\max_{1 \leqslant i \leqslant n} z_n^{-1} |X_i^n| - \mu_n\big| \geqslant x - \mu_n)$$

$$\leqslant 2\mathscr{P}(\big|\max_{1 \leqslant i \leqslant n} z_n^{-1} |X_i^n| - \max_{1 \leqslant i \leqslant n} z_n^{-1} |\hat{X}_i^n|\big| \geqslant x - \mu_n) \leqslant 4\mathscr{P}(\max_{1 \leqslant i \leqslant n} z_n^{-1} |U_i^n| \geqslant x - \mu_n).$$

From this, (3.19), (3.18) and the uniform integrability of $\{\max_{1 \leqslant i \leqslant n} z_n^{-2} (U_i^n)^2\}_n$ the equality (3.3) holds true.

Proof of Corollary 1. By (2.6), (3.5), (2.4) it suffices to prove that

$$\{b_n^{-2} \max_{1 \leqslant i \leqslant n} (X_i^* I(|X_i^*| < b_n))^2\}_n$$

is uniformly integrable, but this follows easily from the iid case.

Proof of Corollary 2. Under the assumptions of the corollary Peligrad [7] proved that for every $k \in N$:

$$\frac{k^2 a_n^2}{\sigma^2(ka_n)} \xrightarrow{n} 0, \quad n \to +\infty,$$

where

$$\sigma^2(ka_n) = \operatorname{Var}\Big(\sum_{i=1}^{n} X_i I(|X_i| < ka_n) - \mathscr{E} X_i I(|X_i| < ka_n)\Big),$$

and $\{a_n\}_n$ is defined in (2.8). So there exists $\{r_n\}_n$, $\lim_n r_n = +\infty$, such that, for every $\{x_n\}_n$, $\lim_n x_n = +\infty$ and $x_n = o(r_n)$,

$$\frac{x_n^2 a_n^2}{\sigma^2(x_n a_n)} \xrightarrow{n} 0, \quad n \to +\infty. \tag{3.20}$$

On the other hand, by Theorem 1.1 in [10], there exists $\{r'_n\}_n$, $\lim_n r'_n = +\infty$, such that, for every $\{x_n\}_n$, $\lim_n x_n = +\infty$ and $x_n = o(r'_n)$,

$$n x_n^2 \mathscr{P}(|X_1| > x_n a_n) \xrightarrow{n} 1, \quad n \to +\infty. \tag{3.21}$$

Now let $b_n = x_n a_n$, where $\lim_n x_n = +\infty$, $x_n = o(r_n \wedge r'_n)$, and $\tau_n = \sigma(x_n a_n)$; then (3.1)–(3.3) are fulfilled, so (3.4) holds. Observe that by (2.9) we have

$$\frac{[nt]}{\tau_n} |\mathscr{E} X_1 I(|X_1| > b_n)| \leqslant \frac{[nt]}{\tau_n} \mathscr{E}|X_1| I(|X_1| > b_n)$$

$$\sim 2 \frac{[nt]}{\sigma(x_n a_n)} x_n a_n \mathscr{P}(|X_1| > x_n a_n), \quad n \to +\infty,$$

so this and (3.20), (3.21) give (3.6). Since $\tau_n \sim \sqrt{\pi/2}\, \mathscr{E}|T_n|$ and

$$\left| \frac{\mathscr{E}|S_n| - \mathscr{E}|T_n|}{\mathscr{E}|T_n|} \right| \leqslant \frac{n \mathscr{E}|X_1| I(|X_1| > b_n)}{\mathscr{E}|T_n|} \sim \frac{2 n b_n \mathscr{P}(|X_1| > b_n)}{\sqrt{2/\pi}\, \tau_n}, \quad n \to +\infty,$$

so, as above, (3.7) holds.

Remark. There are strictly stationary random sequences with infinite variance, φ-mixing, satisfying CLT and not satisfying WIP (i.e. (3.6)). As an example one can use a 1-dependent sequence in Example 2 of [6]. For this sequence, (3.5) does not hold.

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