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A NOTE ON CONVERGENCE RATES IN THE STRONG LAW FOR STRONG MIXING SEQUENCES

BY

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Abstract. For partial sums $\{S_n\}$ of a stationary ergodic sequence $\{X_n\}$ with zero mean we find conditions for

$$\sum_{n=1}^{\infty} n^{\gamma-1} \Pr\{\sup_{k\geqslant n}(S_k/k) > \varepsilon\} < \infty$$

in terms of the strong mixing coefficients $\{\alpha_n\}$ and moments of certain functions of the marginal incremental variable X_1.

1. Introduction. The ergodic theorem for a stationary ergodic sequence $\{X_j\colon j = 0, \pm 1, \ldots\}$ with $\mathrm{E}|X_j| < \infty$ states that the partial sums $S_k = X_1 + \ldots + X_k$ $(k = 1, 2, \ldots)$ satisfy

$$\Pr\{|S_k/k - \mathrm{E}(X_1)| > \varepsilon \text{ (infinitely many } k)\} = 0 \quad (\text{any } \varepsilon > 0). \tag{1.1}$$

This statement gives us no information as to the rate at which these averages of partial sums converge to their limit. Baum and Katz [1] showed that when the random variables (r.v.s) X_j are not merely stationary (hence, identically distributed) but moreover independent, the tails of the distributions of S_k satisfy the finiteness criterion

$$\sum_{k=1}^{\infty} k^{\alpha\gamma-2} \Pr\left\{\frac{|S_k|}{k^{\alpha}} \geqslant \varepsilon\right\} < \infty \quad (\text{any } \varepsilon > 0) \tag{1.2}$$

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for $\alpha\gamma > 1$ and $\alpha > \frac{1}{2}$ if and only if

$$\mathrm{E}(|X_j|^{\gamma}) < \infty \tag{1.3}$$

and, when $\alpha \leqslant 1$,

$$\mathrm{E}(X_j) = 0. \tag{1.4}$$

Further, in this case of independence, (1.2) is equivalent to the apparently stronger statement (e.g. Lai [7])

$$\sum_{n=1}^{\infty} n^{\alpha\gamma-2} \Pr\left\{\sup_{k\geqslant n} \frac{|S_k|}{k^{\alpha}} \geqslant \varepsilon\right\} < \infty \qquad (\text{some } \varepsilon > 0). \tag{1.5}$$

For a textbook account see e.g. Chow and Teicher [3] (§§ 5.2, 10.3–4).

There are various ways of relaxing independence while retaining ergodicity and stationarity: Berbee [2] and Peligrad [8] show that (1.5) and (1.3)–(1.4) remain equivalent, as in the independent case, when the X_j form a ϕ-mixing sequence, i.e. for sets A, B in the respective σ-fields

$$A \in \mathscr{A} \equiv \sigma(\{X_j\colon j = 0, -1, \ldots\}) \quad \text{and} \quad B \in \mathscr{B} \equiv \sigma(\{X_{n+j}\colon j = 0, 1, \ldots\}), \tag{1.6}$$

$$\phi_n \equiv \sup_{A,B} |P(B|A) - P(B)| \to 0 \qquad (n \to \infty).$$

Lai [7] has a range of results under a condition that is weaker than mixing in the sense of using smaller families of sets than $\mathscr{A}$ or $\mathscr{B}$ but stronger in requiring that more than (1.6) be satisfied for these smaller families.

Berbee [2] also showed that under the weaker condition of strong mixing, meaning that the strong mixing coefficients

$$\alpha_n \equiv \sup_{A,B} |P(A \cap B) - P(A)P(B)| \to 0 \qquad (n \to \infty), \tag{1.7}$$

conditions (1.5) and (1.3) are no longer equivalent. What he was able to show – and this paper is about extending his result – is that for *bounded* stationary ergodic sequences with zero mean and whose strong mixing coefficients satisfy, for $\gamma \geqslant 2$,

$$\sum_{n=1}^{\infty} n^{\gamma-2} \alpha_n < \infty, \tag{1.8}$$

the sum in (1.5) in the case $\alpha = 1$ is finite, i.e.

$$\sum_{n=1}^{\infty} n^{\gamma-2} \Pr\left\{\sup_{k\geqslant n} \frac{|S_k|}{k} > \varepsilon\right\} < \infty \qquad (\text{any } \varepsilon > 0). \tag{1.9}$$

(Berbee actually gives results for bounded r.v.s that are not necessarily stationary but for which the supremum over all coefficients at lag n defined as in (1.7), satisfy (1.8).) Our result replaces the assumption of boundedness by a weaker assumption of a finite moment of appropriate order and a compensatingly stronger condition than (1.8), and looks at a one-sided rate result for integer $\gamma \geqslant 2$ of the form

$$\sum_{n=1}^{\infty} n^{\gamma-2} \Pr\left\{\sup_{k \geqslant n} \frac{S_k}{k} > \varepsilon\right\} < \infty \quad (\text{any } \varepsilon > 0). \tag{1.10}$$

The moment concerns only the positive tail, much as a classical result of Kiefer and Wolfowitz [6] likewise gives a one-sided moment condition for a one-sided bound, albeit in a random walk or queueing theory setting. We note that this latter setting prompted Daley and Rolski [5] to study this question which is more general than Berbee addressed.

In Daley and Rolski [5] and Daley et al. [4] we relate results of the type (1.10) to the existence of moments of the waiting time in single-server queues. Example 2 of [4] exhibits a particular $D/G/1$ queue (with dependent stationary service times) such that the $(\gamma-1)$-st moment of the waiting time is finite if and only if the service times have a finite moment of order $\gamma+1$ (cf. Kiefer and Wolfowitz's classical condition requires order γ instead of $\gamma+1$). Also, their Example 3 gives a sequence $\{X_n\}$ satisfying $\mathrm{E}(X_n^+)^2 < \infty$ and $\sum_n \alpha_n < \infty$ but which for some $\varepsilon > 0$ has

$$\sum_{n=1}^{\infty} \Pr\left\{\sup_{k \geqslant n} \frac{S_k}{k} > \varepsilon\right\} = \infty.$$

These examples, together with the equivalence of (1.3) and (1.5) under ϕ-mixing, show that ϕ-mixing can be too close to independence to differentiate among sequences with stronger dependence.

Section 2 states the Theorem that extends Berbee's result. Two particular cases are given as corollaries as they help to illustrate the interplay between moment conditions and mixing conditions which, when combined, suffice to imply (1.10). Section 3 contains some preliminary results; the proof is in Section 4.

2. One-sided convergence rate results. We write $x^+ = \max(0, x)$, while $\lfloor \beta^n \rfloor$ denotes the least integer $\leqslant \beta^n$. Recall that a monotonic increasing function $g(\cdot)$ has its generalized inverse defined by

$$g^{-1}(y) = \inf\{x \geqslant 0\colon\ g(x) \geqslant y\}.$$

THEOREM. *Let $\{X_j\colon j = 0, \pm 1, \ldots\}$ be a zero mean ergodic stationary sequence which for some integer $\gamma \geqslant 2$ and some non-negative monotonic increasing function $g(\cdot)$ has*

$$\mathrm{E}\left([g^{-1}(X_j^+)]^{\gamma}\right) < \infty \tag{2.1}$$

and for some $\beta > 1$

$$\sum_{k=1}^{\infty} k^{2\gamma-2} \alpha_k \sum_{n:\ \lfloor \beta^n \rfloor \geqslant k} \beta^{-n\gamma} [g(\lfloor \beta^n \rfloor)]^{2\gamma} < \infty, \tag{2.2}$$

$$\sum_{m\geqslant 0} \beta^{-m}[g(\lfloor\beta^m\rfloor)]^{2\gamma} < \infty. \tag{2.3}$$

Then, for any $\varepsilon > 0$, *the partial sums* $S_k = X_1 + \ldots + X_k$ *satisfy* (1.10) *for such integers* $\gamma \geqslant 2$.

We give two corollaries, each entailing the use of a different function $g(\cdot)$, to illustrate the possible trade-off between constraints on the positive tails X_j^+ of the increments and the rate of convergence to zero of the strong mixing coefficients. In the first we substitute $g(x) = \delta \log x$ for some $\delta > 0$. In the second we put $g(x) = x^\mu$ for some $0 \leqslant \mu < \frac{1}{2}/\gamma$; note that the moment condition (2.6) is the same as in Theorem 2(i) of Lai [7] but that our condition (2.7) is weaker than his.

COROLLARY 1. *Sufficient conditions for* (1.10) *to hold are that*

$$\mathrm{E}(\exp\{\theta X_j\}) < \infty \qquad (\textit{some } \theta > 0) \tag{2.4}$$

and

$$\sum_{k=2}^{\infty} k^{\gamma-2}(\log k)^{2\gamma}\alpha_k < \infty. \tag{2.5}$$

COROLLARY 2. *Sufficient conditions for* (1.10) *to hold are that, for some* μ *in* $0 \leqslant \mu < \frac{1}{2}/\gamma$,

$$\mathrm{E}[(X_j^+)^{\gamma/\mu}] < \infty \tag{2.6}$$

and

$$\sum_{k=1}^{\infty} k^{\gamma-2+2\mu\gamma}\alpha_k < \infty. \tag{2.7}$$

Remark 1. At an intuitive level, there are two "causes" that may result in the events $\{S_k > \varepsilon k\}$ having probabilities too large for (1.10) to hold, namely (i) an occasional X_j that is very large positive, and (ii) a large number of consecutive X_j which on average exceed ε. Conditions (2.1) and (2.2) ensure that the respective probabilities of (i) and (ii) are sufficiently small. Condition (2.3) is a technical one implied by our truncation of the X_j via the function $g(\cdot)$ in Section 4.

Remark 2. The crux of Berbee's proof that (1.8) implies (1.9), which proof we generally follow, is a bound on moments of S_k obtained from his result, quoted at Lemma 3 below, that is largely a counting exercise; a similar counting exercise occurs in the proof of Lemma 3.1 in Sen [9]. This bound is used in Markov inequalities to bound probabilities on groups of consecutive S_k. The assumption in the theorem that γ should be an integer is partly technical, arising from the combinatorial nature of Berbee's derivation of (3.3) below, and partly of convenience (cf. the Remark to Lemma 3).

Remark 3. A sufficient condition for (1.10) to hold is that $\{X_j\}$ be a sequence of i.i.d. random variables with mean 0 and $\mathrm{E}(X_j^\gamma) < \infty$. Unfortunately, if $\gamma = 2$, then $g(x) = x$ but then (2.3) is divergent. Thus our sufficient conditions for (1.10) to hold are certainly stronger than necessary in the i.i.d. case.

3. Some preliminary results. The proof of the Theorem uses some auxiliary results which we collect here for convenience. First, there is a classical convergence result based on a condensation technique.

LEMMA 1 (Cauchy's condensation test). *Let $\{a_n\}$ be a non-increasing non-negative sequence. Then for $\gamma > 0$*

$$(3.1)\quad \sum_{n\geqslant 1} (\lfloor\beta^n\rfloor)^\gamma a_{\lfloor\beta^n\rfloor} < \infty \text{ for any } \beta > 1 \quad \text{if and only if} \quad \sum_{n\geqslant 1} n^{\gamma-1} a_n < \infty .$$

Recall that this result, if true for some $\beta > 1$, is then true for all finite $\beta > 1$. Some results below that are stated under the condition "for some $\beta > 1$" thus hold for all $\beta > 1$. Application of Lemma 1 gives the following version of a standard property of moments of a non-negative r.v. Y.

LEMMA 2. *For any $\gamma > 1$, $\mathrm{E}(Y^\gamma) < \infty$ if and only if for some $\beta > 1$*

$$(3.2)\qquad \sum_{n\geqslant 1} (\lfloor\beta^n\rfloor)^\gamma \Pr\{Y > \lfloor\beta^n\rfloor\} < \infty .$$

Next we have the following bound on moments of partial sums S_n of bounded r.v.s; its nature is largely combinatoric.

LEMMA 3 (Berbee [2]; cf. the proof of Lemma 3.1 in Sen [9]). *Let $\{X_j\}$ be a stationary sequence of zero mean, uniformly bounded r.v.s, $|X_j| \leqslant_{a.s.} c$ say, whose strong mixing coefficients α_n as in (1.7) satisfy $\sum_{k=1}^{\infty} k^{p-2}\alpha_k < \infty$ for some $p \geqslant 2$. Then, for any integer $m \geqslant 2$,*

$$(3.3)\qquad \mathrm{E}(S_n^{2m}) \leqslant Cc^{2m}\big(n^{m+(m-p)^+} + n\sum_{j=1}^{n} j^{2m-2}\alpha_{j-1}\big),$$

where the constant $C \equiv C(m, \{\alpha_j\})$ depends on m and the mixing coefficients.

Remark. This lemma can be extended to non-integers $m' = m+\delta$ with $0 \leqslant \delta < 1$, by replacing $Cc^{2m}\ldots$ by $Cn^{2\delta}c^{2(m+\delta)}\ldots$ Its use in proving an amended Theorem would lead to greater complexity in the statement for no particularly great gain.

Another property of partial sums is as below; it is a simple elaboration of the relation, for sets of real numbers $\{x_i\}$ and $\{y_i\}$, that

$$\sup_i \{x_i + y_i\} \leqslant \sup_i \{x_i\} + \sup_i \{y_i\}.$$

LEMMA 4 (decomposition). *For a set of r.v.s S_k expressible as $S_k = S'_k + S''_k$,*

$$\sup_k \left\{\frac{S_k}{k}\right\} \leqslant \sup_k \left\{\frac{S'_k}{k}\right\} + \sup_k \left\{\frac{S''_k}{k}\right\}. \tag{3.4}$$

This decomposition lemma is applied in the present context, much as done by e.g. Sacks and Wolfson (see Daley and Rolski [5] for similar use and references), to yield inequalities such as

$$\sum_n f(n) \Pr\left\{\sup_{k\geqslant n} \frac{S_k}{k} > 2\varepsilon\right\} \tag{3.5}$$

$$\leqslant \sum_n f(n) \Pr\left\{\sup_{k\geqslant n} \frac{S'_k}{k} > \varepsilon\right\} + \sum_n f(n) \Pr\left\{\sup_{k\geqslant n} \frac{S''_k}{k} > \varepsilon\right\}$$

for non-negative functions $f(n)$.

The proof of the next result is adapted from Berbee [2]. Berbee has bounded random variables and analyzes two-sided convergence. Our random variables are not bounded, but they are one-sided, and we are only interested in one-sided convergence.

LEMMA 5. *Let the incremental r.v.s X_j contributing to the partial sums S_k have $X_j \geqslant_{a.s.} -a$ for a finite positive constant a. Given $\gamma > 2$, if for all $\varepsilon > 0$ and for some $\beta > 1$*

$$\sum_{m\geqslant 1} \beta^{m(\gamma-1)} \Pr\{S_{\lfloor \beta^m \rfloor} > \varepsilon\beta^m\} < \infty, \tag{3.6}$$

then for all $\varepsilon' > 0$

$$\sum_{n\geqslant 1} n^{\gamma-2} \Pr\left\{\sup_{k\geqslant n} \frac{S_k}{k} > \varepsilon'\right\} < \infty. \tag{3.7}$$

Proof. Choose $\varepsilon' > 0$. Let $\beta > 1$ be given, to be determined shortly, and write $S_k(a) = S_k + ka$ for the sum with non-negative summands $X_j + a$. Observe that, whether finite or infinite, the sum in (3.7) is bounded as in

$$\sum_{n\geqslant 1} n^{\gamma-2} \Pr\left\{\sup_{k\geqslant n} \frac{S_k}{k} > \varepsilon'\right\} \leqslant \sum_{k=1}^{\infty} \sum_{\beta^{k-1}\leqslant n<\beta^k} n^{\gamma-2} \Pr\left\{\sup_{j\geqslant \beta^{k-1}} \frac{S_j}{j} > \varepsilon'\right\} \tag{3.8}$$

$$\leqslant \sum_{k=1}^{\infty} \sum_{n:\beta^{k-1}\leqslant n<\beta^k} n^{\gamma-2}$$

$$\times \sum_{i=k-1}^{\infty} \Pr\left\{\sup_{\beta^i\leqslant j<\beta^{i+1}} \frac{S_j(a)}{j} > a+\varepsilon'\right\}$$

$$= \sum_{i=0}^{\infty} \Pr\left\{\sup_{\beta^i \leqslant j < \beta^{i+1}} \frac{S_j(a)}{j} > a+\varepsilon'\right\} \sum_{k=1}^{i+1} \sum_{n:\beta^{k-1} \leqslant n < \beta^k} n^{\gamma-2}$$

$$\leqslant \sum_{i=0}^{\infty} \frac{[\lfloor \beta^{i+1} \rfloor + 1]^{\gamma-1}}{\gamma-1} \Pr\left\{\frac{S_{\lfloor \beta^{i+1} \rfloor}(a)}{\lfloor \beta^i \rfloor} > a+\varepsilon'\right\},$$

using the non-negativity of the summands. Then, apart from a bounding coefficient enabling us to replace $\lfloor \beta^{i+1} \rfloor + 1$ by β^{i+1}, the sum in (3.7) is bounded above by

$$\sum_{m \geqslant k-1} \sum_{i=1}^{\infty} \frac{\beta^{i(\gamma-1)}}{\gamma-1} \Pr\left\{S_{\lfloor \beta^i \rfloor}(a) > \beta^i \frac{a+\varepsilon'}{\beta}\right\}$$

$$= \sum_{i=1}^{\infty} \frac{\beta^{i(\gamma-1)}}{\gamma-1} \Pr\left\{S_{\lfloor \beta^i \rfloor} > \beta^i \frac{\varepsilon' - a(\beta-1)}{\beta}\right\},$$

which is of the form (3.6) provided $\beta > 1$ is small enough to ensure that $\varepsilon' > a(\beta-1)$. The lemma is proved.

Remark. If we knew that ultimately $\Pr\{S_n/n > \varepsilon\} \to 0$ *monotonically*, then by the Cauchy condensation test we should now have the equivalence of (3.7) with $\sum_{n=1}^{\infty} n^{\gamma-2} \Pr\{S_n/n > \varepsilon\} < \infty$ for stationary ergodic zero mean $\{X_j\}$ that are bounded below a.s., and hence by Lemma 4 (cf. (1°) in Section 4), for any such $\{X_j\}$ (cf. also (1.2) and (1.5)).

LEMMA 6 (strong mixing coefficients over sub-σ-fields). *If the strong mixing coefficients $\{\varrho_n^0\}$ are defined over σ-fields $\{\mathscr{A}_0\}$ and $\{\mathscr{B}_0\}$ that are sub-σ-fields of the corresponding σ-fields $\{\mathscr{A}\}$ and $\{\mathscr{B}\}$ over which the strong mixing coefficients $\{\varrho_n\}$ are defined, then $\varrho_n^0 \leqslant \varrho_n$ $(n = 1, 2, \ldots)$.*

Proof (essentially due to M. R. Leadbetter; see Daley and Rolski [5]). We have

$$\sup_{A \in \mathscr{A}_0, B \in \mathscr{B}_0} |P(A \cap B) - P(A)P(B)| \leqslant \sup_{A \in \mathscr{A}, B \in \mathscr{B}} |P(A \cap B) - P(A)P(B)|.$$

4. Proof of the Theorem. (1°) Use the decomposition $X_j = (X_j^+ - a) + (a - X_j^-)$ to express the partial sums S_j in (1.10) in the form

$$S_k = S_k^{(+)} + S_k^{(-)} \equiv \sum_{j=1}^{k} [(X_j^+ - a) + (a - X_j^-)], \tag{4.1}$$

where $a = \mathrm{E}(X_j^+) = \mathrm{E}(X_j^-)$ (all j). Note that the summands in these latter two sums are one-sided r.v.s with zero means. When (4.1) is used in conjunction with Lemma 4 we have, for $\varepsilon \geqslant 0$,

$$\text{(4.2)}\qquad \Pr\left\{\sup_{k\geqslant n}\frac{S_k}{k}>2\varepsilon\right\}=\Pr\left\{\sup_{k\geqslant n}\left(\frac{S_k^{(+)}}{k}-\varepsilon+\frac{S_k^{(-)}}{k}-\varepsilon\right)>0\right\}$$

$$\leqslant \Pr\left\{\sup_{k\geqslant n}\frac{S_k^{(+)}}{k}>\varepsilon\right\}+\Pr\left\{\sup_{k\geqslant n}\frac{S_k^{(-)}}{k}>\varepsilon\right\}.$$

Thus we have reduced the problem entailed in showing that (1.10) holds to the two cases where $S_k^{(-)}/k \leqslant_{\text{a.s.}} a$, and $S_k^{(+)}/k >_{\text{a.s.}} -a$, with $a \equiv \mathrm{E}(X_j^+)$ as above. The case $\sup_k S_k^{(-)}/k$ is simpler.

(2°) Two simpler cases. Let us note first that the case Berbee treated, that $\{X_j\}$ is a bounded stationary sequence, is included in the Theorem. To see this, let $g(\cdot)$ be a bounded continuous monotonic function for which $\sup_{0<x<\infty} g(x) > \operatorname{ess\,sup}|X_j|$. Then (2.1) and (2.3) are trivially satisfied, while the boundedness of (2.2) reduces to the boundedness of $\sum_{k=1}^{\infty} k^{2(\gamma-1)}\alpha_k k^{-\gamma}$, i.e. the condition (1.8).

The truth of the theorem for bounded $\{X_j\}$ also extends easily to the case where $X_j \leqslant M$ a.s. for some finite M. To see this, let $\varepsilon>0$ be given, and observe that, because $\mathrm{E}X_j=0$, we can find finite positive A such that the stationary sequence $\{Y_j^A\}\equiv\{\max(X_j,-A)\}$ has $\mathrm{E}(Y_j^A)<\frac{1}{2}\varepsilon$. Such a sequence is bounded, and its mixing coefficients are dominated by $\{\alpha_n\}$ much as in Lemma 6. So, Berbee's result implies that

$$\infty > \sum_{n=1}^{\infty} n^{\gamma-2}\Pr\left\{\sup_{k\geqslant n}\frac{1}{k}\sum_{j=1}^{k}\left(\max(X_j,-A)-\mathrm{E}Y_1^A\right)>\tfrac{1}{2}\varepsilon\right\}$$

$$\geqslant \sum_{n=1}^{\infty} n^{\gamma-2}\Pr\left\{\sup_{k\geqslant n}\frac{1}{k}\sum_{j=1}^{k}\max(X_j,-A)>\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon\right\}\qquad(\text{because } \mathrm{E}Y_j^A\leqslant\tfrac{1}{2}\varepsilon)$$

$$\geqslant \sum_{n=1}^{\infty} n^{\gamma-2}\Pr\left\{\sup_{k\geqslant n}\frac{1}{k}\sum_{j=1}^{k}X_j>\varepsilon\right\},$$

i.e. the conclusion (1.10) holds.

(3°) Truncation and the moment bound. Let S_k denote a partial sum of summands $Y_j \equiv X_j^+ - a$. Construct a sequence of truncations of the Y_j's via

$$Y_{j,n}=X_j^+ I_{\{X_j^+\leqslant g(n)\}}-a$$

for some increasing function $g(\cdot)$ to be determined, and let

$$T_n=\sum_{j=1}^{n}Y_{j,n},\qquad X_{j,n}=X_j^+ I_{\{X_j^+\leqslant g(n)\}}.$$

Since

$$\text{(4.3)}\qquad \Pr\{S_n>x\}\leqslant\Pr\{T_n>x\}+\Pr\{S_n\neq T_n\}$$

$$\text{(4.4)}\qquad \leqslant\Pr\{T_n>x\}+n\Pr\{X_j^+>g(n)\},$$

it is enough, by (4.3) applied to Lemma 5 after substituting from (4.4), to show that both

$$\sum_{j\geqslant 1} \beta^{j(\gamma-1)} \Pr\{T_{\lfloor\beta^j\rfloor} > \varepsilon\beta^j\} < \infty \tag{4.5}$$

and

$$\sum_{j\geqslant 1} \beta^{j\gamma} \Pr\{X_i^+ > g(\lfloor\beta^j\rfloor)\} = \sum_{j=1}^{\infty} \beta^{j\gamma} \Pr\{g^{-1}(X_i^+) > \lfloor\beta^j\rfloor\} < \infty. \tag{4.6}$$

Because of (2.1) the summation in (4.6) is finite by Lemma 2.

(4°) To show that the summation in (4.5) is finite we write, for some integer $m \geqslant 2$,

$$\begin{aligned}\sum_{j\geqslant 1} \beta^{j(\gamma-1)} \Pr\{T_{\lfloor\beta^j\rfloor} > \varepsilon\beta^j\} &= \sum_{j\geqslant 1} \beta^{j(\gamma-1)} \Pr\Big\{\sum_{i=1}^{\lfloor\beta^j\rfloor} X_{i,\lfloor\beta^j\rfloor}^+ - a\lfloor\beta^j\rfloor > \varepsilon\beta^j\Big\} \\ &\leqslant \sum_{j\geqslant 1} \beta^{j(\gamma-1)} \Pr\Big\{\sum_{i=1}^{\lfloor\beta^j\rfloor} (X_{i,\lfloor\beta^j\rfloor}^+ - \mathrm{E}X_{j,\lfloor\beta^j\rfloor}^+) > \varepsilon\beta^j\Big\} \\ &\leqslant \sum_{j\geqslant 1} \beta^{j(\gamma-1)} \frac{\mathrm{E}\Big[\Big(\sum_{i=1}^{\lfloor\beta^j\rfloor} (X_{i,\lfloor\beta^j\rfloor}^+ - \mathrm{E}X_{i,\lfloor\beta^j\rfloor}^+)\Big)^{2m}\Big]}{\varepsilon^{2m}\beta^{2jm}},\end{aligned} \tag{4.7}$$

where the last inequality follows from the Markov inequality. From Lemma 3, with p as there, we know that for integers $m \geqslant 2$

$$\mathrm{E}\Big[\Big(\sum_{i=1}^{\lfloor\beta^j\rfloor} (X_{i,\lfloor\beta^j\rfloor}^+ - \mathrm{E}X_{i,\lfloor\beta^j\rfloor}^+)\Big)^{2m}\Big] \leqslant Cg(n)^{2m}\Big(n^{m+(m-p)^+} + n\sum_{k=1}^{n} k^{2m-2}\alpha_{k-1}^{(+)}\Big)\Big|_{n=\lfloor\beta^j\rfloor},$$

where the constant $C \equiv C(m, \{\alpha_k^{(+)}\})$ depends on m and the mixing coefficients of $\{X_j^+\}$. Finiteness of the sum in (4.7) is thus ensured when, firstly,

$$\infty > \sum_{j=1}^{\infty} \frac{\beta^{j(\gamma-1)}[g(\lfloor\beta^j\rfloor)]^{2m}\beta^{j(m+(m-p)^+)}}{\beta^{2jm}} = \sum_{j=1}^{\infty} \beta^{j(\gamma-1-\min(m,p))}[g(\lfloor\beta^j\rfloor)]^{2m},$$

and, secondly, using Lemma 6 as well,

$$\begin{aligned}\infty &> \sum_{j=1}^{\infty} \beta^{j(\gamma-2m)}\big(g(\lfloor\beta^j\rfloor)\big)^{2m} \sum_{k=1}^{\lfloor\beta^j\rfloor} k^{2m-2}\alpha_{k-1} \\ &= \sum_{k=1}^{\infty} k^{2m-2}\alpha_{k-1} \sum_{j:\,\lfloor\beta^j\rfloor\geqslant k}^{\infty} \beta^{j(\gamma-2m)}\big(g(\lfloor\beta^j\rfloor)\big)^{2m}.\end{aligned}$$

When $p = m = \gamma$, so, γ is an integer because m is, these relations reduce to (2.3) and (2.2), respectively, thereby proving the Theorem.

Note that this choice of p and m implies that in the corollaries the assumption stated in Lemma 3 that $\sum_{k=1} k^{p-2}\alpha_k < \infty$ is automatically satisfied.

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