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ON THE MOMENT THEOREM OF MEERSCHAERT

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Abstract. Let q be a full operator-stable measure on R^N, and B an exponent of q. Write $m = \min\{\operatorname{Re} x\}$ and $M = \max\{\operatorname{Re} x\}$, where x ranges over the eigenvalues of B. Suppose that the distribution of a random vector X belongs to the domain of attraction of q, $m \neq \frac{1}{2}$ and $\Theta \in R^N - \{0\}$. The object of this note is to show that some results of Hudson et al. [2] can be proved in a simpler way (and somewhat extended) by using the method presented in Meerschaert [4]. Namely, we prove that $E|\langle X, \Theta\rangle|^\alpha$ is finite for $\alpha \in (0, 1/M)$, and infinite for $\alpha > 1/m$. Basing ourselves on this, we can easily obtain a moment theorem which is near the result of Meerschaert [5].

1. Introduction. Let R^N be an N-dimensional Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\| \ \|$. Suppose $\{X_n\}$ is a sequence of independent R^N-valued random vectors with a common distribution p. Consider the sums

$$Y_n = A_n(X_1 + \ldots + X_n) + x_n,$$

where $x_n \in R^N$ and $\{A_n\}$ is a sequence of linear operators on R^N. The distributions of the vectors Y_n may be written in the form

$$A_n p^n * \delta(x_n), \tag{1}$$

where the power p^n is taken in the sense of convolution, $\delta(x)$ denotes the unit mass at $x \in R^N$, and Ap is a measure defined by $Ap(Z) = p(A^{-1}Z)$ for each Borel subset Z of R^N. A probability measure on R^N is said to be *operator-stable* if it is a weak limit of sequence (1). We call a measure on R^N *full* if its support is not contained in any $(N-1)$-dimensional subspace of R^N. Sharpe [6] gave a characterization of full operator-stable measures. Namely, a full measure q on R^N is operator-stable if and only if there exists a nonsingular linear operator B on R^N such that

$$q^t = t^B q * \delta(x_t) \quad \text{for all } t > 0, \tag{2}$$

where $t^B = \exp(B \log t)$, $x_t \in R^N$.

Operator-stable laws are infinitely divisible with the Lévy representation (x_0, D, Φ), where $x_0 \in R^N$, D is a linear self-adjoint positive operator on R^N, and Φ is a Lévy spectral measure. From (2) we have

$$t\Phi = t^B \Phi \quad \text{for all } t > 0. \tag{3}$$

If all eigenvalues of B have real parts equal to $\frac{1}{2}$, then q is a Gaussian measure with the representation $(x_0, D, 0)$. If they have real parts greater than $\frac{1}{2}$, then q is a purely Poissonian measure with the representation $(x_0, 0, \Phi)$. The class of all distributions p for which (at a suitable choice of operators A_n and vectors x_n) sequence (1) converges weakly to q is called the *domain of attraction* of q.

2. Lemmas.

LEMMA 1. *Suppose that E is a linear operator on R^N and that all eigenvalues of E have positive real parts. Then there exist positive numbers K, L, ε, κ such that*

$$\|t^E x\| \geqslant Kt^\varepsilon \|x\| \quad \textit{and} \quad \|t^{-E} x\| \leqslant Lt^{-\kappa} \|x\| \quad \textit{for all } x \in R^N \textit{ and } t > 1,$$

and, consequently,

$$\lim_{t\to\infty} \|t^E x\| = +\infty, \quad \lim_{t\to 0} \|t^E x\| = 0 \quad \textit{for each } x \in R^N - \{0\}. \tag{4}$$

Proof. The lemma follows from Theorems 1 and 2 in Hirsch and Smale [1].

Let $q = (0, 0, \Phi)$ be a full operator-stable measure on R^N, and B an exponent of q. Define a function on $R^N - \{0\}$ by setting

$$g(\Theta) = \Phi\{x \in R^N \colon |\langle x, \Theta\rangle| > 1\}.$$

LEMMA 2. *The function g defined above is continuous, $g > 0$, and*

$$\lim_{n\to\infty} g(\Theta_n) = +\infty \quad \textit{as } \|\Theta_n\| \to +\infty.$$

Proof. The continuity of g is proved in Meerschaert [4]. Suppose $g(\Theta_0) = 0$ for $\Theta_0 \neq 0$. By (4), we have

$$\lim_{n\to\infty} \|t_n^{B^*} \Theta_0\| = +\infty \quad \text{as } t_n \to +\infty.$$

From (3) it follows that $g(t_n^{B^*} \Theta_0) = 0$. Thus, we can find y_0 such that

$$\|y_0\| = 1 \quad \text{and} \quad \Phi\{x \in R^N \colon |\langle x, y_0\rangle| > 0\} = 0.$$

Consequently, $q = (0, 0, \Phi)$ is not a full measure, which contradicts the assumption. Let now $\|\Theta_n\| \to +\infty$. By (4), we can write $\Theta_n = s_n^{B^*} y_n$, where $\|y_n\| = 1$ and $s_n \to +\infty$. Moreover, we have $\inf_n g(y_n) > 0$. Hence

$$\lim_{n\to\infty} g(\Theta_n) = \lim_{n\to\infty} s_n g(y_n) = +\infty.$$

Let γ be a probability measure on the real line R and $\alpha > 0$. We can easily obtain the following equivalence:

LEMMA 3. *The integral* $\int_R |x|^\alpha \gamma(dx)$ *is finite if and only if the series* $\sum_{n=1}^{\infty} \lambda^{n\alpha} \gamma\{x \in R: |x| > \lambda^n\}$ *is convergent for some* $\lambda > 1$.

LEMMA 4. *Let* $\tau_1, \ldots, \tau_l \in R$. *Then*

$$\Big(\sum_{j=1}^{l} |\tau_j|\Big)^\alpha \leqslant C \sum_{j=1}^{l} |\tau_j|^\alpha,$$

where

$$C = \begin{cases} 1 & \text{if } \alpha \in (0, 1), \\ l^{\alpha-1} & \text{if } \alpha \geqslant 1. \end{cases}$$

Proof. This lemma follows from the convexity of the function $y \to y^\alpha$ $(y > 0, 0 < \alpha < 1)$ and from the Hölder inequality.

3. Theorems. Let q be a full operator-stable measure on R^N, and B an exponent of q. We define numbers m and M such that

$$m = \min\{\operatorname{Re} x\colon\ x \in \operatorname{Sp} B\}, \qquad M = \max\{\operatorname{Re} x\colon\ x \in \operatorname{Sp} B\}. \tag{5}$$

THEOREM 1. *Suppose q is a full operator-stable measure on R^N, the numbers m and M are given by (5), and $m > \frac{1}{2}$. If a measure p on R^N belongs to the domain of attraction of q, then, for each $\Theta \in R^N - \{0\}$, the integral $\int_{R^N} |\langle x, \Theta\rangle|^\alpha p(dx)$ is finite for $\alpha \in (0, 1/M)$, and infinite for $\alpha > 1/m$.*

Proof. By assumption, we can find sequences $\{A_n\}$ and $\{x_n\}$ such that

$$\lim_{n \to \infty} A_n p^n * \delta(x_n) = q = (0, 0, \Phi).$$

Thus, by Lemma 2, we have, for all $z \in R^N - \{0\}$,

$$\lim_{n \to \infty} np\{x \in R^N\colon\ |\langle A_n x, z\rangle| > 1\} = \Phi\{x \in R^N\colon\ |\langle x, z\rangle| > 1\} = g(z) > 0, \tag{6}$$

and furthermore this convergence is uniform on compact subsets of $R^N - \{0\}$.

Let $\Theta \in R^N \setminus \{0\}$ be arbitrarily fixed. We define on a half-line $(r_0, +\infty)$ a function L by setting

$$L(r) = \max\{n\colon\ \|A_n^{*-1} \Theta\| \leqslant r\}. \tag{7}$$

We know that $\|A_n\| \to 0$, and that $\|A_{n+1} A_n^{-1}\|$ remains bounded away from zero and infinity as $n \to \infty$. Hence $\lim_{r \to +\infty} L(r) = +\infty$ and the set $\{A_{L(r)}^{*-1}(\Theta/r)\colon\ r \geqslant r_0\}$ is compactly contained in $R^N - \{0\}$. Let $\mathscr{D}$ denote the set of all points y such that

$$\lim_{n \to \infty} A_{L(r_n)}^{*-1}(\Theta/r_n) = y \quad \text{for a sequence } r_n \to +\infty.$$

From (6) we then have

$$\lim_{n\to+\infty} L(r_n)p\{x\in R^N\colon |\langle x, \lambda\Theta\rangle| > r_n\} = g(\lambda y) \quad \text{for each } \lambda > 0.$$

Consequently, if we put $f(\Theta) = p\{x\in R^N\colon |\langle x, \Theta\rangle| > 1\}$, we get

$$\lim_{n\to+\infty} \frac{\lambda^\alpha f(r_n^{-1}\Theta)}{f(\lambda r_n^{-1}\Theta)} = \frac{\lambda^\alpha g(y)}{g(\lambda y)} = \frac{g(y)}{g(\lambda^{I-\alpha B^*}y)}, \tag{8}$$

where I denotes the identity operator on R^N.

For given $\alpha > 0$ and $\lambda > 1$, consider the following function:

$$r \to \frac{\lambda^\alpha f(r^{-1}\Theta)}{f(\lambda r^{-1}\Theta)}, \quad r \geqslant r_0. \tag{9}$$

By (8), the set of all limit points of (9) at $+\infty$ takes the form

$$\left\{\frac{g(y)}{g(\lambda^{I-\alpha B^*}y)}\colon\ y\in\mathscr{D}\right\}.$$

Let now $\alpha\in(0, 1/M)$. In this case, the spectrum of the operator $I-\alpha B^*$ is contained in the half-plane $\operatorname{Re} z > 0$. Thus, by Lemma 1, we have immediately

$$\lim_{\lambda\to+\infty} \inf_{y\in\mathscr{D}} \|\lambda^{I-\alpha B^*}y\| = +\infty,$$

and consequently, by Lemma 2,

$$\lim_{\lambda\to+\infty} \inf_{y\in\mathscr{D}} g(\lambda^{I-\alpha B^*}y) = +\infty.$$

Hence we can find $\lambda_0 > 1$ such that

$$\inf_{y\in\mathscr{D}} g(\lambda_0^{I-\alpha B^*}y) > \sup_{y\in\mathscr{D}} g(y).$$

This means that

$$\limsup_{r\to+\infty} \frac{\lambda_0^\alpha f(r^{-1}\Theta)}{f(\lambda_0 r^{-1}\Theta)} < 1.$$

In particular, if we put λ_0^n instead of r, we obtain the sufficient condition for the series $\sum_{n=1}^{\infty} \lambda_0^{n\alpha} f(\lambda_0^{-n}\Theta)$ to be convergent. By Lemma 3, the first part of the theorem is proved.

We now consider the function

$$r \to \frac{\lambda^\alpha f(\lambda^{-1}r^{-1}\Theta)}{f(r^{-1}\Theta)}, \quad r \geqslant r_0.$$

The set of all its limit points at $+\infty$ takes the form

$$\left\{\frac{g(\lambda^{\alpha B^*-I}y)}{g(y)}:\ y\in\mathcal{D}\right\}.$$

Suppose $\alpha > 1/m$. Then the spectrum of the operator αB^*-I lies in the half-plane $\operatorname{Re} z > 0$. In the same way as in the first part we can find $\gamma_0 > 1$ such that

$$\liminf_{r\to+\infty}\frac{\gamma_0^\alpha f(\gamma_0^{-1}r^{-1}\Theta)}{f(r^{-1}\Theta)} < 1.$$

Putting γ_0^n instead of r, we obtain

$$\liminf_{n\to+\infty}\frac{\gamma_0^\alpha f(\gamma_0^{-n-1}\Theta)}{f(\gamma_0^{-n}\Theta)} < 1.$$

It now suffices to make use of the d'Alembert criterion and Lemma 3. The theorem is proved.

Let V be a subspace of R^N, and W the dual space of V. Theorem 1 is also true if we consider the spaces V and W, respectively, instead of R^N. In particular, we obtain the following corollary:

COROLLARY 1. *Suppose q is a full operator-stable measure on V with an exponent B and $\{\operatorname{Re} x:\ x\in \operatorname{Sp} B\} = \{a\}$. If a measure p on V belongs to the domain of attraction of q, then, for each $\Theta\in W-\{0\}$, the integral $\int_V |\langle x, \Theta\rangle|^\alpha p(dx)$ is finite for $\alpha\in(0, 1/a)$, and infinite for $\alpha > 1/a$ and $a > \frac{1}{2}$.*

Proof. The case $a > \frac{1}{2}$ needs no proof. If $a = \frac{1}{2}$, then q is a Gaussian measure. Let p_Θ be a distribution defined by $p_\Theta(Z) = p\{x\in V:\ \langle x, \Theta\rangle\in Z\}$ for each Borel subset Z of the real line R. The result of Kłosowska [3] states that, for all $\Theta\in W-\{0\}$, the measure p_Θ belongs to the domain of attraction of the nondegenerate Gaussian law on R. Thus p_Θ has absolute moments of order α if $\alpha\in(0, 2)$.

We now give a few preliminaries and some notation which will be needed in the next theorem. Let q be again a full operator-stable measure on R^N with an exponent B. We put $\{\operatorname{Re} x:\ x\in\operatorname{Sp} B\} = \{a_1, \ldots, a_s\}$ and assume that $1/a_1 < 1/a_2 < \ldots < 1/a_s$. Basing ourselves on the results from Hirsch and Smale [1], we can find a basis $\{b_1, \ldots, b_N\}$ with respect to which the matrix representing B has a block diagonal form. Due to this form, we obtain a direct sum decomposition of R^N into B-invariant subspaces $V_1, \ldots, V_s$ such that

$$\{a_j\} = \{\operatorname{Re} x:\ x\in\operatorname{Sp} B|V_j|\}.$$

Furthermore, there exists a dual basis $\{c_1, \ldots, c_N\}$ ($\langle b_i, c_j\rangle = 1$ if $i=j$ and 0 otherwise) with respect to which the matrix representing B^* is just the transpose of the matrix for B with respect to $\{b_1, \ldots, b_N\}$. If $\{b_r, \ldots, b_l\}$

span V_j, then the subspace $W_j = \operatorname{Span}\{c_r, \ldots, c_l\}$ is the dual space for V_j, W_j is B^*-invariant,

$$R^N = W_1 \oplus \ldots \oplus W_s \quad \text{and} \quad \{a_j\} = \{\operatorname{Re} x \colon x \in \operatorname{Sp} B | W_j\}.$$

Now, for $x \in R^N - \{0\}$, we can write $x = x_1 + \ldots + x_s$ — the unique direct sum decomposition with respect to $\{V_j\}$, and $x = x_1^* + \ldots + x_s^*$ — the same with respect to $\{W_j\}$. Suppose that p belongs to the domain of attraction of q, and that we can find norming operators A_n which are V_j-invariant ($j = 1, 2, \ldots, s$). We shall then say that *p belongs to the domain of direct attraction of q*. Define now the function α^* as in Meerschaert [5] by setting

$$\alpha^*(x) = \min\{1/a_j \colon x_j^* \neq 0\}, \quad x \in R^N - \{0\}. \tag{10}$$

As a consequence of Corollary 1 we obtain

THEOREM 2. *Suppose p belongs to the domain of direct attraction of a full operator-stable measure q on R^N with an exponent B. Define a^* as in* (10). *Then, for all $\Theta \in R^N - \{0\}$, the integral $\int_{R^N} |\langle x, \Theta\rangle|^\alpha p(dx)$ is finite if $\alpha \in (0, \alpha^*(\Theta))$, and infinite if $\alpha > \alpha^*(\Theta)$ and $\alpha^*(\Theta) < 2$.*

Proof. Let V denote one of the spaces $V_1, \ldots, V_s$, let the space W be the dual space of V, and the number a be the real part of all eigenvalues of $B | V$ and $B^* | W$. Denote by Π_V the natural projection map onto V. Since Π_V commutes with A_n, we have

$$\lim_{n\to\infty} (A_n | V) \Pi_V p^n * \delta(y_n) = \Pi_V q, \quad \text{where } y_n \in V.$$

The measure $\Pi_V q$ is a full operator-stable measure on V with the exponent $B|V$ and $\{\operatorname{Re} x \colon x \in \operatorname{Sp} B|V\} = \{a\}$. Moreover, for all $y \in W$, we have the equality

$$\int_{R^N} |\langle x, y\rangle|^\alpha p(dx) = \int_V |\langle x, y\rangle|^\alpha \Pi_V p(dx).$$

Using Corollary 1, we infer that, for all $y \in W - \{0\}$, the integral $\int_{R^N} |\langle x, y\rangle|^\alpha p(dx)$ is finite if $\alpha \in (0, 1/a)$, and infinite if $\alpha > 1/a$ and $a > \frac{1}{2}$. Taking $\Theta \in R^N - \{0\}$, we can write $\Theta = \Theta_1^* + \ldots + \Theta_s^*$, where $\Theta_j^* \in W_j$. Without loss of generality we may assume that $\Theta_j^* \neq 0$ for $j = 1, 2, \ldots, l$ ($l = 2, 3, \ldots, s$) and $\Theta_j^* = 0$ for $j = l+1, \ldots, s$. In this case,

$$\alpha^*(\Theta) = \min\{1/a_1, \ldots, 1/a_l\} = 1/a_1.$$

Let $\alpha \in (0, 1/a_1)$. It is seen that then

$$\int_{R^N} |\langle x, \Theta_j^*\rangle|^\alpha p(dx) < +\infty \quad \text{for all } j = 1, 2, \ldots, l$$

and, by Lemma 4, also

$$\int_{R^N} |\langle x, \Theta\rangle|^\alpha p(dx) < +\infty.$$

Let $\alpha \in (1/a_1, 1/a_2)$ and $1/a_1 < 2$. We notice that now

$$\int_{R^N} |\langle x, \Theta_1^* \rangle|^\alpha p(dx) = +\infty \quad \text{and} \quad \sum_{j=2}^{l} \int_{R^N} |\langle x, \Theta_j^* \rangle|^\alpha p(dx) < +\infty.$$

Using Lemma 4 once again, we can find numbers K, $L > 0$ such that

$$|\langle x, \Theta_1^* \rangle|^\alpha \leqslant K |\langle x, \Theta \rangle|^\alpha + L \sum_{j=2}^{l} |\langle x, \Theta_j^* \rangle|^\alpha,$$

and, consequently, we obtain $\int_{R^N} |\langle x, \Theta \rangle|^\alpha p(dx) = +\infty$. The Hölder inequality implies that the above integral is infinite for all $\alpha > 1/a_1$. The theorem has thus been proved.

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