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DEFORMATIONS OF THE SEMICIRCLE LAW DERIVED FROM RANDOM WALKS ON FREE GROUPS

BY

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Abstract. New 1-parameter families of central limit distributions are investigated by means of random walks on trees associated with free groups under two kinds of states: one is Haagerup's function and the other is a spherical function associated with unitary representations of the principal series. Those families give rise to deformations of Wigner's semicircle distribution by non-symmetric probability measures.

1. Introduction. In classical probability theory a significant role is played by the central limit theorem which asserts that statistical properties of independent identically distributed random variables are subject to the Gaussian distribution. In quantum probability theory (or non-commutative probability theory) there have been made a considerable number of studies on quantum analogues of the classical central limit theorems. Quantum Brownian motions on Fock spaces are subjects for non-commutative central limit theorems. It is well known that the Gaussian distribution appears in a quantum Brownian motion on the boson Fock space [4], the Fermion distribution appears in one on the Fermion Fock space [14] and Wigner's semicircle distribution appears in one on the free Fock space [10]–[12]. An interesting new example is studied in [1] and [9], where the arcsine distribution appears in a study of a chronological Fock space. Random walks on Cayley graphs associated with discrete groups offer good motivation to study central limit theorems in a quantum context. Typically it is known that the limit distribution obtained from regular representations of free Abelian groups is the Gaussian distribution and that from free groups Wigner's semicircle distribution [13]. Motivated by generalizing [13], a systematic study for general discrete groups has been initiated in [7]. From a combinatorial point of view, all these results so far discussed are obtained by counting the number of particular type of pairwise partitions or, equivalently, the number of walks on a graph which start from an origin and return to it. As a result, all odd moments of limit distributions vanish, and hence the limit distributions are always symmetric with respect to 0.

In this paper, we consider random walks on trees associated with free groups and investigate their limit distributions under two particular states: one is Haagerup's function, which depends exponentially on the length of an element of a free group; and the other is a spherical function associated with a unitary representation of the principal series. They are defined on finitely generated free groups. Under these states, all types of walks on trees contribute to the moments of limit distributions. Therefore the limit distributions become non-symmetric with respect to 0 (see Figs. A and B). This is a noticeable difference from usual non-commutative central limit theorems. Moreover, since these states depend on the number of generators of free groups, there appear various limit distributions by changing the proportion of subtrees on which random walks wander to whole trees associated with free groups (see Theorem 2.2). This is also a new phenomenon.

In Section 2, we introduce preliminary notions and state the main results in Theorems 2.1 and 2.2. In Section 3 we prove Theorem 2.1 by calculating all moments of limit distributions using a 'non-commutative binomial expansion' formula. By modifying the proof of Theorem 2.1, we prove Theorem 2.2 in Section 4.

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2. Preliminaries and results. Let F_n be the free group generated by $\{g_1, g_2, \dots, g_n\}$. Every element $g \neq e$ admits a reduced word expression $g = g_{i_1}^{\varepsilon_1} g_{i_2}^{\varepsilon_2} \cdots g_{i_l}^{\varepsilon_l}$, where $\varepsilon_j \neq 0$, $i_1 \neq i_2 \neq \dots \neq i_l$. We call $|g| = \sum |\varepsilon_j|$ the *length* of g and put $|e| = 0$. We consider a random variable $X_k \in C(F_n)$:

$$X_k = g_1 + g_1^{-1} + g_2 + g_2^{-1} + \dots + g_k + g_k^{-1},$$

called a *random walk*, where $k \leqslant n$, and observe its distribution under the following states:

(a) Haagerup's function: Haagerup [5] proved that for $0 \leqslant a \leqslant 1$ the function ϕ_a: $F_n \to C$ defined by $\phi_a(g) := a^{|g|}$, $g \in F_n$, is positive definite, hence ϕ_a is considered as a state on $C(F_n)$.

(b) A spherical function associated with a unitary representation of the principal series: Figà-Talamanca and Picardello [3] constructed a 1-parameter family $\{\pi_z\}$ of unitary representations of F_n, called the *principal series*. We consider the state corresponding to $z = 1/2$, which is given as

$$\psi_n(g) := (1 + |g|(n-1)/n)/(\sqrt{2n-1})^{|g|}.$$

Our goal is to obtain central limit theorems for the random walks $\{X_k\}$ under the states $\{\phi_a\}$ and $\{\psi_n\}$, so we need to rescale $\{X_k\}$. In the case of $\{\phi_a\}$, since the average is equal to

$$\phi_a(g_i + g_i^{-1}) = 2a$$

and the variance is

$$\phi_a\big((g_i+g_i^{-1}-2a)^2\big)=2(1-a^2),$$

it follows that

$$Y_k:=\frac{X_k-2ak}{\sqrt{2k(1-a^2)}} \tag{2.1}$$

is a normalized random walk with average 0 and variance 1 under ϕ_a. Similarly, as

$$\psi_n(g_i+g_i^{-1})=2\sqrt{2n-1}/n$$

and

$$\psi_n\big((g_i+g_i^{-1}-2\sqrt{2n-1}/n)^2\big)=2\big(1-2(2n-1)/n^2+(3n-2)/n(2n-1)\big),$$

it follows that

$$Z_k:=\frac{X_k-2k(\sqrt{2n-1}/n)}{\sqrt{2k\left(1-\dfrac{2(2n-1)}{n^2}\right)+\dfrac{3n-2}{n(2n-1)}}} \tag{2.2}$$

is a normalized random walk with average 0 and variance 1 under ψ_n.

The main results are stated as follows.

THEOREM 2.1. *Let $\mu_{a,k}$ be the distribution associated with the random walk $Y_k\in C(F_n)$ $(k\leqslant n)$ under the state ϕ_a. Assume that $a\approx A(2k)^\alpha$ as $k\to\infty$. Then*:

(i) *If $\alpha<-1/2$, $d\mu_{a,k}$ converges weakly to the normalized semicircle distribution*:

$$\lim_{k\to\infty} d\mu_{a,k}(x)=\frac{1}{2\pi}\chi_{[-2,2]}(x)\sqrt{4-x^2}\,dx. \tag{2.3}$$

(ii) *If $\alpha=-1/2$ and $0\leqslant A\leqslant 1$, $d\mu_{a,k}$ converges weakly to a probability measure with a parameter A*:

$$\lim_{k\to\infty} d\mu_{a,k}(x)=\frac{1}{2\pi}\chi_{[-2-A,2-A]}(x)\frac{\sqrt{(2+A+x)(2-A-x)}}{1-Ax}dx. \tag{2.4}$$

Remark. (1) Denote the right-hand side of (2.4) by $d\mu_A(x)$. Note that μ_0 is the semicircle distribution which maximizes Voiculescu's free entropy among the probability measures supported in $\boldsymbol{R}$ with $\int x^2\,d\mu(x)\leqslant 1$, and $T_*\mu_1$ also maximizes the entropy among the probability measures supported in $\boldsymbol{R}_{\geqslant 0}$ with $\int x\,d\mu(x)\leqslant 1$, where $T(x)=1-x$ (see [7]).

(2) Note also that $T_*\mu_1$ is a kind of 'one-sided' Ullman distributions. The density function of μ_A is plotted in Fig. A.

(3) In [2], we see the 1-parameter family μ_A in a different context. Bożejko et al. [2] found a 2-parameter family $\pi_{\alpha,\beta}$ which is derived from the 'c-free' Poisson law. The relation is given as

$$\pi_{\beta,\beta}=\beta S^*\mu_{\sqrt{\beta}}+(1-\beta)\,\delta_0,\qquad \text{where } S(x)=(1-x)/\sqrt{\beta}.$$

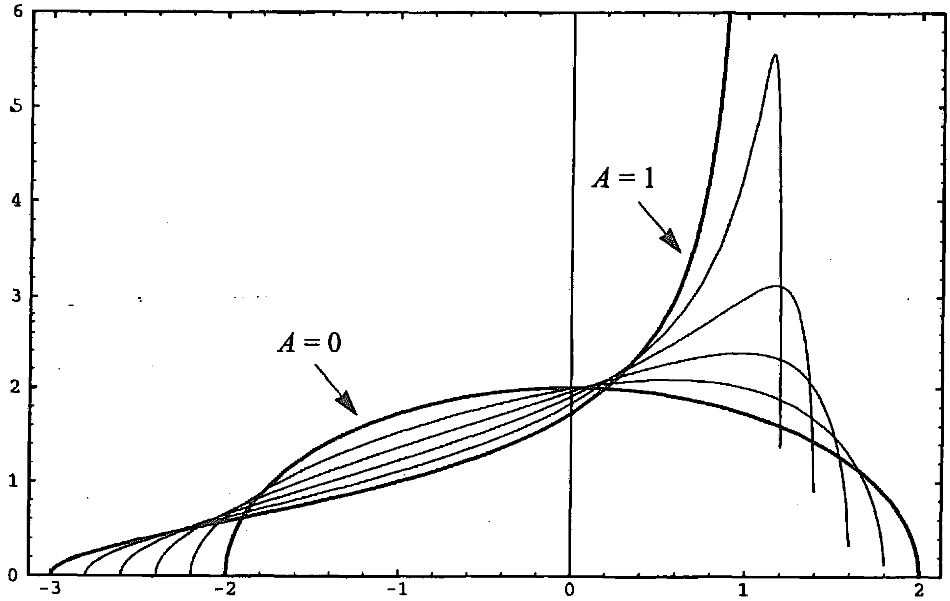

Fig. A. The density function of μ_A

THEOREM 2.2. *Let $\nu_{n,k}$ be the distribution associated with the random walk $Z_k \in C(F_n)$ $(k \leqslant n)$ under the state ψ_n. Assume that $n \approx B^2 k^\beta$ as $k \to \infty$. Then:*

(i) *If $\beta > 1$, $d\nu_{n,k}$ converges weakly to the normalized semicircle distribution:*

$$\lim_{k\to\infty} d\nu_{n,k}(x) = \frac{1}{2\pi}\chi_{[-2,2]}(x)\sqrt{4-x^2}\,dx. \tag{2.5}$$

(ii) *If $\beta = 1$ and $B > 1$, $d\nu_{n,k}$ converges weakly to a probability measure with a parameter B:*

$$\lim_{k\to\infty} d\nu_{n,k}(x) \tag{2.6}$$

$$= \frac{1}{2\pi}\chi_{[-2-2/B,\,2-2/B]}(x)\frac{(B^2-1)\sqrt{(2+2/B+x)(2-2/B-x)}}{(B-1/B-x)^2}dx.$$

Remark. Denote the right-hand side of (2.6) by $d\nu_B(x)$. Note that, as $B \to \infty$, ν_B converges to the normalized semicircle distribution. On the other hand, as $B \searrow 1$, ν_B converges to the Dirac measure concentrated at 0. The density function of ν_B is plotted in Fig. B.

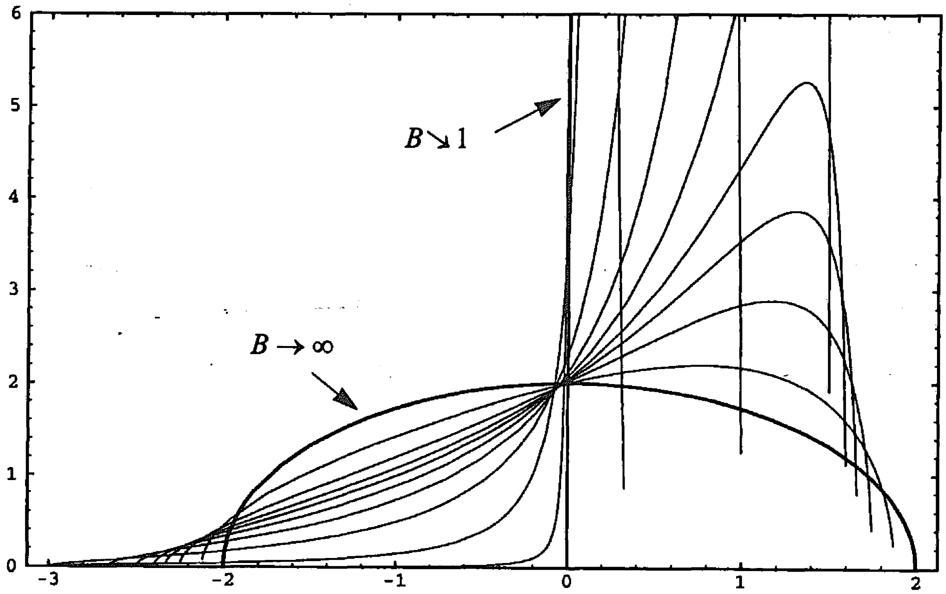

Fig. B. The density function of ν_B

3. Proof of Theorem 2.1. We first compute $\phi_a(X_k^l)$ explicitly. Let $N(l, r)$ be the number of reduced words with length r appearing in the expansion of X_k^l, putting $N(0, 0) = 1$ and $N(l, r) = 0$ for $l < r$ or $r < 0$. Then we consider a polynomial

$$f_l(w) = \sum_{r=0}^{l} N(l, r) w^r.$$

By the definition of ϕ_a, we see that $\phi_a(X_k^l) = f_l(a)$. Note also that $\{N(l, r)\}$ satisfy the recursion formula

$$N(l, r) = (2k-1) N(l-1, r-1) + N(l-1, r+1) \quad \text{for } l > 1 \tag{3.1}$$

with $N(1,1) = 2k$ and $N(1, 0) = 0$. Let S, D: $\mathbf{C}[w] \to \mathbf{C}[w]$ be linear operators defined by $D[1] = 0$, $D[w^l] = w^{l-1}$ and $S[w^l] = w^{l+1}$. Then (3.1) yields another expression of f_l,

$$f_l(w) = \frac{2k}{2k-1} [(2k-1) S + D]^l [1] \ (l > 0), \qquad f_0(w) = 1. \tag{3.2}$$

We call a sequence $\mathscr{E} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l) \in \{0, 1\}^l$ *Catalanian* if $\sum_{j=1}^{r} \varepsilon_j \geqslant r/2$ for all $1 \leqslant r \leqslant l$. Let $C_{p,q}$ be the number of Catalanian sequences of length $l = p+q$ with $\sum_{j=1}^{l} \varepsilon_j = p$. It is known (see, e.g., [8]) that

$$C_{p,q} = \binom{p+q}{p} - \binom{p+q}{p+1},$$

called the *Catalan number*.

LEMMA 3.1. *For* $u, v \in \boldsymbol{C}$,

$$[uS+vD]^l[1] = \sum_{\substack{p+q=l \\ p \geqslant q}} C_{p,q} u^p v^q w^{p-q}.$$

Proof. Let $T^1 := S$, $T^0 := D$. For a sequence $\mathscr{E} = (\varepsilon_1, \ldots, \varepsilon_l)$,

$$T^{\varepsilon_l} T^{\varepsilon_{l-1}} \ldots T^{\varepsilon_1}[1] = \begin{cases} w^{p-q} & \text{if } \mathscr{E} \text{ is Catalanian with } \sum_{j=1}^{l} \varepsilon_j = p, \\ 0 & \text{otherwise.} \end{cases}$$

Then the assertion follows immediately. ■

By this lemma, (3.2) becomes

$$f_l(w) = \frac{2k}{2k-1} \sum_{\substack{p+q=l \\ p \geqslant q}} C_{p,q}(2k-1)^p w^{p-q} \qquad (l>0). \tag{3.3}$$

Now, we obtain an explicit formula of the moments of $\mu_{a,k}$. By (3.2),

$$\begin{aligned}
\phi_a(Y_k^m) &= \left(\frac{1}{\sqrt{2k(1-a^2)}}\right)^m \sum_{l=0}^{m} \binom{m}{l} (-2ka)^{m-l} \phi_a(X_k^l) \\
&= \frac{2k}{2k-1} \left(\frac{1}{\sqrt{2k(1-a^2)}}\right)^m \sum_{l=0}^{m} \binom{m}{l} (-2ka)^{m-l} f_l(a) \\
&= \frac{2k}{2k-1} \left(\frac{1}{\sqrt{2k(1-a^2)}}\right)^m [(2k-1)S+D-2ka]^m[1](a) \\
&\qquad - \frac{1}{2k-1} \left(\frac{-2ka}{\sqrt{2k(1-a^2)}}\right)^m.
\end{aligned} \tag{3.4}$$

Hence by (3.3) we obtain

$$\begin{aligned}
&\phi_a(Y_k^m) \\
&= \frac{2k}{2k-1} \left(\frac{1}{\sqrt{2k(1-a^2)}}\right)^m \sum_{l=1}^{m} \binom{m}{l} (-2ka)^{m-l} \sum_{\substack{p+q=l \\ p \geqslant q}} C_{p,q}(2k-1)^p a^{p-q} \\
&\qquad - \frac{1}{2k-1} \left(\frac{-2ka}{\sqrt{2k(1-a^2)}}\right)^m.
\end{aligned} \tag{3.5}$$

Suppose that a is a function of k such that $\lim_{k \to \infty} a/k^{\alpha} = A < \infty$. Then, each term on the right-hand side of (3.5) is of order $(1+2\alpha)(m/2+p-l)$ or $(1+2\alpha)m/2-1$ in k. Therefore, we need $\alpha \leqslant -1/2$ in order to obtain a meaningful limit.

Now we prove Theorem 2.1. We first discuss the case of $\alpha = -1/2$. Without loss of generality, we can assume $a = A/\sqrt{2k}$. From (3.4) it follows that

$$\int_{\boldsymbol{R}} e^{itx}\, d\mu_{a,k}(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \phi_a((-itY_k)^m)$$
$$= \frac{2k}{2k-1} \exp\left(\frac{it}{\sqrt{2k(1-a^2)}}[(2k-1)S+D-A\sqrt{2k}]\right)[1]\left(\frac{A}{\sqrt{2k}}\right)$$
$$-\frac{1}{2k-1}\exp\left(-\frac{itA}{2k\sqrt{1-a^2}}\right).$$

Since the multiplication operator commutes with S and D, we have

$$\exp\left(\frac{it}{\sqrt{2k(1-a^2)}}[(2k-1)S+D-A\sqrt{2k}]\right)[1] \tag{3.6}$$
$$= \exp\left(-\frac{itA}{\sqrt{1-a^2}}\right)\exp\left(\frac{it}{\sqrt{2k(1-a^2)}}[(2k-1)S+D]\right)[1].$$

Then, using (3.3),

$$\int_{\boldsymbol{R}} e^{itx}\, d\mu_{a,k}(x) = \frac{2k}{2k-1}\exp\left(-\frac{itA}{\sqrt{1-a^2}}\right)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{it}{\sqrt{2k(1-a^2)}}\right)^m$$
$$\times \sum_{\substack{p+q=m\\ p\geq q}} C_{p,q}(2k-1)^p\left(\frac{A}{\sqrt{2k}}\right)^{p-q} - \frac{1}{2k-1}\exp\left(-\frac{itA}{2k\sqrt{1-a^2}}\right).$$

Letting $k\to\infty$, we have

$$\int_{\boldsymbol{R}} e^{itx}\, d\mu_A(x) = e^{-itA}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\sum_{\substack{p+q=m\\ p\geq q}} C_{p,q}A^{p-q}. \tag{3.7}$$

We here need the Bessel functions $J_r(z)$:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{m=0}^{\infty}\frac{(iz/2)^{2m}}{(m+r)!\,m!}, \qquad r\in \boldsymbol{N},\ z\in\boldsymbol{C}. \tag{3.8}$$

The following integral formula is well known:

$$J_r(z) = \frac{1}{\pi i^r}\int_0^{\pi} e^{iz\cos\theta}\cos r\theta\, d\theta. \tag{3.9}$$

Since

$$|J_r(z)| \leqslant \left(\frac{|z|}{2}\right)^r \frac{1}{r!} \exp\left(\frac{|z|^2}{4}\right),$$

the series $\sum_{r=0}^{\infty} C^r J_r(z)$ converges absolutely for any $C \in \boldsymbol{C}$.

Now we return to (3.7). Since

$$C_{p,q} \leqslant \binom{p+q}{p},$$

we see easily that the right-hand side converges absolutely for all A. Then we can decompose the sum (3.7) as

$$\sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{\substack{p+q=m \\ p \geqslant q}} C_{p,q} A^{p-q} = \sum_{r=0}^{\infty} A^r \sum_{m=0}^{\infty} \frac{(it)^{2m+r}}{(2m+r)!} C_{m+r,m}$$

$$= \sum_{r=0}^{\infty} (iA)^r \left(\sum_{m=0}^{\infty} t^r \frac{(it)^{2m}}{(m+r)!\, m!} + \sum_{m=1}^{\infty} t^{r+2} \frac{(it)^{2(m-1)}}{(m+r+1)!\,(m-1)!} \right).$$

In view of (3.8) and (3.9), the last expression equals to

$$(3.10) \qquad \sum_{r=0}^{\infty} (iA)^r \big(J_r(2t) + J_{r+2}(2t)\big) = \frac{1}{\pi} \int_0^{\pi} e^{2it\cos\theta} \sum_{r=0}^{\infty} A^r \big(\cos r\theta - \cos(r+2)\theta\big)\, d\theta.$$

With the help of the formula

$$\sum_{r=0}^{\infty} A^r \cos r\theta = \frac{1 - A\cos\theta}{1 - 2A\cos\theta + A^2} \qquad \text{for } |A| < 1,$$

we see easily

$$(3.11) \qquad \sum_{r=0}^{\infty} A^r \big(\cos r\theta - \cos(r+2)\theta\big) = \frac{2\sin^2\theta}{1 - 2A\cos\theta + A^2}, \qquad 0 \leqslant A < 1,$$

and the series is absolutely convergent. Consequently, we have

$$(3.12) \qquad \int_{\boldsymbol{R}} e^{itx}\, d\mu_A(x) = \frac{e^{-itA}}{\pi} \int_0^{\pi} e^{2it\cos\theta} \frac{2\sin^2\theta}{1 - 2A\cos\theta + A^2}\, d\theta$$

$$= e^{-itA} \int_{-2}^{2} e^{itx} \frac{1}{2\pi} \frac{\sqrt{4-x^2}}{1 - Ax + A^2}\, dx = \int_{-2-A}^{2-A} e^{itx} \frac{1}{2\pi} \frac{\sqrt{(2+A+x)(2-A-x)}}{1 - Ax}\, dx.$$

Now we consider the case of $A = 1$. We return to (3.10). In this case, we see that

$$\sum_{r=0}^{\infty} i^r \big(J_r(2t) + J_{r+2}(2t)\big) = J_0(2t) + iJ_1(2t) = \frac{1}{\pi} \int_0^{\pi} e^{2it\cos\theta} (1 + \cos\theta)\, d\theta,$$

which coincides with (3.12) with $A = 1$. We have thus completed the proof of (ii).

Next we show (i). Assume $\alpha < -1/2$ and go back to (3.5). Then only the terms with $l = m = 2p = 2q$ remain in the limit as $k \to \infty$, that is,

$$\lim_{k\to\infty} \phi_a(Y_k^m) = \begin{cases} C_{p,q} & \text{when } m = 2p, \\ 0 & \text{when } m \text{ is an odd number,} \end{cases}$$

which is reduced to (ii) with $A = 0$. We have established Theorem 2.1.

4. Proof of Theorem 2.2. We prove Theorem 2.2 by modifying the proof in Section 3. We first consider the polynomial

$$h_l(w) = \sum_{r=0}^{l} N(l, r)\left(1+\frac{n-1}{n}r\right)w^r,$$

where $N(l, r)$ is defined in the previous section. The relation between $h_l(w)$ and $f_l(w)$ is given by

$$h_l(w) = \left[1+\frac{n-1}{n}w\frac{d}{dw}\right] f_l(w).$$

Then, by (3.2) and (3.3),

$$h_l(w) = \frac{2k}{2k-1}\left[1+\frac{n-1}{n}w\frac{d}{dw}\right][(2k-1)S+D]^l[1] \tag{4.1}$$

$$= \frac{2k}{2k-1}\sum_{\substack{p+q=l\\ p\geqslant q}} C_{p,q}(2k-1)^p\left(1+\frac{n-1}{n}(p-q)\right)w^{p-q}.$$

Let

$$b_n = \sqrt{2n-1}/n \quad \text{and} \quad \sigma_n = 2\big(1-2(2n-1)/n^2+(3n-2)/n(2n-1)\big).$$

From the definition of ψ_n we see that $\psi_n(X_k^l) = h_l(1/\sqrt{2n-1})$. We also obtain explicit formulae for the moments of $\nu_{n,k}$,

$$\psi_n(Z_k^m) = \frac{2k}{2k-1}\left(\frac{1}{\sqrt{k\sigma_n}}\right)^m\left(1+\frac{n-1}{n}w\frac{d}{dw}\right) \tag{4.2}$$

$$\times[(2k-1)S+D-2kb_n]^m[1]\left(\frac{1}{\sqrt{2n-1}}\right)-\frac{1}{2k-1}(-2kb_n)^m$$

$$= \frac{2k}{2k-1}\left(\frac{1}{\sqrt{k\sigma_n}}\right)^m\sum_{l=0}^{m}\binom{m}{l}(-2kb_n)^{m-l} \tag{4.3}$$

$$\times\sum_{\substack{p+q=l\\ p\geqslant q}} C_{p,q}\left(1+\frac{n-1}{n}(p-q)\right)(2k-1)^p(2n-1)^{-(p-q)/2}-\frac{1}{2k-1}(-2kb_n)^m.$$

Suppose that n is a function of k and $\lim_{k\to\infty} n/k^{\beta} = \infty$. Then, repeating an argument similar to that in Section 3, we obtain the condition $\beta \geqslant 1$.

First we show (ii) of Theorem 2.2. We can assume $n = B^2 k$. Applying a method as in (3.6) and using (4.2) and (4.1), we have

$$\int_{R} e^{itx}\, d\nu_{n,k}(x) = \frac{2k}{2k-1}\left(1+\frac{n-1}{n} w\frac{d}{dw}\right)\exp\left(-it\frac{2kb_n}{\sqrt{k\sigma_n}}\right)$$

$$\times \exp\left(\frac{it}{\sqrt{k\sigma_n}}[(2k-1)S+D]\right)[1]\left(\frac{1}{\sqrt{2n-1}}\right)-\frac{1}{2k-1}\exp\left(-it\frac{2kb_n}{\sqrt{k\sigma_n}}\right)$$

$$= \frac{2k}{2k-1}\exp\left(-it\frac{2kb_n}{\sqrt{k\sigma_n}}\right)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{it}{\sqrt{k\sigma_n}}\right)^m$$

$$\times \sum_{\substack{p+q=m\\ p\geqslant q}} C_{p,q}\left(1+\frac{n-1}{n}(p-q)\right)(2k-1)^p(2n-1)^{-(p-q)/2}-\frac{1}{2k-1}\exp\left(-it\frac{2kb_n}{\sqrt{k\sigma_n}}\right).$$

Then, letting $k\to\infty$, we have

$$\int_{R} e^{itx}\, d\nu_B(x) = e^{-(2it)/B}\sum_{m=0}^{\infty}\frac{1}{m!}(it)^m \sum_{\substack{p+q=m\\ p\geqslant q}} C_{p,q}(1+p-q)B^{q-p}. \tag{4.4}$$

In a manner as in the previous section, we have

$$\int_{R} e^{itx}\, d\nu_B(x) = e^{-(2it)/B}\frac{1}{\pi}\int_0^{\pi} e^{2it\cos\theta}\sum_{r=0}^{\infty}(1+r)B^{-r}(\cos r\theta-\cos(r+2)\theta)\,d\theta.$$

For $B>1$ we apply (3.11) to obtain

$$\sum_{r=0}^{\infty}(1+r)B^{-r}(\cos r\theta-\cos(r+2)\theta)$$

$$= \frac{\partial}{\partial B^{-1}}B^{-1}\sum_{r=0}^{\infty}B^{-r}(\cos r\theta-\cos(r+2)\theta)$$

$$= \frac{\partial}{\partial B^{-1}}\left(\frac{2B^{-1}\sin^2\theta}{1-2B^{-1}\cos\theta+B^{-2}}\right) = \frac{2(B^2-1)\sin^2\theta}{(B+B^{-1}-2\cos\theta)^2}.$$

Therefore,

$$\int_{R} e^{itx}\, d\nu_B(x) = e^{-(2it)/B}\frac{1}{\pi}\int_0^{\pi} e^{2it\cos\theta}\frac{2(B^2-1)\sin^2\theta}{(B+B^{-1}-2\cos\theta)^2}d\theta$$

$$= \frac{1}{2\pi}\int_{-2-2/B}^{2-2/B} e^{itx}\frac{(B^2-1)\sqrt{(2-2/B-x)(2+2/B+x)}}{(B-1/B-x)^2}dx.$$

In the case of $B = 1$, since

$$\sum_{\substack{p+q=m \\ p \geqslant q}} C_{p,q}(1+p-q) = 2^m,$$

we see from (4.4) that

$$\int_{\mathbf{R}} e^{itx}\, d\nu_1(x) = e^{-2it} \sum_{m=0}^{\infty} \frac{1}{m!}(2it)^m = 1.$$

Namely, ν_1 is the Dirac measure concentrated at 0.

Now we show (i). Assuming $\beta > 1$, from (4.3) we see that

$$\lim_{k\to\infty} \psi_a(Z_k^m) = \begin{cases} C_{p,q} & \text{when } m = 2p, \\ 0 & \text{when } m \text{ is an odd number,} \end{cases}$$

which turns out to be the case of (ii) with $B = \infty$. This completes the proof.

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