PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 14, Fasc. 2 (1993), pp. 313–326

ON THE RATE OF CONVERGENCE FOR DISTRIBUTIONS OF INTEGRAL TYPE FUNCTIONALS FOR SUMS OF INFIMA OF INDEPENDENT RANDOM VARIABLES

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Abstract. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on [0, 1]. Put

$$X_m^* = \inf(X_1, X_2, \ldots, X_m), \ m \geqslant 1, \quad \text{and} \quad S_n = \sum_{m=1}^{n} X_m^*, \ n \geqslant 1.$$

In this paper the convergence rate for distributions of integral type functionals for sums S_n, $n \geqslant 1$, is obtained.

1. Introduction and results. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on [0. 1].

Let us put

$$X_m^* = \inf(X_1, X_2, \ldots, X_m), \ m \geqslant 1, \quad \tilde{S}_n = \sum_{m=0}^{n} X_m^*, \ n \geqslant 1, \quad \tilde{S}_0 = 0$$

and define

$$\tilde{S}_{n,k} = \Big(\tilde{S}_k - \sum_{i=1}^{k} i^{-1}\Big)\Big(2\sum_{m=1}^{n} m^{-1}\Big)^{-1/2}, \ 1 \leqslant k \leqslant n, \quad \tilde{S}_{n,0} = 0. \tag{1}$$

Let $\{S_n(t), \ t \in \langle 0, 1\rangle\}$ be a random function defined as follows:

$$S_n(t) = \tilde{S}_{n,k} + \frac{t - t_k}{t_{k+1} - t_k}(\tilde{S}_{n,k+1} - \tilde{S}_{n,k}) \ \text{for} \ t \in \langle t_k, t_{k+1}), \quad S_n(0) = 0, \tag{2}$$

where $t_k = \sum_{i=1}^{k} i^{-1} \left(\sum_{m=1}^{n} m^{-1}\right)^{-1}$, $1 \leqslant k \leqslant n$, $t_0 = 0$.

Let $f(t, x)$ be a continuous function which has continuous partial derivatives on the set $\langle 0, 1\rangle \times \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the set of real numbers. We assume that there exist positive constants α and Ω such that

$$|Df(t, x)| \leqslant \Omega(1 + |x|^{\alpha}) \quad \text{for} \ (t, x) \in \langle 0, 1\rangle \times \boldsymbol{R}, \tag{3}$$

where D denotes either the identity operator I or partial derivative operators $\partial/\partial t$ and $\partial/\partial x$.

It is known from Corollary 1 (cf. [7]) that $S_n \xrightarrow{D} W$ as $n \to \infty$, where $W = \{W(t),\ t \in \langle 0, 1\rangle\}$ is a Wiener process. Hence, if Φ is a continuous functional defined on $C_{\langle 0,1\rangle}$, where $(C_{\langle 0,1\rangle}, \mathscr{B}_C)$ is the space of continuous functions, then (cf. [1], p. 30)

$$\Phi(S_n) \xrightarrow{D} \phi(W) \quad \text{as } n \to \infty. \tag{4}$$

The main purpose of this paper is to give the rate of convergence in (4) for the functional

$$\Phi(x) = \int_0^1 f(t, x(t))\, dt, \qquad x(\cdot) \in C_{\langle 0,1\rangle}, \tag{5}$$

where $f(t, x)$ is a function satisfying (3).

We can prove the following

THEOREM 1. *Let $\{S_{n,k},\ 1 \leqslant k \leqslant n\}$, $n \geqslant 1$, be a sequence given by* (1). *Assume that Φ is a functional defined by* (5) *and such that the distribution of the random variable $\Phi(W)$ satisfies the Lipschitz condition with a positive constant L, i.e.*

$$P\left[x - \delta \leqslant \int_0^1 f(t, W(t))\, dt \leqslant x + \delta\right] \leqslant 2L\delta$$

for any $x \in \mathbf{R}$ and $\delta > 0$. If we define $\{Z_n, n \geqslant 1\}$ as

$$Z_n = \sum_{k=0}^{n-1} f(t_k, \tilde{S}_{n,k})(t_{k+1} - t_k), \tag{6}$$

where $\tilde{S}_{n,k}$ and t_k, $0 \leqslant k \leqslant n$, are given in (1) *and* (2), *respectively, then*

$$\sup_x |P[Z_n \leqslant x] - P[\Phi(W) \leqslant x]| = O\left(\frac{(\log_2 n)^{7\alpha}}{(\log n)^{2/5}}\right) \quad \text{as } n \to \infty, \tag{7}$$

where $\log_2 n = \log(\log n)$.

THEOREM 2. *Suppose the assumptions of Theorem* 1 *hold. Then in* (7) *we can put $\Phi(S_n)$ instead of Z_n, where $S_n = \{S_n(t),\ t \in \langle 0, 1\rangle\}$ is given by* (2).

This type of theorems for independent random variables and for martingales has been obtained in [2] and [14], respectively.

Let $\mathscr{L}_C$ denote the Lévy–Prohorov distance, i.e., for any two measures P and Q on $(C, \mathscr{B}_C)$

$$\mathscr{L}_C(P, Q) < \varepsilon \quad \textit{iff} \quad P(B) \leqslant Q(G_\varepsilon(B)) + \varepsilon \ \textit{and}\ Q(B) \leqslant P(G_\varepsilon(B)) + \varepsilon$$

for all $B \in \mathscr{B}$, where

$$G_\varepsilon(B) = \{x: \bigvee_{y \in B} \varrho(x, y) < \varepsilon\},$$

and ϱ is the uniform metric on $C_{\langle 0,1\rangle}$.

We can prove the following

THEOREM 3. *Let P_n denote the distribution of $S_n = \{S_n(t),\ t \in \langle 0, 1\rangle\}$ in the space $(C, \mathscr{B}_C)$. Then*

$$\mathscr{L}_C(P_n, W) = O\big((\log_2 n)^{1/2} (\log n)^{-1/3}\big) \tag{8}$$

as $n \to \infty$, where W is the Wiener measure on $C_{\langle 0,1\rangle}$.

Let us observe that in the case where $\{X_n, n \geqslant 1\}$ are i.r.vs. uniformly distributed on $[0, 1]$, Theorem 3 gives the estimate on $\mathscr{L}_C(P_n, W)$ stronger than that in [7] where the relation $\mathscr{L}_C(P_n, W) = O\big((\log n)^{-1/8}\big)$ has been obtained.

2. Proof of the results. In the proofs of Theorems 1–3 we apply some lemmas given by Dehéuvels ([3], [4], lemmes 3.1–3.3), Grenander ([5], Lemma 3.4) and the Skorokhod representation theorem (see [16] and [17]) which we state as a lemma in Section 3 for the sake of clarity.

Proof of Theorem 1. Let us write

$$c_n = \Big(2 \sum_{m=1}^{n} m^{-1}\Big)^{1/2} \tag{9}$$

and set

$$\begin{aligned} V_{n,k} &= [\tau_{k+1} - \tau_k - \mathrm{E}(\tau_{k+1} - \tau_k)]/kc_n, \quad && 1 \leqslant k \leqslant n, \\ V_{n,0} &= 0, && n \geqslant 1, \end{aligned} \tag{10}$$

and put

$$U_{n,k} = \sum_{m=1}^{k} V_{n,k}, \quad 1 \leqslant k \leqslant n,$$

where the random variables τ_n, $n \geqslant 1$, are given in Section 3 by (3.1) $(\varepsilon(n) = n^{-1})$.

Observe that $V_{n,k}$, $1 \leqslant k \leqslant n$, are independent random variables (Lemma 3.2) and

$$\mathrm{E}V_{n,k} = 0, \quad \sigma^2 V_{n,k} = 2/kc_n^2, \quad \sigma^2 U_{n,k} = t_k, \quad \sigma^2 U_{n,n} = 1. \tag{11}$$

Let us write

$$L_n^{(s)} = \sum_{k=1}^{n} \mathrm{E}\,|V_{n,k}|^s, \quad s \geqslant 2.$$

By Lemma 3.2 we can see that

$$L_n^{(s)} = O\big(s!\,(\log n)^{-s/2+1}\big) \quad \text{for } s \geqslant 2, \tag{12}$$

and putting $s = 6$, we get

$$L_n^{(6)} = O\big(6!\,(\log n)^{-2}\big). \tag{12'}$$

Let us define

$$Z_n^{(1)} = \sum_{k=0}^{m-1} f(t_k, U_{n,k})(t_{k+1}-t_k), \tag{13}$$

where t_k, $0 \leqslant k \leqslant n$, are given in (2).

It is easy to notice that by (11) and (12′) the sequence $\{V_{n,k},\ 1 \leqslant k \leqslant n\}$, $n \geqslant 1$, satisfies the conditions of Theorem 1 (cf. [2]). Applying this theorem to the sequence of random variables $\{V_{n,k},\ 1 \leqslant k \leqslant n\}$ we have

$$\sup_x |P[Z_n^{(1)} \leqslant x] - P[\Phi(W) \leqslant x]| = O\left((\log(L_n^{(6)})^{-1})^{(\alpha+1)/2} (L_n^{(6)})^{1/4}\right) \tag{14}$$

$$= O\left((\log_2 n)^{(\alpha+1)/2} (\log n)^{-1/2}\right).$$

Now, by the Skorokhod representation result applied to the sequence $V_n = \{V_{n,1}, V_{n,2}, \ldots, V_{n,n}\}$, there is a standard Wiener process $\{W(t),\ t \in \langle 0, 1\rangle\}$ together with a sequence of nonnegative independent random variables $z_1, z_2, \ldots, z_n$ on a new probability space such that

$$\{U_{n,1}, U_{n,2}, \ldots, U_{n,n}\} \overset{\mathrm{D}}{=} \{W(T_1), W(T_2), \ldots, W(T_n)\}, \quad n \geqslant 1, \tag{15}$$

where $T_k = \sum_{m=1}^{k} z_m$, $1 \leqslant k \leqslant n$, and $\overset{\mathrm{D}}{=}$ means the equivalence in joint distribution. Moreover,

$$\mathrm{E} z_k = \mathrm{E} V_{n,k}^2, \tag{16}$$

and, for each real number $r \geqslant 1$,

$$\mathrm{E}|z_k|^r \leqslant C_r \mathrm{E}(V_{n,k})^{2r}, \quad 1 \leqslant k \leqslant n, \tag{17}$$

where

$$C_r = 2(8/\pi^2)^{r-1} \Gamma(r+1),$$

and

$$V_{n,k} \overset{\mathrm{D}}{=} W(T_k) - W(T_{k-1}). \tag{18}$$

Let us define $Z_n^{(2)}$, $n \geqslant 1$, as follows:

$$Z_n^{(2)} = \sum_{k=0}^{n-1} f(t_k, \tilde{S}_{n,\tau_k})(t_{k+1}-t_k), \quad n \geqslant 1,$$

where

$$\tilde{S}_{n,\tau_k} = \Big(\sum_{i=1}^{\tau_k} X_i^* - \sum_{i=1}^{k} i^{-1}\Big)/c_n.$$

Write

$$\tilde{S}_{\tau_k} = \sum_{i=1}^{\tau_k} X_i^* \quad \text{and} \quad U_k = \sum_{m=1}^{k} (\tau_{m+1} - \tau_m) m^{-1}.$$

Let us estimate

$$P\big[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \geqslant \delta_n\big],$$

where $\{\delta_n, n \geqslant 1\}$ is a sequence of positive real numbers decreasing to zero such that $\delta_n c_n \to \infty$ as $n \to \infty$.

By (3.8) in Lemma 3.2 and simple evaluations, we get

$$(19)\quad P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \geqslant \delta_n] = P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{\tau_k} - U_k| \geqslant \delta_n c_n]$$
$$\leqslant P[\max_{1\leqslant k\leqslant n} \max(\tilde{S}_{\tau_1}, 2-\tilde{S}_{\tau_1} + U_k - U_k') \geqslant \delta_n c_n]$$
$$\leqslant P[U_n - U_n' + 2 \geqslant \delta_n c_n] \leqslant P[|U_n - U_n' - \mathrm{E}(U_n - U_n')| \geqslant \delta_n c_n - 3]$$
$$\leqslant \frac{\mathrm{E}[U_n - U_n' - \mathrm{E}(U_n - U_n')]^4}{(\delta_n c_n - 3)^4} \leqslant \frac{C}{\delta_n^4 c_n^4},$$

where C is a positive constant independent of n, and U_n' is given in (3.4).

If we put $\delta_n = (\log n)^{-2/5}$, we also have

$$(20)\quad P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \geqslant (\log n)^{-2/5}] = O((\log n)^{-2/5})$$

because $c_n \sim (2\log n)^{1/2}$.

Now, observe that from the construction of z_i, $i \geqslant 1$, relations (11), (12), (16), (17) and Kolmogorov's type inequality, (3.2) and (3.7) we obtain

$$(21)\quad P[\max_{1\leqslant k\leqslant n} |T_k - t_k| \geqslant g(n)] = P[\max_{1\leqslant k\leqslant n} |T_k - \mathrm{E}T_k| \geqslant g(n)]$$
$$\leqslant [\mathrm{E}(T_n - \mathrm{E}T_n)^4]/g^4(n) = [\mathrm{E}(\sum_{m=1}^{n} (z_m - \mathrm{E}z_m))^4]/g^4(n)$$
$$\leqslant [\sum_{m=1}^{n} \mathrm{E}(z_m - \mathrm{E}z_m)^4 + 2(\sum_{m=1}^{n} \mathrm{E}(z_m - \mathrm{E}z_m)^2)^2]/g^4(n)$$
$$\leqslant [2^3 \sum_{m=1}^{n} (\mathrm{E}z_m^4 + (\mathrm{E}z_m)^4) + 2(\sum_{m=1}^{n} \mathrm{E}z_m^2)^2]/g^4(n)$$
$$\leqslant C[2^3 \sum_{m=1}^{n} (\mathrm{E}V_{n,m}^8 + (\sigma^2 V_{n,m})^4) + 2(\sum_{m=1}^{n} V_{n,m}^4)^2]/g^4(n)$$
$$= O((g^4(n)\log^3 n)^{-1}),$$

where $g(n) \to 0$ as $n \to \infty$, so that $g^4(n)\log^3 n \to \infty$ as $n \to \infty$.

Putting $g(n) = (\log n)^{-3/5}$, we get

$$(21')\quad P[\max_{1\leqslant k\leqslant n} |T_k - t_k| \geqslant (\log n)^{-3/5}] = O((\log n)^{-3/5}).$$

Now, we shall estimate $P[|Z_n^{(2)} - Z_n^{(1)}| > \delta_n]$.

Let us set

$$(22)\quad B_n^{(1)} = [\sup_{0\leqslant t\leqslant 1+g(n)} |W(t)| < a_n,\ |T_n - 1| < g(n),\ \max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_n} - U_{n,k}| < \delta_n],$$

where

$$a_n = (\log_2 n)^{1/2}, \quad g(n) \to 0, \quad \delta_n \to 0 \quad \text{as } n \to \infty$$

in such a way that

$$g^4(n) \log^3 n \to \infty, \quad \delta_n c_n \to \infty \quad \text{as } n \to \infty.$$

It is easy to see that

$$(23) \quad P[|Z_n^{(2)} - Z_n^{(1)}| > \delta_n] \leqslant P[|Z_n^{(2)} - Z_n^{(1)}| > \delta_n, B_n^{(1)}] + P[\sup_{0 \leqslant t \leqslant 1+g(n)} |W(t)| \geqslant a_n] + P[|T_n - 1| \geqslant g(n)] + P[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,\tau_n} - U_{n,k}| \geqslant \delta_n].$$

It is well known that

$$(24) \quad P[\sup_{0 \leqslant t \leqslant 1+g(n)} |W(t)| \geqslant a_n] \leqslant 4P\left[|W(1)| > \frac{a_n}{\sqrt{1+g(n)}}\right] \leqslant \frac{8}{\sqrt{2\pi}} \frac{\sqrt{1+g(n)}}{a_n} \exp\left[\frac{-a_n^2}{1+g(n)}\right] = O\left(((\log_2 n)^{1/2} \log n)^{-1}\right).$$

On the other hand, by the mean value theorem, (3) and (19) one can note that on the set $B_n^{(1)}$ we get

$$P[|Z_n^{(2)} - Z_n^{(1)}| > \delta_n, B_n^{(1)}]$$
$$\leqslant P\left[\sum_{k=1}^{n-1} \left|\frac{\partial f}{\partial x}(t_k, U_{n,k} + \theta_k(\tilde{S}_{n,\tau_k} - U_{n,k}))(\tilde{S}_{n,\tau_k} - U_{n,k})(t_{k+1} - t_k)\right| > \delta_n, B_n^{(1)}\right]$$
$$\leqslant P[\Omega_0(a_n)^\alpha \max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| \sum_{k=0}^{n-1} (t_{k+1} - t_k) > \delta_n]$$
$$= P\left[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,\tau_k} - U_{n,k}| > \frac{\delta_n}{\Omega_0(a_n)^\alpha}\right] \leqslant \frac{C\Omega_0^4 (a_n)^{4\alpha}}{\delta_n^4 c_n^4},$$

where $0 < \theta_k < 1$, Ω_0 is a positive constant depending only on the function f, and $C > 0$ is independent of n.

Hence, using (19)–(21′), (23), (24) and putting $\delta_n = (\log n)^{-2/5}$ and $g(n) = (\log n)^{-3/5}$, we obtain

$$(25) \quad P[|Z_n^{(2)} - Z_n^{(1)}| > (\log n)^{-2/5}] = O\left((\log_2 n)^{2\alpha} (\log n)^{-2/5}\right).$$

Now, we are going to estimate $P[|Z_n - Z_n^{(2)}| > \delta_n]$, where $\{Z_n, n \geqslant 1\}$ is given by (6).

Observe that

$$(26) \quad P[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| > \delta_n] = P[\max_{1 \leqslant k \leqslant n} |\tilde{S}_k - \tilde{S}_{\tau_k}| > \delta_n c_n].$$

Notice that for $k \geqslant \tau_k$, by definition (3.1), we have

$$\inf(X_1, X_2, \ldots, X_{\tau_k+i}) \leqslant \varepsilon(k) = 1/k \quad \text{for } i \geqslant 1,$$

and in this case

$$|\tilde{S}_k - S_{\tau_k}| = \sum_{m=\tau_k+1}^{k} X_m^* \leqslant k\varepsilon(k) = 1 \quad \text{for } k \geqslant 1.$$

So, we can get

$$(27) \quad P\big[\max_{1 \leqslant k \leqslant n} |\tilde{S}_k - \tilde{S}_{\tau_k}| > \delta_n c_n\big] \leqslant P\big[\max_{\substack{1 \leqslant k \leqslant n \\ \tau_k > k}} \sum_{m=k+1}^{\tau_k} X_m^* > \delta_n c_n\big]$$

$$\leqslant P\big[\max_{\substack{1 \leqslant k \leqslant n \\ \tau_k > k}} \sum_{m=k+1}^{\tau_n} X_m^* > \delta_n c_n\big] \leqslant P\big[\sum_{m=1}^{\tau_n} X_m^* > \delta_n c_n, \tau_n > n\big]$$

because $\tau_k \leqslant \tau_{k+1}$ for $k \geqslant 1$.

Moreover, by Lemmas 3.4 and 3.5, we obtain

$$P\big[\sum_{m=1}^{\tau_n} X_m^* > \delta_n c_n, \tau_n > n\big] = \sum_{k=n+1}^{\infty} P\big[\sum_{m=1}^{\tau_n} X_m^* > \delta_n c_n, \tau_n = k\big]$$

$$= \sum_{k=n+1}^{\infty} P\big[\sum_{m=1}^{k} X_m^* > \delta_n c_n | \tau_n = k\big] P[\tau_n = k]$$

$$\leqslant \sum_{k=n+1}^{\infty} \frac{\mathrm{E}\big[(\sum_{m=1}^{k} X_m^*)^p | \tau_n = k\big]}{(\delta_n c_n)^p} P[\tau_n = k]$$

$$\leqslant \frac{C}{(\delta_n c_n)^p} \sum_{k=n+1}^{\infty} (\log k) P[\tau_n = k]$$

$$= \frac{C}{(\delta_n c_n)^p} \mathrm{E}\big((\log \tau_n) I[\tau_n > n]\big) = O\Big(\frac{\log n}{(\delta_n c_n)^p}\Big),$$

where C is a positive constant independent of n.

Hence, by (26) and (27) we get

$$(28) \quad P\big[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| > \delta_n\big] = O\Big(\frac{\log n}{(\delta_n c_n)^p}\Big),$$

and putting $\delta_n = (\log n)^{-2/5}$ and $p = 14$ we obtain

$$(29) \quad P\big[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| > (\log n)^{-2/5}\big] = O\big((\log n)^{-2/5}\big).$$

Let us write

$$B_n^{(2)} = B_n^{(1)} \cap \big[\max_{1 \leqslant k \leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| < \delta_n\big].$$

Observe that on the set $B_n^{(2)}$, by (3), (28) and (29), we get

$$(30)\quad P[|Z_n - Z_n^{(2)}| > \delta_n, B_n^{(2)}]$$
$$\leqslant P\Big[\Big|\sum_{k=1}^{n-1} \frac{\partial f}{\partial x}(t_k, U_{n,k} + (\tilde{S}_{n,\tau_k} - U_{n,k}) + \theta_k(\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}))$$
$$\times (\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k})(t_{k+1} - t_k)\Big| > \delta_n\Big]$$
$$\leqslant P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k} - \tilde{S}_{n,\tau_k}| > \delta_n/\Omega_0(a_n)^\alpha] = O\big(\Omega_0^p(a_n)^{p\alpha}(\log n)/(\delta_n c_n)^p\big)$$
$$= O\big(\Omega_0^{14}(\log_2 n)^{7\alpha}/(\log n)^{2/5}\big)$$

if we put $p = 14$, $a_n = (\log_2 n)^{1/2}$, and $\delta_n = (\log n)^{-2/5}$.

By (20)–(22), (24) and (29), (30), we obtain

$$(31)\qquad P[|Z_n - Z_n^{(2)}| > (\log n)^{-2/5}] = O\big((\log_2 n)^{7\alpha}(\log n)^{-2/5}\big).$$

Hence, by (14), (25) and (31) we get (7), and the proof of Theorem 1 is completed.

Proof of Theorem 2. Observe that

$$(32)\quad \sup_x |P[\Phi(S_n) \leqslant x] - P[\Phi(W) \leqslant x]| \leqslant \sup_x |P[\Phi(S_n) \leqslant x] - P[Z_n \leqslant x]|$$
$$+ \sup_x |P[Z_n \leqslant x] - P[\Phi(W) \leqslant x]| = I_1 + I_2.$$

The estimation of I_2 gives Theorem 1.

Moreover, we can write

$$(33)\quad \sup_x |P[\Phi(S_n) \leqslant x] - P[Z_n \leqslant x]| \leqslant P[|\Phi(S_n) - Z_n| \geqslant \delta_n]$$
$$+ \sup_x P[x - \delta_n < Z_n \leqslant x + \delta_n] \leqslant P[|\Phi(S_n) - Z_n| \geqslant \delta_n] + 2I_2 + 2\delta_n L$$

because $P[\Phi(W) \leqslant x]$ satisfies the Lipschitz condition with a positive constant L.

Hence, taking into account the proof of Theorem 1, we see that the proof of Theorem 2 will be completed if we show that

$$(34)\qquad P[|\Phi(S_n) - Z_n| \geqslant \delta_n] = O\big((\log n)^{-2/5}\big).$$

Now, observe that on the set $B_n^{(3)} = \{\sup_{0\leqslant t\leqslant 1} |S_n(t)| < a_n\}$, where $\{a_n\}$ is as in (22), we have

$$|\Phi(S_n) - Z_n| I(B_n^{(3)}) = \Big|\int_0^1 f(t, S_n(t))\,dt - \sum_{k=0}^{n-1} f(t_k, \tilde{S}_{n,k})(t_{k+1} - t_k)\Big| I(B_n^{(3)})$$
$$= \Big|\sum_{k=0}^{n-1} \int_{t_k}^{t_k+1} \big(f(t, S_n(t)) - f(t_k, \tilde{S}_{n,k})\big)\,dt\Big| I(B_n^{(3)})$$

$$\leqslant \Omega_0 a_n^\alpha \sum_{k=0}^{n-1} \Big(\int_{t_k}^{t_k+1} (s-t_k)\,ds + \int_{t_k}^{t_{k+1}} |S_n(t)-\tilde{S}_{n,k}|\,dt\Big)$$

$$\leqslant \Omega_0 a_n^\alpha \left(\sum_{k=0}^{n-1} \frac{(t_{k+1}-t_k)^2}{2} + \frac{1}{c_n} \max_{1\leqslant k\leqslant n} \left| X_{k+1}^* - \frac{1}{k+1} \right| \sum_{k=0}^{n-1} (t_{k+1}-t_k) \right)$$

$$\leqslant \Omega_0 a_n^\alpha \left(\frac{1}{2} \max_{1\leqslant k\leqslant n} (t_{k+1}-t_k) + \frac{1}{c_n}(X_1^*+1) \right) \leqslant \frac{\Omega_0 a_n^2}{c_n} \left(\frac{3}{2c_n} + X_1 \right)$$

by the definitions of t_k and c_n (cf. (2) and (9)).

Hence

$$P[|\Phi(S_n)-Z_n| \geqslant \delta_n,\ B_n^{(3)}] \leqslant P\left[X_1 \geqslant \frac{\delta_n c_n}{\Omega_0 a_n^\alpha} - \frac{3}{2c_n}\right] = 0 \tag{35}$$

for sufficiently large n such that

$$\frac{\delta_n c_n}{\Omega_0 a_n^\alpha} - \frac{3}{2c_n} \sim \frac{(\log n)^{1/10}}{\Omega_0 (\log_2 n)^\alpha} \geqslant 1.$$

Moreover, for sufficiently large n we can get

$$P(\overline{B_n^{(3)}}) = P[\max_{0\leqslant k\leqslant n-1} \sup_{t\in\langle t_k,t_{k+1})} |S_n(t)| \geqslant a_n]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} (|\tilde{S}_{n,k}| + |X_{k+1}^* - (k+1)^{-1}|/c_n) \geqslant a_n]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,k}| \geqslant a_n/2] + P[X_1+1 \geqslant (a_n c_n)/2]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} |U_{n,k}| + |U_{n,k} - \tilde{S}_{n,\tau_k}| + |\tilde{S}_{n,\tau_k} - \tilde{S}_{n,k}| \geqslant a_n/2]$$

$$\leqslant P[\max_{1\leqslant k\leqslant n} |U_{n,k}| \geqslant a_n/2 - 2\delta_n] + P[\max_{1\leqslant k\leqslant n} |U_{n,k} - \tilde{S}_{n,\tau_k}| \geqslant \delta_n]$$

$$+ P[\max_{1\leqslant k\leqslant n} |\tilde{S}_{n,\tau_k} - \tilde{S}_{n,k}| \geqslant \delta_n]$$

$$\leqslant P[\max_{0\leqslant k\leqslant n} |W(T_k)| \geqslant a_n/2 - 2\delta_n] + 2c(\log n)^{-2/5}$$

$$\leqslant P[\sup_{0\leqslant t\leqslant 1+g(n)} |W(t)| \geqslant a_n/2 - 2\delta_n]$$

$$+ P[\max_{0\leqslant k\leqslant n} |T_k - t_k| \geqslant g(n)] + 2c(\log n)^{-2/5} = O\big((\log n)^{-2/5}\big)$$

by (15), (19)–(21′), (24) and (29). Hence, using (35), we get (34). Combining this with (7), (32) and (33) we complete the proof of Theorem 2.

Proof of Theorem 3. Let us define a random function $\{U_n(t),\ t\in\langle 0, 1\rangle\}$ as follows:

(36) $$U_n(t) = U_{n,k} + \frac{t-t_k}{t_{k+1}-t_k}(U_{n,k+1}-U_{n,k}) \quad \text{for } t \in \langle t_k, t_{k+1}),$$
$$U_n(0) = 0, \quad 0 \leqslant k \leqslant n-1,\ n \geqslant 1,$$

where $U_{n,k}$ and t_k are as in the proof of Theorem 1. Let $P_n^{(1)}$ be the distribution of $\{U_n(t)\}$ in $(C, \mathscr{B}_C)$. At first, we show that

(37) $$\mathscr{L}_C(P_n^{(1)}, W) = O\big((\log_2 n)^{1/2}(\log n)^{-1/3}\big).$$

Let us observe that by (15) and a simple evaluation we obtain

$$P\big[\sup_{0\leqslant t\leqslant 1} |U_n(t)-W(t)| \geqslant \delta_n\big]$$
$$\leqslant P\big[\max_{0\leqslant k\leqslant n-1}\ \sup_{t\in\langle t_k,t_{k+1})} |U_{n,k}-W(t)| + \max_{0\leqslant k\leqslant n-1} |V_{n,k+1}| \geqslant \delta_n\big]$$
$$\leqslant P\big[\max_{0\leqslant k\leqslant n} |W(T_k)-W(t_k)| \geqslant \delta_n/3\big]$$
$$+P\big[\max_{0\leqslant k\leqslant n-1}\ \sup_{t\in\langle t_k,t_{k+1})} |W(t)-W(t_k)| \geqslant \delta_n/3\big]$$
$$+P\big[\max_{0\leqslant k\leqslant n} |V_{n,k+1}| \geqslant \delta_n/3\big].$$

Putting

$$B_n = \{\max_{0\leqslant k\leqslant n} |T_k-t_k| < g(n)\},$$

where $g(n)\to 0$, $n\to\infty$, so that $g^4(n)\log^3 n\to\infty$, by the invariance property of the Wiener process and the form of $\{t_k, 0\leqslant k\leqslant n\}$ we obtain

$$P\big[\max_{0\leqslant k\leqslant n} |W(T_k)-W(t_k)| \geqslant \delta_n/3,\ B_n\big]$$
$$\leqslant P\big[\max_{0\leqslant k\leqslant n}\ \sup_{t_k-g(n)\leqslant t\leqslant t_k+g(n)} |W(t)-W(t_k)| \geqslant \delta_n/3\big]$$
$$\leqslant P\big[\max_{0\leqslant k\leqslant n}\big(\sup_{0\leqslant t\leqslant g(n)} |W(t_k-t)-W(t_k)| + \sup_{0\leqslant t\leqslant g(n)} |W(t_k+t)-W(t_k)|\big) \geqslant \delta_n/3\big]$$
$$\leqslant P\big[2\max_{0\leqslant t\leqslant g(n)} |W(t)| > \delta_n/3\big] \leqslant 4P\big[|W(1)| > \delta_n/6\sqrt{g(n)}\big]$$
$$\leqslant \frac{8}{2\pi}\frac{6\sqrt{g(n)}}{\delta_n}\exp\left[-\frac{\delta_n^2}{36g(n)}\right],$$

where $g(n)$ and δ_n are such that $\delta_n/\sqrt{g(n)}\to\infty$ as $n\to\infty$.

Moreover, we can get

$$P\big[\max_{0\leqslant k\leqslant n-1}\ \sup_{t\in\langle t_k,t_{k+1})} |W(t)-W(t_k)| \geqslant \delta_n/3\big]$$
$$= P\big[\max_{0\leqslant k\leqslant n-1}\ \sup_{t\in\langle 0,t_{k+1}-t_k)} |W(t_k+t)-W(t_k)| > \delta_n/3\big]$$

$$\leqslant P\big[\sup_{t\in\langle 0,1/c_n^2)} |W(t)| > \delta_n/3\big] \leqslant 4P\,[|W(1)| > (\delta_n c_n)/3]$$

$$\leqslant \frac{8}{\sqrt{2\pi}}\frac{3}{\delta_n c_n}\exp\,[(\delta_n^2 c_n^2)/9],$$

where δ_n is taken so that $\delta_n c_n \to \infty$ as $n \to \infty$.

We can see that, by our Lemma 3.2 and Theorem 10 in [11] (p. 247),

$$P\big[\max_{0\leqslant k\leqslant n-1} |V_{n,k+1}| \geqslant \delta_n/3\big] = P\big[\max_{1\leqslant k\leqslant n} [(\tau_{k+1}-\tau_k-1)/k] \geqslant (\delta_n c_n)/3\big]$$

$$= 1-\prod_{k=1}^{n}\left(1-P\,[\tau_{k+1}-\tau_k \geqslant 1+(k\delta_n c_n)/3]\right)$$

$$= 1-\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)^{(k\delta_n c_n)/3}\right)$$

$$\sim 1-\exp\left[-\sum_{k=1}^{n}\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)^{(k\delta_n c_n)/3}\right] \sim \sum_{k=1}^{n}\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)^{(k\delta_n c_n)/3}$$

$$\sim \frac{\log n}{\exp\,[(\delta_n c_n)/3]}.$$

Hence, using (21) and Lemma 1.2 of [13], we get

$$\mathscr{L}_C(P_n^1, W) = O\Bigg(\max\Bigg(\delta_n, \frac{48}{\sqrt{2\pi}}\frac{\sqrt{g(n)}}{\delta_n}\exp\left[-\frac{\delta_n^2}{36g(n)}\right], \frac{\log n}{\exp\,[(\delta_n c_n)/3]},$$
$$\frac{8}{\sqrt{2\pi}}\frac{3}{\delta_n c_n}\exp\,[(\delta_n^2 c_n^2)/9],\ g(n),\ \frac{1}{g^4(n)\log^3 n}\Bigg)\Bigg).$$

Putting $\delta_n = (\log_2 n)^{1/2}(\log n)^{-1/3}$ and $g(n) = (\log n)^{-2/3}$, we obtain (37). Using (19) and (28), we get (8).

3. Lemmas. In this section we give some lemmas we needed in the proofs of Theorems 1–3.

Let $\{\varepsilon(n),\, n \geqslant 1\}$ be a sequence of positive numbers strictly decreasing to zero.

By $\{\tau_n = \tau(\varepsilon(n)),\, n \geqslant 1\}$ we denote the sequence of random variables such that

$$\tau_n = \inf\{m\colon \inf(X_1, X_2, \ldots, X_m) \leqslant \varepsilon(n)\}, \tag{3.1}$$

where $\{X_n,\, n \geqslant 1\}$ is a sequence of i.r.vs. u.d. on [0, 1].

LEMMA 3.1. *The sequence* $\{\tau_n,\, n \geqslant 1\}$ *increases with probability one and* $\tau_n \to \infty$ *a.s. as* $n \to \infty$.

LEMMA 3.2. *The random variables* $\tau_{n+1}-\tau_n$, $n \geqslant 1$, *are independent, and if* $\varepsilon(n) = n^{-1}$, *then*

(3.2) $$\mathrm{E}(\tau_{n+1}-\tau_n) = 1, \quad \sigma^2(\tau_{n+1}-\tau_n) = 2n, \quad n \geqslant 1,$$

(3.3) $$P[(\tau_{n+1}-\tau_n) \geqslant r] = \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^{r-1} \quad \textit{for any } r>0,\ n\geqslant 1.$$

Let us put

(3.4) $$U_n = \sum_{k=1}^{n-1}(\tau_{k+1}-\tau_k)\frac{1}{k}, \quad U_n' = \sum_{k=1}^{n-1}(\tau_{k+1}-\tau_k)\frac{1}{k+1}.$$

Then

(3.5) $$\mathrm{E}U_n - \log n = O(1), \quad \mathrm{E}U_n' - \log n = O(1),$$

(3.6) $$\sigma^2 U_n - 2\log n = O(1), \quad \sigma^2 U_n' - 2\log n = O(1),$$

(3.7) $$\sum_{k=1}^{n} \mathrm{E}(\tau_{k+1}-\tau_k)^p/k^p \sim \sum_{k=1}^{n} \mathrm{E}(\tau_{k+1}-\tau_k)^p/(k+1)^p \sim p!\log n,$$

(3.8) $$\mathrm{E}(U_n - U_n') = O(1), \quad \sigma^2(U_n-U_n') = O(1),$$
$$\mathrm{E}\big(U_n-U_n'-\mathrm{E}(U_n-U_n')\big)^4 = O(1),$$

where $b_n = O(1)$ means that the sequence $\{b_n, n\geqslant 1\}$ is bounded as $n\to\infty$.

LEMMA 3.3. *Let U_n, U_n' be given by* (3.4). *Then*

(3.9) $$-2+U_n' \leqslant \tilde{S}_{\tau_n} - \tilde{S}_{\tau_1} \leqslant U_n \ \textit{a.s.}, \quad n\geqslant 2,$$

(3.10) $$\tilde{S}_{\tau_{n-1}} \leqslant \tilde{S}_m \leqslant \tilde{S}_{\tau_m} \quad \textit{for } m\in\langle\tau_{n-1},\tau_n),$$

where

$$\tilde{S}_n = \sum_{k=1}^{n} X_k^*, \quad \tilde{S}_{\tau_n} = \sum_{m=1}^{\tau_n} X_m^*, \quad n\geqslant 1,\ X_k^* = \inf(X_1, X_2, \ldots, X_k),\ k\geqslant 1.$$

LEMMA 3.4. *If we put $S_{N,m} = \sum_{k=m}^{N} X_k^*$, then for all $p\geqslant 1$*
$$\mathrm{E}S_{N,m}^p = O(\log N - \log m).$$

LEMMA 3.5. *If $\varepsilon(n) = 1/n$, then*

(3.11) $$\mathrm{E}\big((\log\tau_n) I[\tau_n > n]\big) = O(\log n).$$

Proof. By definition (3.1), we have

$$\mathrm{E}\big((\log\tau_n) I[\tau_n>n]\big) = \sum_{k=n+1}^{\infty}(\log k) P[\tau_n = k] = n^{-1}\sum_{k=n+1}^{\infty}(1-n^{-1})^{k-1}\log k.$$

Let us put $q = 1-n^{-1}$ and write $A_n = \int_{n+1}^{\infty}(\log x) q^{x-1}\,dx$.
By a simple evaluation we get

$$A_n = \frac{1}{\log q}\int_{n+1}^{\infty}(\log x)[q^{x-1}]'\,dx = \frac{q^n\log(n+1)}{\log(1/q)} + \frac{1}{\log(1/q)}\int_{n+1}^{\infty}\frac{q^{x-1}}{x}\,dx$$

$$\leqslant \frac{q^n \log(n+1)}{\log(1/q)} + \frac{1}{(n+1)\log(1/q)} \int_{n+1}^{\infty} q^{x-1}\, dx = \frac{q^n \log(n+1)}{\log(1/q)} + \frac{q^n}{(n+1)\log^2(1/q)},$$

and so

$$\frac{1}{n} A_n = \left(1-\frac{1}{n}\right)^n \left[\frac{\log(n+1)}{n\log(1+1/(n-1))} + \frac{1}{n(n+1)\log^2(1+1/(n-1))}\right] = O(\log n)$$

because $\log(1/q) = \log(1+1/(n-1)) = 1/(n-1) + O(1/(n-1))$.

Hence, using the integrable type criterion of series convergence, we have (3.11).

LEMMA 3.6 (the Skorokhod representation theorem [14]). *Let $Y_1, Y_2, \ldots, Y_n$ be mutually independent random variables with zero means and $\sigma^2 Y_i = \sigma_i^2$, $1 \leqslant i \leqslant n$. Then there exists a sequence of nonnegative, mutually independent random variables $z_1, z_2, \ldots, z_n$ with the following properties:*

The joint distributions of the r.vs. $Y_1, Y_2, \ldots, Y_n$ are identical to the joint distributions of the r.vs. $W(z_1)$, $W(z_1+z_2)-W(z_1), \ldots, W(z_1+\ldots+z_n) - W(z_1+\ldots+z_{n-1})$, $\mathrm{E} z_i = \sigma_i^2$ and $\mathrm{E}|z_i|^k \leqslant C_k \mathrm{E}(Y_i)^{2k}$, $k \geqslant 1$, where $C_k = 2(8/\pi^2)^{k-1}\Gamma(k+1)$.

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Received on 29.6.1992;
revised version on 25.5.1993