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TESTING GAUSSIAN SEQUENCES AND ASYMPTOTIC INVERSION OF TOEPLITZ OPERATORS

BY

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Abstract. This paper is motivated by the statistical problem of testing a zero-mean stationary Gaussian probability measure P on $\boldsymbol{R}^{\boldsymbol{Z}}$ against a similiar probability Q. The method uses a sequence of Neyman–Pearson's tests of the finite sections P_n of P against the corresponding sections Q_n of Q. First, following D. Dacunha-Castelle, we discuss the behaviour of the power achieved for levels approaching zero exponentially fast at a suitable rate. Then, considering the likelihood ratio of Q_n w.r.t. P_n, we ask whether there exist approximate inverses of the covariance matrices of these probabilities, and approximates of their determinants, which preserve the asymptotics of the tests considered. It turns out that these matrices are finite sections of the Toeplitz operators whose symbols are the spectral densities of P and Q. Using results of H. Widom on this class of operators we point out that such approximations exist and work under some factorisation condition for spectral densities. It is also shown that the same approximation method works for asymptotic solving of a class of discrete Wiener–Hopf equations.

1. Introduction. Let P and Q be two probability measures on $\boldsymbol{R}^{\boldsymbol{Z}}$, and let X_t: $\boldsymbol{R}^{\boldsymbol{Z}} \to \boldsymbol{R}$, $t \in \boldsymbol{Z}$, be the canonical coordinate process. For each positive integer n, let $X(n) = (X_0, \ldots, X_{n-1})'$ and denote by P_n (resp. Q_n) the probability distribution of $X(n)$ under P (resp. Q).

The problem considered here is the statistical problem of testing P against Q using a sequence of tests of P_n against Q_n for given observations $X(n)$ and significance levels α_n. The main question is the asymptotic behaviour of the sequence of powers achieved by the most powerful tests for suitable sequences of levels α_n. Let us first recall the main facts. Assume that P_n and Q_n are equivalent for each n, and put

$$\Lambda_n = \log dQ_n/dP_n.$$

Given a significance level α_n in the open interval (0, 1), the most powerful tests of P_n against Q_n at level α_n are the Neyman–Pearson tests of size α_n, namely those tests τ_n that satisfy

(1.1) $$I(\Lambda_n > a_n) \leqslant \tau_n \leqslant I(\Lambda_n \geqslant a_n), \qquad E_{P_n}(\tau_n) = \alpha_n.$$

Here $I(A)$ stands for the indicator function of the set A, and a_n is a real non-random constant. Such tests exist (see [16], p. 74). The power of the test τ_n is by definition $\beta_n = E_{Q_n}(\tau_n)$. It was observed by Dacunha-Castelle [7], [8], [10] that the power problem indicated above can be made precise and solved by the method of large deviations for Λ_n. The following remark drawn from [9] and [21] gives some information concerning this method.

Let S_n be a sequence of real-valued random variables defined on some probability spaces $(\Omega_n, \mathscr{F}_n, \boldsymbol{P}_n)$. Denote by ϕ_n the moment generating function of S_n:

$$\phi_n(\theta) = E_{\boldsymbol{P}_n} \exp \theta S_n, \qquad \theta \in \boldsymbol{R}.$$

Suppose that $\phi_n(\theta_0) < \infty$ for some $\theta_0 > 0$, so that ϕ_n exists and is finite on the closed interval $[0, \theta_0]$, by Hölder's inequality. By rescaling S_n, if necessary, we can assume that $\theta_0 = 1$. It is proved in Lemma 2.1 that if

(1.2) $$(1/n)\log \phi_n(\theta) \to \psi(\theta), \qquad 0 \leqslant \theta \leqslant 1,$$

(1.3) ψ is differentiable and strictly convex on $(0, 1)$,

then for all a in $\psi'(0, 1)$ we have

(1.4) $$(1/n)\log \boldsymbol{P}_n(S_n > na) \to -\psi^{\#}(a).$$

Here $\psi^{\#}$ is the Fenchel–Legendre conjugate of ψ ([19], pp. 28–35) given by

(1.5) $$\psi^{\#}(a) = \sup_{0 \leqslant \theta \leqslant 1} \big(a\theta - \psi(\theta)\big), \qquad a \in \boldsymbol{R}.$$

The derivative ψ' is then strictly increasing and continuous ([19], pp. 5–7), and the restriction of $\psi^{\#}$ to $\psi'(0, 1)$ is given by solving the equations ([19], p. 34):

(1.6) $$\psi'(\theta) = a, \qquad \psi^{\#}(a) + \psi(\theta) = a\theta.$$

Now look at the pair Λ_n, P_n in place of S_n, $\boldsymbol{P}_n$. Note that in this case $\phi_n(1) = 1$. Assume that (1.2) and (1.3) hold, fix a as above and put $\alpha_n = \exp\big(-n\psi^{\#}(a)\big)$. Using (1.4) it can be shown (cf. [7], [8], [10]) (see also Theorem 2.4) that

(1.7) $$(1/n)\log(1-\beta_n) \to a - \psi^{\#}(a), \qquad a_n \sim na.$$

It is therefore of interest to find conditions on the pair P, Q under which (1.2) and (1.3) hold and then identify ψ. Such conditions have been found by Dacunha-Castelle [7], [8], [10] and refined by Coursol and Dacunha-Castelle [6] in the Gaussian case. We discuss this now.

Let $\mathscr{G}$ be the set of probability measures on $\boldsymbol{R}^{\boldsymbol{Z}}$ that are Gaussian, stationary, with mean zero and absolutely continuous spectral measure (with respect to normalized Lebesgue measure on the unit circle). Denote by $\mathscr{S}$ the

Szegö class, that is the set of non-negative functions on the unit circle with integrable logarithm. A theorem of Coursol and Dacunha-Castelle [6] states that if P, Q are in $\mathscr{G}$ with spectral densities u, v in $\mathscr{S}$, then (1.2) holds for the pair Λ_n, P_n with

$$\psi(\theta) = \frac{1}{2}\int \big(\theta \log u + (1-\theta)\log v - \log(\theta u + (1-\theta)v)\big)\frac{dx}{2\pi}. \tag{1.8}$$

A short proof of this is given in Theorem 2.2 below. Note that, under these conditions, P_n, Q_n are equivalent for each n (see, e.g., (1.9) below and Proposition 3.5). The condition (1.3) can be assured by assuming, for example, that the spectral densities u, v are (essentially) bounded away from zero and infinity and they differ on a set of positive measure.

Fix a pair P, Q in $\mathscr{G}$ with spectral densities u, v and look at the form of Λ_n in this case. The covariance matrix of P_n has (s, t) entries

$$E_P X_s X_t = \hat{u}(s-t) = \int_{-\pi}^{\pi} e^{-i(s-t)x} u(e^{ix})\frac{dx}{2\pi}, \qquad s, t = 0, \ldots, n-1.$$

This is the n-th order Toeplitz matrix with symbol u, usually denoted by $T_n(u)$. Note that for w in L^1, and ξ, η in C^n (the product of n copies of the complex plane) we have

$$\eta^* T_n(w)\xi = \int \Big(\sum_0^{n-1} \xi_t e^{itx}\Big)\Big(\sum_0^{n-1} \eta_t e^{itx}\Big)^* w(e^{ix})\frac{dx}{2\pi}. \tag{1.9}$$

This shows that if $w \geqslant 0$ and if w is non-zero, that is $w > 0$ on a set of positive measure, then $T_n(w)$ is invertible for all n (Proposition 3.5). Thus assuming this for u, v we have

$$\Lambda_n = \frac{1}{2}\left(\log\frac{\det T_n(u)}{\det T_n(v)} + X(n)'\,[T_n^{-1}(u) - T_n^{-1}(v)]\,X(n)\right). \tag{1.10}$$

Since the treatment of the power problem is asymptotic, it seems reasonable to expect that the asymptotics of the tests defined by (1.1) will not be violated if the inverses (and determinants) of the matrices $T_n(u)$ and $T_n(v)$ are approximated in some appropriate way. This question is closely related to that of asymptotic inversion of Toeplitz operators as follows.

(a) For every (linear and bounded) operator A on a separable complex Hilbert space, denote by $\|A\|_\infty$ the operator norm of A and by $\|A\|_1$ the trace of the (positive) square root of $A^* A$. Such an operator is said to be of *trace class* if $\|A\|_1$ is finite (see the last paragraph of this section).

First, we show that if the spectral densities u, v are bounded away from zero and infinity and they differ on a set of positive measure, then any real approximate inverses $T_n^{(-1)}(u)$ and $T_n^{(-1)}(v)$ for $T_n(u)$ and $T_n(v)$ that satisfy

$$\|T_n^{-1}(w) - T_n^{(-1)}(w)\|_\infty \to 0 \tag{1.11}$$

preserve the limit formula of Coursol and Dacunha-Castelle (1.8) and the asymptotics of the corresponding approximate tests at exponential levels (Theorems 2.3 and 2.4). We show in addition that these asymptotics are also preserved if the determinant of $T_n(u)$ (resp. $T_n(v)$) is approximated by the n-th power of the geometric mean of u (resp. v). This is made precise in Section 2. Now note that if w is non-negative and bounded away from zero and infinity, then the sequence $T_n(w)$ and that of the inverses are bounded in operator norm. This follows from (1.9). Thus, in this case, (1.11) is equivalent to

$$\|T_n^{(-1)}(w)\|_\infty \ll 1, \quad \|1_n - T_n(w)\, T_n^{(-1)}(w)\|_\infty \to 0, \tag{1.12}$$

where $a_n \ll b_n$ stands for the usual Landau notation $a_n = O(b_n)$, and 1_n is the $n \times n$ unit matrix. The conditions (1.12) have three remarkable features. First, they are more manageable than (1.11). Second, if a pair M_n, $M_n^{(-1)}$ of operators on C^n satisfy (1.12), then M_n is invertible for all sufficiently large n and the sequence of inverses is bounded in operator norm ([23], p. 193). Third, they have an operator meaning which is related to the problem of asymptotic inversion of invertible Toeplitz operators by the so-called finite section method (see (c) below). Note that (1.12) is implied by

$$\|T_n^{(-1)}(w)\|_\infty \ll 1, \quad \|1_n - T_n(w)\, T_n^{(-1)}(w)\|_1 \to 0 \tag{1.13}$$

since the trace class norm is greater than the operator norm. We now describe a class of symbols w and approximate inverses that satisfy (1.12) or (1.13).

(b) For w in L^∞, denote by $T(w)$ and $H(w)$ the usual Toeplitz and Hankel operators on H^2 with symbol w. Here H^p, $1 \leqslant p \leqslant \infty$, stands for the usual Hardy space of the unit circle [20]. The matrix entries of $T(w)$ and $H(w)$ in the natural basis of H^2 are given by (see [4])

$$T(w)(s, t) = \hat{w}(s-t), \qquad H(w)(s, t) = \hat{w}(s+t+1), \qquad 0 \leqslant s, t < \infty.$$

Let Π_n denote the natural projection of H^2 onto the subspace of polynomials of degree at most $n-1$. Then $T_n(w)$ can also be viewed as an operator on the range of Π_n and identified with $\Pi_n T(w) \Pi_n$. Similarly, the finite Hankel operators are defined by $H_n(w) = \Pi_n H(w) \Pi_n$. Denote by $\mathscr{C}_+$ the Banach algebra of those continuous functions with vanishing Fourier coefficients on the negative integers, and by $H_{1/2}$ the class of those w in L^2 that satisfy

$$\|w\|_{1/2} = \Big(\sum_{t=-\infty}^{+\infty} |t|\, |\hat{w}(t)|^2\Big)^{1/2} < \infty.$$

Note that $L^\infty \cap H_{1/2}$ is a Banach algebra under the norm $\|w\|_\infty + \|w\|_{1/2}$ (cf. [15]; see also [4]), and that $H^\infty \cap H_{1/2}$ is a closed subalgebra of $L^\infty \cap H_{1/2}$. Finally, put $\check{a}(z) = a(1/z)$.

A theorem of Widom [23] (see also [22] and [24]) states that if a, b are in the range of the exponential function acting on $H^\infty \cap H_{1/2}$, then $T_n(\check{a}b)$ is invertible for all sufficiently large n and the approximate inverse

$$(1.14) \qquad W_n(\check{a}b) = T_n(\check{a}^{-1})\,T_n(b^{-1}) - \Pi_n H(b^{-1})\,H(a^{-1})\,\Pi_n$$

satisfies (1.13). We show in Theorem 3.3 that for every pair a, b of invertible elements in $\mathscr{C}_+$ (resp. $H^\infty \cap H_{1/2}$), the approximate inverse (1.14) and its modified version

$$(1.15) \qquad \tilde{W}_n(\check{a}b) = T_n(\check{a}^{-1})\,T_n(b^{-1}) - H_n(b^{-1})\,H_n(a^{-1})$$

satisfy (1.12) (resp. (1.13)). The interest of the latter version lies in the fact that it involves only finite matrices.

(c) Here is another relevant application of the asymptotic inversion conditions (1.12). This deals with the problem of asymptotic inversion of Toeplitz operators. Let w be in L^∞ and suppose that $T(w)$ is invertible. This is so if, for example, w has a factorisation $\check{a}b$ with a, b in H^∞ both invertible (cf. [4] and [5]; see also (3.2)). So the equation

$$(1.16) \qquad T(w)\,x = y$$

can be solved for x uniquely whatever be y in H^2. In terms of coordinates, this equation is equivalent to the following system of discrete Wiener–Hopf equations:

$$\sum_{t \geqslant 0} \hat{w}(s-t)\,\hat{x}(t) = \hat{y}(s), \qquad 0 \leqslant s < \infty.$$

For practical reasons, one needs to transform the infinite dimensional equation (1.16) into a sequence of finite dimensional equations whose solution converges to the exact solution of (1.16). This can be done with the so-called reduction, projection or finite section method (see [3], [4], [11], [17]) as follows. Solve, if possible, the finite dimensional equation

$$(1.17) \qquad T_n(w)\,\xi = y_n \qquad (y_n = \Pi_n y)$$

and ask whether the resulting solution ξ_n converges in H^2 to the exact solution of (1.16). Baxter [3] has proved this convergence under the condition that w lies in the range of the exponential function acting on the Banach algebra of those f that satisfy $\sum_t v(t)\,|\hat{f}(t)| < \infty$ for some fixed sequence v which is submultiplicative and bounded from below by 1.

The point is that stated as above the problem does not require to invert $T_n(w)$ exactly. One tries some approximate and manageable inverse $T_n^{(-1)}(w)$ for $T_n(w)$ and defines

$$(1.18) \qquad \tilde{\xi}_n = T_n^{(-1)}(w)\,y_n.$$

So the problem is turned into the question of finding conditions on w under which such an approximate inverse exists and makes the approximate solution $\tilde{\xi}_n$ close to ξ_n and convergent to the exact solution of (1.16). It turns out as in the power problem that every sequence $T_n^{(-1)}(w)$ satisfying (1.12) works. This is made precise and proved in Theorem 3.4.

We conclude this section with some remarks on the materials used in the statements and proofs of theorems. These are the von Neumann–Schatten norm ideals [12], [18]. Let $\mathscr{H}$ be a separable complex Hilbert space. Denote by $\mathscr{I}_\infty(\mathscr{H})$ or simply $\mathscr{I}_\infty$ the Banach algebra of (linear and bounded) operators on $\mathscr{H}$. The operator norm is denoted by $\|\cdot\|_\infty$. For each real number $p \geqslant 1$, the von Neumann–Schatten class $\mathscr{I}_p$ consists of those A in $\mathscr{I}_\infty$ that satisfy $\|A\|_p = [\operatorname{tr}(A^* A)^{p/2}]^{1/p} < \infty$, where tr stands for trace. Note that every positive operator has a well-defined trace (eventually infinite). In addition, every member of $\mathscr{I}_1$ has finite trace. The class $\mathscr{I}_1$ (resp. $\mathscr{I}_2$) is the trace class (resp. Hilbert–Schmidt class). Each $\mathscr{I}_p$ is a two-sided ideal of $\mathscr{I}_\infty$ and a Banach space with the norm $\|\cdot\|_p$. For finite p, the members of $\mathscr{I}_p$ are compact operators and the finite rank operators are dense in $\mathscr{I}_p$. There are two basic inequalities. The first one is the following:

$$|\operatorname{tr}(A)| \leqslant \|A\|_1, \qquad A \in \mathscr{I}_1.$$

If $A \in \mathscr{I}_p$ and $B \in \mathscr{I}_{p'}$ with $1/p+1/p' = 1$, then AB and BA are in $\mathscr{I}_1$ and the pair A, B satisfies the Hölder inequality for operators:

$$\|AB\|_1 \leqslant \|A\|_p \|B\|_{p'}.$$

2. Testing Gaussian sequences. This section deals with three items. The first is a remark on large deviations drawn from [9] and [21] and adapted to our situation. The second gives a short proof of a theorem of Coursol and Dacunha-Castelle [6]. The third investigates the asymptotics of the Neyman–Pearson tests (1.1) when the inverses (and determinants) of the finite Toeplitz matrices associated with the spectral densities u, v are suitably approximated.

Let S_n be a sequence of real-valued random variables defined on some probability spaces $(\Omega_n, \mathscr{F}_n, P_n)$. Denote by ϕ_n the moment generating function of S_n.

LEMMA 2.1 ([9], [21]). *Suppose that*
(a) $\phi_n(1) < \infty$ *for all (sufficiently large)* n;
(b) $(1/n)\log\phi_n(\theta) \to \psi(\theta)$ *for all* θ *in* $[0, 1]$;
(c) ψ *is differentiable and strictly convex on* $(0, 1)$.
Then

$$\forall a \in \psi'(0, 1), \; (1/n)\log P_n(S_n > na) \to -\psi^{\#}(a),$$

where $\psi^{\#}$ *is the Fenchel–Legendre conjugate of* ψ *defined by* (1.5) *and* (1.6). *In addition, the same is true for* $(S_n \geqslant na)$.

Proof. Put $dP_{n,\theta} = [e^{\theta S_n}/\phi_n(\theta)]\, dP_n$, $\psi_n = (1/n)\log\phi_n$, and denote by $E_{n,\theta}$ the expectation operator under $P_{n,\theta}$. The main step of the proof is the following law of large numbers:

$$(1/n)\, S_n \to \psi'(\theta) \text{ in } P_{n,\theta} \text{ probability.}$$

To prove this we shall show that

$$P_{n,\theta}(S_n/n < \alpha \text{ or } S_n/n > \beta) \to 0 \quad \text{if } \alpha < \psi'(\theta) < \beta.$$

We prove that $P_n(S_n/n < \alpha) \to 0$. The other half of the proof is similar to the first. Fix θ' such that $0 \leqslant \theta' < \theta$. Using Markov's inequality, we get

$$\begin{aligned} P_{n,\theta}(S_n < n\alpha) &\leqslant E_{n,\theta} \exp[(\theta'-\theta)(S_n - n\alpha)] \\ &= \exp\left[-n(\theta-\theta')\left(\frac{\psi_n(\theta)-\psi_n(\theta')}{\theta-\theta'} - \alpha\right)\right]. \end{aligned}$$

Now

$$\frac{\psi_n(\theta)-\psi_n(\theta')}{\theta-\theta'} \to \frac{\psi(\theta)-\psi(\theta')}{\theta-\theta'}.$$

Thus there exists a θ' as above such that the right-hand quotient is bounded from below by, say, half of $\psi'(\theta)+\alpha$, and therefore so is the left-hand term for all sufficiently large n. This proves that $P_n(S_n < n\alpha) \to 0$. Next, we prove the convergence property stated in the lemma. Fix θ in (0, 1) and put $a = \psi'(\theta)$, so that $\psi^{\#}(a)+\psi(\theta) = a\theta$. The inequality

$$\limsup (1/n)\log P_n(S_n > na) \leqslant -\psi^{\#}(a)$$

follows from a standard argument [1] using only Markov's inequality. To prove that

$$\liminf (1/n)\log P_n(S_n > na) \geqslant -\psi^{\#}(a)$$

fix $\varepsilon > 0$ and $\tau \in (0, 1)$ such that $a < \psi'(\tau) < a+\varepsilon$. Thus $\tau \to \theta$ as $\varepsilon \to 0$, since ψ' is strictly increasing and continuous on (0, 1) ([19], pp. 5–7). Now

$$\begin{aligned} P_n(S_n > na) &= \phi_n(\tau)\, E_{n,\tau} \exp(-\tau S_n)\, I(S_n/n > a) \\ &\geqslant \phi_n(\tau) \exp[-n\tau(a+\varepsilon)]\, P_{n,\tau}(a < S_n/n < a+\varepsilon). \end{aligned}$$

Taking logarithms, dividing by n, using the above law of large numbers, and letting $\varepsilon \to 0$ we get the first part of the lemma. Replacing $(S_n > na)$ by $(S_n \geqslant na)$ does not violate the above properties of the lim sup and lim inf, so the conclusion remains true also in this case. This completes the proof. ∎

We now give a short proof of formula (1.8) for the pair Λ_n, P_n.

THEOREM 2.2 ([6]). *If P, Q are in $\mathcal{G}$ with spectral densities u, v in $\mathcal{S}$, then* (1.2) *holds for the pair Λ_n, P_n with ψ as in* (1.8).

Proof. Note first that if w is in $\mathscr{S}$, then $w > 0$ on a set of positive measure. Thus (see Proposition 3.5) $T_n(w)$ is invertible for all n. Using formula (1.10) for Λ_n, and noting that, under P, the $\boldsymbol{R}^n$-valued random vector $X(n)$ is Gaussian with mean zero and covariance $T_n(u)$, we get

$$\phi_n(\theta) = \left(\frac{\det T_n(u)}{\det T_n(v)}\right)^{\theta/2} F_n(\theta), \tag{2.1}$$

$$F_n(\theta) = [\det T_n(u)]^{-1/2} \int_{\boldsymbol{R}^n} \exp[-\tfrac{1}{2}x' A_n(\theta) x] \frac{dx}{(2\pi)^{n/2}}, \tag{2.2}$$

$$A_n(\theta) = (1-\theta) T_n^{-1}(u) + \theta T_n^{-1}(v). \tag{2.3}$$

Now, for every $\theta \in [0, 1]$, the matrix $A_n(\theta)$ is real, symmetric and positive. Using the Gaussian integral formula

$$\int_{\boldsymbol{R}^n} \exp[-\tfrac{1}{2}x' Ax]\, dx = (2\pi)^{n/2} (\det A)^{-1/2}$$

which is valid for every $n \times n$ real, symmetric and positive matrix A, we get

$$\begin{aligned} F_n(\theta) &= [\det T_n(u)]^{-1/2} [\det A_n(\theta)]^{-1/2} \\ &= [\det T_n(v)]^{1/2} \left[\det T_n\big(\theta u + (1-\theta) v\big)\right]^{-1/2}, \end{aligned}$$

$$\phi_n(\theta) = [\det T_n(u)]^{\theta/2} [\det T_n(v)]^{(1-\theta)/2} \left[\det T_n\big(\theta u + (1-\theta) v\big)\right]^{-1/2}.$$

The conclusion follows now from the first Szegö limit theorem ([13], p. 44):

$$[\det T_n(w)]^{1/n} \to G(w) = \exp \int \log w(e^{ix}) \frac{dx}{2\pi}, \qquad w \in \mathscr{S}. \tag{2.4}$$

Since $\mathscr{S}$ is convex, $\theta u + (1-\theta) v$ is also in $\mathscr{S}$. This completes the proof. ∎

We now turn to the power problem outlined in Section 1. Our objective is to show that the asymptotics (1.7) of the tests (1.1) are not violated if the inverses (and determinants) of the Toeplitz matrices occurring in the definition of these tests are suitably approximated.

Suppose that the spectral densities u, v are bounded away from zero and infinity. Let $T_n^{(-1)}(u)$ (resp. $T_n^{(-1)}(v)$) be any real $(n \times n)$-matrix which approximates $T_n^{-1}(u)$ (resp. $T_n^{-1}(v)$) in the sense of (1.11). Approximate the log-determinant of $T_n(u)$ (resp. $T_n(v)$) by $\log[G(u)]^n$ (resp. $\log[G(v)]^n$), where $G(w)$ stands for the geometric mean of the function w as in (2.4). Define the approximate log-likelihood ratio of Q_n w.r.t. P_n by

$$\Lambda_n' = \tfrac{1}{2}\left\{\log\big(G(u)/G(v)\big)^n + X(n)' [T_n^{(-1)}(u) - T_n^{(-1)}(v)] X(n)\right\}. \tag{2.5}$$

Put $\tilde{\phi}_n(\theta) = E_{P_n} \exp(\theta \Lambda_n')$. We begin with the limit formula of Coursol and Dacunha-Castelle for $\tilde{\phi}_n$.

THEOREM 2.3. *Suppose that the spectral densities u, v are bounded away from zero and infinity, and that the approximate inverses $T_n^{(-1)}(u)$, $T_n^{(-1)}(v)$ satisfy* (1.11). *Then*

$$(1/n)\log \tilde{\phi}_n(\theta) \to \psi(\theta), \quad 0 \leqslant \theta \leqslant 1,$$

with ψ as in (1.8).

Proof. Following the method of proof of Theorem 2.2, we have

$$\tilde{\phi}_n(\theta) = \big(G(u)/G(v)\big)^{n\theta/2}\,\tilde{F}_n(\theta),$$

$$\tilde{F}_n(\theta) = \big(\det T_n(u)\big)^{-1/2} \int_{R^n} \exp\big[-\tfrac{1}{2}x'\tilde{A}_n(\theta)x\big]\frac{dx}{(2\pi)^n},$$

$$\tilde{A}_n(\theta) = T_n^{-1}(u) - \theta\big[T_n^{(-1)}(u) - T_n^{(-1)}(v)\big].$$

Since $\tilde{F}_n(\theta)$ is unchanged if $\tilde{A}_n(\theta)$ is replaced by its symmetric, symmetrizing $T_n^{(-1)}(u)$ and $T_n^{(-1)}(v)$, if necessary, we can assume that the last two matrices are symmetric. Define $A_n(\theta)$ as in (2.3) and put

$$R_n(w) = T_n^{-1}(w) - T_n^{(-1)}(w), \quad Z_n(\theta) = A_n(\theta)^{-1/2}\big[R_n(u) - R_n(v)\big]A_n(\theta)^{-1/2}.$$

Thus

$$\tilde{A}_n(\theta) = A_n(\theta) + \theta\big[R_n(u) - R_n(v)\big] = A_n(\theta)^{1/2}\big[1_n + \theta Z_n(\theta)\big]A_n(\theta)^{1/2}.$$

Therefore, if we prove that $\|Z_n(\theta)\|_\infty < 1$, and actually we shall prove that this norm converges to zero, then $1_n + \theta Z_n(\theta)$ and $\tilde{A}_n(\theta)$ are positive for all sufficiently large n. Thus the pair $\tilde{\phi}_n$, $\tilde{F}_n$ is related to the pair ϕ_n, F_n of (2.1) and (2.2) by the following relations:

$$\tilde{F}_n(\theta) = F_n(\theta)\big[\det\big(1_n + \theta Z_n(\theta)\big)\big]^{-1/2}, \quad \tilde{\phi}_n(\theta) = \phi_n(\theta)\,\varepsilon_n(\theta),$$

$$\varepsilon_n(\theta) = \left[\left(\frac{G(u)}{G(v)}\right)^n \left(\frac{\det T_n(u)}{\det T_n(v)}\right)^{-1}\right]^{\theta/2} \big[\det\big(1_n + \theta Z_n(\theta)\big)\big]^{-1/2}.$$

By Theorem 2.2, the proof will be complete if we prove that

(a) $\|Z_n(\theta)\|_\infty \to 0$;

(b) if U_n is a sequence of $n\times n$ real matrices satisfying (a), then

$$(1/n)\log\det(1_n + U_n) \to 0.$$

To prove (a), observe that, by (1.9), the eigenvalues of both $T_n(u)$ and $T_n(v)$ are uniformly bounded away from zero and infinity. Thus the operator norm of $A_n(\theta)^{-1/2}$ is uniformly bounded (in both n and θ). Now (a) follows from the approximation condition (1.11).

To prove (b), put $f_n(\theta) = \log\det(1_n + \theta U_n)$, $0 \leqslant \theta \leqslant 1$, and use the mean-value theorem to estimate $f_n(1)$. Note that $1_n + \theta U_n$ is positive for all sufficiently large n, since U_n satisfies (a). Now the derivative of f_n is $\operatorname{tr}\big[(1_n + \theta U_n)^{-1}U_n\big]$ (see, e.g., [12], p. 163). By Hölder's inequality for operators, this is bounded by $\|U_n\|_\infty \|(1_n+\theta U_n)^{-1}\|_1$, and we have

$$\|(1_n+\theta U_n)^{-1}\|_1 \leqslant n\,\|(1_n+\theta U_n)^{-1}\|_\infty \leqslant n(1-\theta\,\|U_n\|_\infty)^{-1}.$$

Thus, by the mean-value theorem, we have, for all sufficiently large n,

$$|\log\det(1_n+U_n)| \leqslant n\,\|U_n\|_\infty(1-\|U_n\|_\infty)^{-1}.$$

This proves (b) and completes the proof. ∎

Next, suppose that the conditions of Theorem 2.3 are satisfied. Fix a in $\psi'(0, 1)$, and consider those tests τ' that satisfy

$$I(\Lambda'_n > a'_n) \leqslant \tau'_n \leqslant I(\Lambda'_n \geqslant a'_n), \qquad E_{P_n}(\tau'_n) = \exp[-n\psi^{\#}(a)], \tag{2.6}$$

where a'_n is a real non-random constant. Of course, such tests exist, since Λ'_n is finite. The power of τ'_n is $\beta'_n = E_{Q_n}(\tau'_n)$.

THEOREM 2.4. *Suppose that*

(a) *the spectral densities u, v are bounded away from zero and infinity and they differ on a set of positive measure;*

(b) *the approximate inverses $T_n^{(-1)}(u)$, $T_n^{(-1)}(v)$ satisfy* (1.11).

Then the power β'_n and the threshold a'_n of the test defined by (2.6) *also satisfy* (1.7).

Proof. We split the proof into three parts:

$$(1/n)\log Q_n(\Lambda_n < na) \to a-\psi^{\#}(a), \tag{2.7}$$

$$\liminf(1/n)\log(1-\beta'_n) \geqslant a-\psi^{\#}(a), \qquad \liminf(1/n)\,a'_n \geqslant a, \tag{2.8}$$

$$\limsup(1/n)\log(1-\beta'_n) \leqslant a-\psi^{\#}(a), \qquad \limsup(1/n)\,a'_n \leqslant a. \tag{2.9}$$

Note first that ψ is differentiable and strictly convex on (0, 1), since u, v are (essentially) bounded away from both zero and infinity, and they differ on a set of positive measure. Thus its derivative maps (0, 1) onto a non-empty open interval. Also ([19], p. 34) $\psi^{\#}$ is differentiable on $\psi'(0, 1)$, and its derivative is the reciprocal of ψ', so $\psi^{\#}$ is strictly increasing on $\psi'(0, 1)$.

To prove (2.7), note that $\log dP_n/dQ_n = -\Lambda_n$, $dQ_n = \exp(\Lambda_n)\,dP_n$. Thus, the moment generating function of $-\Lambda_n$ under Q_n is $\theta \mapsto \phi_n(1-\theta)$ with $0 \leqslant \theta \leqslant 1$. Because of this, the Fenchel–Legendre conjugate of the limiting function of $(1/n)\log\phi_n(1-\theta)$ is given at every b by

$$\sup_{0\leqslant\theta\leqslant 1}\big(b\theta-\psi(1-\theta)\big) = b+\psi^{\#}(-b).$$

Thus, by Lemma 2.1 and Theorem 2.2, $(1/n)\log Q_n(\Lambda_n < na) \to a-\psi^{\#}(a)$. This proves (2.7).

Consider now the first part of (2.8). Fix $\varepsilon > 0$ sufficiently small such that $a\pm\varepsilon\in\psi'(0, 1)$ and $\psi^{\#}(a-\varepsilon) < \psi^{\#}(a) < \psi^{\#}(a+\varepsilon)$. We shall prove that

$$Q_n\big(\Lambda_n < n(a-\varepsilon)\big) \leqslant 1-\beta'_n \tag{2.10}$$

for all sufficiently large n. Assume this for a moment. Then, using (2.7), it follows that

$$a-\varepsilon-\psi^{\#}(a-\varepsilon) \leqslant \liminf (1/n)\log(1-\beta_n').$$

Letting $\varepsilon \to 0$, we get the first part of (2.8). Now (2.10) can be justified as follows. Consider the test of P_n against Q_n which rejects P_n when $\Lambda_n \geqslant n(a-\varepsilon)$. This is a Neyman–Pearson test of size $\alpha_n(\varepsilon) = P_n(\Lambda_n \geqslant n(a-\varepsilon))$. It is thus most powerful at level $\alpha_n(\varepsilon)$ (see [16], p. 74). By Lemma 2.1 and Theorem 2.2, $(1/n)\log\alpha_n(\varepsilon) \to -\psi^{\#}(a-\varepsilon)$. Since $\psi^{\#}(a-\varepsilon) < \psi^{\#}(a)$, it follows that for all sufficiently large n we have $\alpha_n(\varepsilon) > \exp[-n\psi^{\#}(a)]$. This implies (2.10) by the Neyman–Pearson lemma [16].

To prove the second part of (2.8), consider the unique θ in (0, 1) such that $\psi'(\theta) = a$, $\psi^{\#}(a)+\psi(\theta) = a\theta$. Using Markov's inequality, we get

$$Q_n(\Lambda_n' \leqslant a_n') \leqslant \exp[(1-\theta)a_n'] E_{Q_n}\exp[(1-\theta)(-\Lambda_n')]. \tag{2.11}$$

Now $(1/n)\log E_{Q_n}\exp[(1-\theta)(-\Lambda_n')] \to \psi(\theta)$, by Theorem 2.3. Using the first part of (2.8), it follows that

$$a-[\psi^{\#}(a)+\psi(\theta)] \leqslant (1-\theta)\liminf(a_n'/n).$$

This proves the second part of (2.8).

To prove (2.9), note that

$$\exp[-n\psi^{\#}(a)] = E_{P_n}(\tau_n') \leqslant P_n(\Lambda_n' \geqslant a_n') \leqslant \exp[-\theta a_n']\tilde{\phi}_n(\theta).$$

Thus $-\psi^{\#}(a) \leqslant -\theta(a_n'/n)+(1/n)\log\tilde{\phi}_n(\theta)$. Letting $n\to\infty$, we get the second part of (2.9). Thus $a_n' \sim na$. Using this and (2.11), we get the first part of (2.9). This completes the proof. ∎

3. Asymptotic inversion of Toeplitz operators. In this section we prove that, for every pair a, b of invertible elements in $\mathscr{C}_+$ (resp. $H^\infty \cap H_{1/2}$), the approximate inverses given by (1.14) and (1.15) satisfy (1.12) (resp. (1.13)). Then we consider the problem of asymptotic inversion of Toeplitz operators. Let us first recall some elementary but basic facts on Toeplitz and Hankel operators on H^2. These operators are linked by the following identity (see [4] and [24]):

$$T(\phi\psi) = T(\phi)T(\psi)+H(\phi)H(\breve{\psi}). \tag{3.1}$$

This shows that

$$T(\breve{\phi}\psi) = T(\breve{\phi})T(\psi) \quad \text{if } \phi\in H^\infty \text{ or } \psi\in H^\infty. \tag{3.2}$$

Thus, if ϕ is in H^∞ and is invertible, then $T(\phi)$ and $T(\breve{\phi})$ are invertible with inverses $T(\phi^{-1})$ and $T(\breve{\phi}^{-1})$. Note also that

$$T_n(\phi\psi) = T_n(\phi)T_n(\psi) \quad \text{if } \phi, \psi\in H^2.$$

Thus, if ϕ is in H^∞ and is invertible, then $T_n(\phi)$ and its transpose $T_n(\breve{\phi})$ are invertible with inverses $T_n(\phi^{-1})$ and $T_n(\breve{\phi}^{-1})$. Finally, recall two important properties of Hankel operators. The first is that

$$\|H(\phi)\|_2 = \Big(\sum_{t>0} t\,|\check{\phi}(t)|^2\Big)^{1/2}. \tag{3.3}$$

Thus Hankel operators with symbol in $L^\infty \cap H_{1/2}$ are Hilbert–Schmidt. Second, a theorem of Hartman [14] states that $H(w)$ is compact if and only if the Fourier transform of w coincides on the positive integers with that of some continuous function.

We begin with two lemmas due to Widom [23], [24]. The first gives some information on the approximate inversion conditions (1.12). The second describes how strong convergence, that is pointwise convergence, of sequences of operators is converted into convergence in $\mathscr{I}_p$-norms under some conditions.

LEMMA 3.1 ([23], p. 193). *Let M_n, $M_n^{(-1)}$ be operators on $\mathbf{C}^n$ satisfying* (1.12). *Then M_n is invertible for all sufficiently large n, and the sequence of inverses is bounded in operator norm.*

LEMMA 3.2 ([24], p. 6). *Let A be a compact operator on a separable complex Hilbert space, and let B_n, B, C_n, C be operators on this space such that*

$$B_n \to B, \; C_n^* \to C^* \; \textit{strongly}.$$

Then $B_n A C_n \to BAC$ in operator norm. If, in addition, A is in $\mathscr{I}_p$, with $1 \leqslant p < \infty$, then this convergence also holds in $\mathscr{I}_p$-norm.

We now come to the main result of this section.

THEOREM 3.3. *Let a, b be a pair of invertible elements in $\mathscr{C}_+$ (resp. $H^\infty \cap H_{1/2}$). Then $T_n(\breve{a}b)$ is invertible for all sufficiently large n, and the approximate inverses* (1.14) *and* (1.15) *satisfy* (1.12) (*resp.* (1.13)).

Proof. The basic argument is a commutativity device of Basor and Helton [2], which leads to a nice factorisation of $T_n(\breve{a}b)$. Here is the precise method. For every w on L^∞, write $T_n(w) = \Pi_n T(w) \Pi_n$. Then, by (3.2),

$$T_n(\breve{a}b) = \Pi_n T(\breve{a}) T(b) \Pi_n.$$

Note also that $\Pi_n T(w) = T_n(w)$ and $T(\breve{w})\Pi_n = T_n(\breve{w})$ if $w \in H^\infty$. Commute $T(\breve{a})$ and $T(b)$ in the expression of $T_n(\breve{a}b)$ and compensate this operation with the multiplicative commutator

$$[T^{-1}(b)\, T(\breve{a})]\,[T(b)\, T^{-1}(\breve{a})] = 1 + A.$$

Thus

$$T_n(\breve{a}b) = T_n(b)\, M_n\, T_n(\breve{a}), \qquad M_n = \Pi_n (1+A) \Pi_n.$$

Note that $1+A$ is invertible, and put $M_n^{(-1)} = \Pi_n (1+A)^{-1} \Pi_n$. We first show that A is compact (resp. trace class). Then identifying the unit operator on the range on Π_n with Π_n, we shall prove that the pair M_n, $M_n^{(-1)}$ of operators on the range of Π_n satisfies (1.12) (resp. (1.13)), and that

(3.4) $$M_n^{-1} - M_n^{(-1)} \to 0 \text{ in operator (resp. trace class) norm,}$$

where M_n^{-1} stands for the inverse of M_n on the range of Π_n. The proof will be completed by showing that the approximate inverse

(3.5) $$T_n^{(-1)}(\breve{a}b) = T_n(\breve{a}^{-1})\, M_n^{(-1)}\, T_n(b^{-1})$$

satisfies (1.12) (resp. (1.13)), and that the two sequences

$$T_n^{(-1)}(\breve{a}b) - W_n(\breve{a}b), \qquad W_n(\breve{a}b) - \tilde{W}_n(\breve{a}b)$$

converge to zero in operator (resp. trace class) norm.

To prove that A is compact (resp. of trace class), write

$$1 + A = 1 + [T(b^{-1})\, T(\breve{a}) - T(\breve{a})\, T(b^{-1})]\, T(b)\, T(\breve{a}^{-1}),$$

use (3.2), and then (3.1), to get $A = -[H(b^{-1})H(a)]\, T(b)\, T(\breve{a}^{-1})$. Thus A is compact by Hartman's theorem (resp. trace class by (3.3) and Hölder's inequality for operators). To prove that the pair M_n, $M_n^{(-1)}$ satisfies (1.12) (resp. (1.13)), note first that the sequence $M_n^{(-1)}$ is bounded in operator norm. On the other hand, since $\Pi_n(1-\Pi_n) = 0$, we have

$$\begin{aligned} M_n M_n^{(-1)} &= \Pi_n(1+A)[1-(1-\Pi_n)](1+A)^{-1}\Pi_n \\ &= \Pi_n - \Pi_n A(1-\Pi_n)(1+A)^{-1}\Pi_n, \\ \Pi_n - M_n M_n^{(-1)} &= [\Pi_n A(1-\Pi_n)]\,[(1+A)^{-1}\Pi_n]. \end{aligned}$$

The desired result now follows from Lemma 3.2, since Π_n converges strongly to the unit operator and is self-adjoint. This also proves (3.4) by Lemma 3.1 and Hölder's inequality for operators. Thus the approximate inverse (3.5) satisfies (1.12) (resp. (1.13)), since for every bounded w the sequence $T_n(w)$ is bounded in operator norm. Using (3.1) and then (3.2), we get

$$(1+A)^{-1} = 1 - T(\breve{a})\, B\, T(b), \qquad B = H(b^{-1})\, H(a^{-1}).$$

Thus the approximate inverse (3.5) is $T_n(\breve{a}^{-1})\, T_n(b^{-1})$ minus

$$R_n = [T_n(\breve{a}^{-1})\, T(\breve{a})]\, B\, [T(b)\, T_n(b^{-1})].$$

It is thus enough to check that R_n, $\Pi_n B \Pi_n$ and $H_n(b^{-1})\, H_n(a^{-1})$ (which is $\Pi_n H(b^{-1})\,\Pi_n H(a^{-1})\,\Pi_n$) all converge to B in operator (resp. trace class) norm. This follows from Lemma 3.2 since, for every bounded w, $T_n(w)$ converges strongly to $T(w)$, and the adjoints of $T_n(w)$, $T(w)$ are obtained by replacing w by its complex conjugate. This completes the proof. ∎

We now turn to the problem of asymptotic inversion of Toeplitz operators by the method of finite section. The notations are as in Section 1.

THEOREM 3.4. *Let w be in L^∞ and suppose that $T(w)$ is invertible. Then for every sequence of approximate inverses $T_n^{(-1)}(w)$ satisfying* (1.12) *we have the following properties*:

(a) $T_n(w)$ *is invertible for all sufficiently large n, and the sequence of inverses is bounded in operator norm.*

(b) *Define* $\tilde{\xi}_n$ *by* (1.18). *If* x *(resp.* ξ_n*) is the unique solution of* (1.16) *(resp.* (1.17)), *then*

$$\|x-\tilde{\xi}_n\|_2 \leqslant \|1_n - T_n(w)\, T_n^{(-1)}(w)\|_\infty + \|(1-\Pi_n)\,x\|_2,$$
$$\|\xi_n-\tilde{\xi}_n\|_2 \leqslant \|1_n - T_n(w)\, T_n^{(-1)}(w)\|_\infty .$$

In particular, $T_n^{-1}(w)$ *and* $T_n^{(-1)}(w)$ *converge strongly to* $T^{-1}(w)$.

Proof. Part (a) follows from Lemma 3.1. To prove (b), fix y in H^2 and consider the equations (1.16), (1.17) and (1.18). Write

$$x-\tilde{\xi}_n = (1-\Pi_n)\,x + \Pi_n(x-\xi_n) + (\xi_n-\tilde{\xi}_n)$$

and note that

$$\begin{aligned} T_n(w)\,\Pi_n(x-\xi_n) &= T_n(w)\,\Pi_n x - \Pi_n y \\ &= \Pi_n\,[T(w)\,\Pi_n x - T(w)\,x] = \Pi_n\, T(w)\,(\Pi_n-1)\,x. \end{aligned}$$

Thus, for all sufficiently large n, we have

$$\Pi_n(x-\xi_n) = -T_n^{-1}(w)\,\Pi_n\, T(w)\,(1-\Pi_n)\,x,$$
$$\|\Pi_n(x-\xi_n)\|_2 \leqslant \|T_n^{-1}(w)\|_\infty\, \|T(w)\|_\infty\, \|(1-\Pi_n)\,x\|_2 .$$

On the other hand, for all sufficiently large n, we have

$$\xi_n-\tilde{\xi}_n = [T_n^{-1}(w) - T_n^{(-1)}(w)]\,\Pi_n y = T_n^{-1}(w)\,[1_n - T_n(w)\,T_n^{(-1)}(w)]\,\Pi_n y.$$

Thus $\|\xi_n-\tilde{\xi}_n\|_2 \leqslant \|T_n^{-1}(w)\|_\infty\, \|1_n - T_n(w)\, T_n^{(-1)}(w)\|_\infty\, \|y\|_2$. This completes the proof. ∎

We close this section with a remark on the invertibility of $T_n(w)$ for a non-negative, integrable and non-zero symbol w. This is a particular case of the following more general remark. Let μ be a non-negative Borel measure on the unit circle. Define the $n\times n$ Toeplitz matrix $T_n(\mu)$ with entries

$$T_n(\mu)(s,t) = \int_{\boldsymbol{T}} z^{-(s-t)}\, d\mu(z), \qquad s, t = 0, \ldots, n-1,$$

where $\boldsymbol{T}$ stands for the unit circle. Observe that, for ξ and η in $\boldsymbol{C}^n$ with components (ξ_t) and (η_t), we have an analog of (1.9):

$$\eta^*\, T_n(\mu)\,\xi = \int_{\boldsymbol{T}} \Big(\sum_0^{n-1} \xi_t z^t\Big)\Big(\sum_0^{n-1} \eta_t z^t\Big)^* d\mu(z), \tag{3.6}$$

so that $T_n(\mu)$ is non-negative. The measure μ is said to be *finitely supported* if there exists a finite subset of $\boldsymbol{T}$ whose complement has zero μ-measure.

PROPOSITION 3.5. *If* μ *is a non-negative Borel measure on the unit circle, and if* μ *is not finitely supported, then* $T_n(\mu)$ *is invertible for all* n.

Proof. Put $f(z) = \sum_0^{n-1} \xi_t z^t$ and denote by $\mathscr{Z}$ the zero set of f on $\boldsymbol{T}$. Suppose that $\xi \neq 0$, so that $\mathscr{Z}$ is finite (possibly empty). Thus $\mu(\boldsymbol{T} \setminus \mathscr{Z}) > 0$. Since μ is regular ([20], p. 50), there exists a compact K contained in $\boldsymbol{T} \setminus \mathscr{Z}$ with positive μ-measure. Using (3.6), we get

$$\xi^* T_n(\mu) \xi \geqslant \int_K |f(z)|^2 \, d\mu(z) \geqslant \mu(K) \inf_{z \in K} |f(z)|^2 > 0,$$

which proves the proposition. ■

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