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HAUSDORFF DIMENSION THEOREMS FOR SELF-SIMILAR MARKOV PROCESSES

BY

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Abstract. Let $X(t)$ $(t \in \boldsymbol{R}_+)$ be an α-self-similar Markov process on $\boldsymbol{R}^d$ or $\boldsymbol{R}^d_+$. The Hausdorff dimension of the image, graph and zero set of $X(t)$ are obtained under certain mild conditions. Similar results are also proved for a class of elliptic diffusions.

1. Introduction. The class of α-self-similar (α-s.s.) Markov processes on $(0, \infty)$ and $[0, \infty)$ were introduced and studied by Lamperti [9], who used the name "semi-stable". The α-s.s. Markov processes on $\boldsymbol{R}^d$ or $\boldsymbol{R}^d \setminus \{0\}$ were investigated by Graversen and Vuolle-Apiala [6]. See also Kiu [8].

A very important and useful result in Lamperti [9] is Theorem 4.1, which relates, through random time change, a $[0, \infty)$-valued self-similar Markov process with a real-valued Lévy process, and hence makes it possible to study sample path properties of α-s.s. Markov processes on $[0, \infty)$ by using known results for Lévy processes. By applying this argument, Liu [11], Xiao and Liu [29] obtained the Hausdorff and packing dimension of the sample paths of $[0, \infty)$-valued α-s.s. Markov processes, and Li et al. [10] studied the exact Hausdorff and packing measure of the image set. See Lamperti [9], Vuolle-Apiala [23] for other applications. Graversen and Vuolle-Apiala [6] and Kiu [8] generalized some of Lamperti's results, including the above-mentioned Theorem 4.1, to $\boldsymbol{R}^d$-valued isotropic α-s.s. Markov processes. Vuolle-Apiala and Graversen [25], based on the results in Graversen and Vuolle-Apiala [6], studied the duality of isotropic α-s.s. Markov processes on $\boldsymbol{R}^d \setminus \{0\}$. However, it seems very difficult to apply the results of Graversen and Vuolle-Apiala [6] to study sample path properties such as the lower functions, the exact Hausdorff measure of the image, graph and level sets as well as the local times of (isotropic) $\boldsymbol{R}^d$-valued α-s.s. Markov processes via corresponding results for Lévy processes, because one has to deal with the angular process. Recently, Xiao [28] has studied the lower functions of certain $\boldsymbol{R}^d$-valued α-s.s. Markov processes by applying general Markov arguments and has generalized many results about Brownian motion and stable Lévy processes to more general α-s.s. Markov processes.

In this paper, we will continue the line of research of Xiao [28] and study the Hausdorff dimension of the image, graph and zero set of certain α-s.s. Markov processes on $\boldsymbol{R}^d$ or $\boldsymbol{R}^d_+$. The advantage of our arguments is that they make the role played by the self-similarity index explicit. Our methods can also be applied to a class of elliptic diffusions which are not necessarily self-similar. There has been a lot of work on the Hausdorff dimension and exact Hausdorff measure of the random sets associated with Brownian motion, stable Lévy processes and Gaussian random fields, in which the methods often depend on special properties of the process that cannot be carried over to more general Markov processes. For surveys of recent developments on these and related topics, we refer to Taylor [21] and Xiao [27].

The rest of the paper is organized as follows. In Section 2 we recall briefly the definition of an α-s.s. Markov processes. In Section 3, we calculate the Hausdorff dimension of the image and graph of certain α-s.s. Markov processes including strictly stable Lévy processes. Our formula for the Hausdorff dimension of the graph seems new even for strictly stable Lévy processes. In Section 4, we study the zero set of these α-s.s. Markov processes. In Section 5, we study analogous problems and escape rates for a class of elliptic diffusions which are not necessarily self-similar.

We will use K to denote unspecific positive and finite constants whose value may be different in each appearance. Specific constants are denoted by $K_1, K_2, \dots$

2. Preliminaries. Throughout this paper, $(S, \mathscr{B})$ denotes $\boldsymbol{R}^d$, $\boldsymbol{R}^d\setminus\{0\}$ or $\boldsymbol{R}^d_+$ with the usual Borel σ-algebra, and Δ a point attached to S as an isolated point. Ω denotes the space of all functions ω from $[0, \infty)$ to $S\cup\{\Delta\}$ having the following properties:

(i) $\omega(t) = \Delta$ for $t \geqslant \tau$, where $\tau = \inf\{t \geqslant 0;\ \omega(t) = \Delta\}$;

(ii) ω is right continuous and has a left limit at every $t \in [0, \infty)$.

Let $\alpha > 0$ be a given constant. A stochastic process $X = (X(t), P^x)$ with state space $S\cup\{\Delta\}$ is called an *α-self-similar (s.s.) Markov process* if there exists a transition function $\boldsymbol{P}(t, x, A)$ which satisfies:

$$\boldsymbol{P}(0, x, A) = I_A(x) \quad \text{for all } x \in S,\ A \in \mathscr{B}, \tag{2.1}$$

and

$$\boldsymbol{P}(t, x, A) = \boldsymbol{P}(at, a^\alpha x, a^\alpha A) \quad \text{for all } t > 0,\ a > 0,\ x \in S,\ A \in \mathscr{B}, \tag{2.2}$$

such that $(X(t), P^x)$ is a time homogeneous Markov process with a transition function $\boldsymbol{P}(t, x, A)$ and for every $x \in S$, $X(t) \in \Omega$ P^x-almost surely. We will call α the *self-similarity index* of X.

For $d > 1$, X is called an *isotropic α-s.s. Markov process* if its transition function further satisfies the following condition:

$$\boldsymbol{P}(t, x, A) = \boldsymbol{P}(t, \phi(x), \phi(A)) \quad \text{for all } t \geqslant 0,\ x \in S,\ A \in \mathscr{B},\ \phi \in O(d), \tag{2.3}$$

where $O(d)$ denotes the group of orthogonal transformations on $\boldsymbol{R}^d$.

Remark. Condition (2.2) is equivalent to the statement that for every $a > 0$ the P^x-distribution of $X(t)$ $(t \geq 0)$ is equal to the $P^{a^\alpha x}$-distribution of $a^{-\alpha} X(at)$ $(t \geq 0)$. We write this self-similar property as

$$(2.4) \qquad \big(X(\cdot), P^x\big) \stackrel{\mathrm{d}}{=} \big(a^{-\alpha} X(a\cdot), P^{a^\alpha x}\big) \quad \text{for every } a > 0.$$

It is easy to see that all $(1/\alpha)$-strictly stable Lévy processes in $\boldsymbol{R}^d$ are α-s.s. Markov processes and $1/\alpha$ symmetric stable Lévy processes are isotropic a-s.s. Markov processes. It is proved by Graversen and Vuolle-Apiala [6] that if $X(t)$ is an isotropic α-s.s. Markov process on S, then $\big(|X(t)|, P^{|x|}\big)$ is an α-s.s. Markov process on $|S|$; and if $X(t)$ is an α-s.s. Markov process on $\boldsymbol{R}^d$, then for every $\gamma > 0$

$$\big(X(t)^{\langle\gamma\rangle}, P^{x^{\langle 1/\gamma\rangle}}\big)$$

is also an $(\alpha\gamma)$-s.s. Markov process on $\boldsymbol{R}^d$, where $0^{\langle\gamma\rangle} = 0$ and $x^{\langle\gamma\rangle} = x|x|^{\gamma-1}$ for $x \neq 0$.

The Bessel processes form exactly the class of (1/2)-s.s. diffusions on $(0, \infty)$. We refer to Revuz and Yor [15] for the definition and properties of Bessel processes.

More examples of α-s.s. Markov processes can be found in Lamperti [9], Graversen and Vuolle-Apiala [6], Stone [18], Vuolle-Apiala and Graversen [25], and Vuolle-Apiala [24].

Throughout this paper, we will only consider α-s.s. Markov processes with the strong Markov property. It was shown by Lamperti [9] and Graversen and Vuolle-Apiala [6] that every self-similar Markov process on $(0, \infty)$ and every isotropic self-similar Markov process on $\boldsymbol{R}^d \setminus \{0\}$ is automatically a strong Markov process with respect to a right-continuous filter of σ-algebras.

We end this section by recalling briefly the definition of Hausdorff measure and Hausdorff dimension. Let Φ be the class of functions ϕ: $(0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leqslant K \quad \text{for } 0 < s < \tfrac{1}{2}\delta.$$

For $\phi \in \Phi$, the *ϕ-Hausdorff measure* of $E \subseteq \boldsymbol{R}^N$ is defined by

$$\phi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \Big\{\sum_i \phi(2r_i):\ E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon\Big\},$$

where $B(x, r)$ denotes the open (or closed) ball of radius r centered at x. It is known that ϕ-m is a metric outer measure and every Borel set in $\boldsymbol{R}^N$ is ϕ-m measurable. The *Hausdorff dimension* of E is defined by

$$\dim E = \inf\{\alpha > 0:\ s^\alpha\text{-}m(E) = 0\}.$$

We refer to Falconer [5] for more properties of Hausdorff measure and Hausdorff dimension.

3. Hausdorff dimension of the image and graph. Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process on S. For the simplicity of notation, we will assume $S = \boldsymbol{R}^d$. For any Borel set $F \subseteq [0, \infty)$, we consider the Hausdorff dimension of the image $X(F)$ and the graph set $\operatorname{Gr} X(F) = \{(t, X(t)):\ t \in F\}$. We first note that it is impossible to have general formulae in terms of α and d for the Hausdorff dimension of the image and graph of all α-s.s. Markov processes, because for any $\gamma > 0$ the Hausdorff dimension of the image and graph of $X(t)^{\langle\gamma\rangle}$ are the same as those of $X(t)$. So later we will restrict our attention to a certain class for α-s.s. Markov processes.

Let $K_1 > 0$ be a fixed constant. Following Pruitt and Taylor [14], a collection $\Lambda(a)$ of cubes of side a in $\boldsymbol{R}^d$ is called *K_1-nested* if no ball of radius a in $\boldsymbol{R}^d$ can intersect more than K_1 cubes of $\Lambda(a)$. Clearly, for each integer $n \geqslant 1$ the collection of dyadic (semi-dyadic) cubes of order n in $\boldsymbol{R}^d$ is K_1-nested with $K_1 = 3^d$.

We first prove the following lemma which generalizes Lemma 6.1 of Pruitt and Taylor [14] and is essential for our purpose.

LEMMA 3.1. *Let $X(t)$ $(t \geqslant 0)$ be a time homogeneous strong Markov process in $\boldsymbol{R}^d$ with transition function $\boldsymbol{P}(t, x, A)$ and let $\Lambda(a)$ be a fixed K_1-nested collection of cubes of side a $(a \leqslant 1)$ in $\boldsymbol{R}^d$. For any $u \geqslant 0$ we denote by $M_u(a, s)$ the number of cubes in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u+s]$. Then*

$$E^0\big(M_u(a, s)\big) \leqslant 2K_1 s \Big[\inf_{x \in \boldsymbol{R}^d} E^x\Big(\int_0^s 1_{B(x,a/3)}\big(X(t)\big)\, dt\Big)\Big]^{-1}, \tag{3.1}$$

where 1_B is the indicator function of the set B.

Proof. Let $\tau_0 = u$. For $k \geqslant 1$, we define stopping times

$$\tau_k = \inf\{t \geqslant \tau_{k-1}:\ |X(t) - X(\tau_j)| > a \text{ for } j = 0, 1, \ldots, k-1\}.$$

Then $|X(\tau_k) - X(\tau_j)| \geqslant a$ for $k \neq j$, and hence the balls $B\big(X(\tau_k), a/3\big)$ $(k \geqslant 0)$ are disjoint. Let

$$T_k = \int_{\tau_k}^{\tau_k+s} 1_{B(X(\tau_k),a/3)}\big(X(t)\big)\, dt$$

be the sojourn time of $X(t)$ in $B\big(X(\tau_k), a/3\big)$ between τ_k and $\tau_k + s$. Put $\eta = \min\{k:\ \tau_k > u+s\}$. Then

$$X([u, u+s]) \subseteq \bigcup_{k=0}^{\eta-1} B\big(X(\tau_k), a\big). \tag{3.2}$$

Let I_k be the indicator of the event

$$\{\eta - 1 \geqslant k\} = \{\tau_k \leqslant u+s\}.$$

Then, by the strong Markov property, we have

$$\begin{aligned} E^0(I_k T_k) &= E^0\Big(I_{\{\tau_k \leqslant u+s\}} E^{X(\tau_k)}\Big(\int_0^s 1_{\{|X(t)-X(\tau_k)|<a/3\}}\, dt\Big)\Big) \\ &\geqslant E^0(I_k) \cdot \inf_{x \in \boldsymbol{R}^d} E^x\Big(\int_0^s 1_{B(x,a/3)}\big(X(t)\big)\, dt\Big). \end{aligned} \tag{3.3}$$

Noticing that

$$\sum_{k=0}^{\infty} I_k T_k \leqslant 2s \quad \text{and} \quad \eta = \sum_{k=0}^{\infty} I_k$$

and using (3.3), we have

$$E^0(\eta)\cdot \inf_{x\in \mathbf{R}^d} E^x\Big(\int_0^s 1_{B(x,a/3)}(X(t))\,dt\Big) \leqslant E^0\Big(\sum_{k=0}^{\infty} I_k T_k\Big) \leqslant 2s. \tag{3.4}$$

Now it is clear that (3.1) follows from (3.2), (3.4) and the *nested* property of $\Lambda(a)$.

We will consider the class of all α-s.s. Markov processes on S satisfying the following two conditions: there exist positive constants K_2, K_3 and $r_0 \in (0, 1/3)$ such that for every $x \in S$ we have

$$\mathbf{P}(1, x, B(x, r_0)) \geqslant K_2; \tag{3.5}$$

and for every $x \in S$ and $r \geqslant 0$ small

$$\mathbf{P}(1, x, B(x, r)) \leqslant K_3 \min\{1, r^d\}. \tag{3.6}$$

We will see that the upper bounds for $\dim X(F)$ and $\dim\mathrm{Gr}\, X(F)$ depend only on (3.5), while the lower bounds depend only on (3.6).

Clearly, condition (3.6) is satisfied by every self-similar Markov process $X(t)$ with bounded density for the P^x-distribution of $X(1)$. Conditions (3.5) and (3.6) are satisfied by a strictly stable Lévy process $X(t)$ on $\mathbf{R}^d$, because its transition function is translation invariant and $X(1)$ has a bounded density function. It is easy to verify that if $X(t)$ is a symmetric stable Lévy process on $\mathbf{R}^d$, then $|X(t)|$ (as a self-similar Markov process on $\mathbf{R}_+$) satisfies (3.5) and (3.6). It should be noticed that the transition function of $|X(t)|$ is not translation invariant. Condition (3.6) also holds for a Bessel process of dimension δ (not necessarily an integer) with $\delta \geqslant 1$. But it is not clear whether (3.5) is true for Bessel processes with non-integer δ. We conjecture that if $X(t)$ $(t \geqslant 0)$ is a Bessel process with non-integer dimension δ, then for every Borel set $F \subseteq [0, \infty)$ almost surely

$$\dim X(F) = \min\{1, 2\dim F\}.$$

Applying Lemma 3.1 to the α-s.s. Markov processes satisfying (3.5), we obtain the following lemma:

LEMMA 3.2. *Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ with transition function $\mathbf{P}(t, x, A)$ satisfying* (3.5). *Assume that $\Lambda(a)$ and $M_u(a, s)$ are as in Lemma* 3.1. *Then there exists a positive constant K such that for all $u \geqslant 0$ and all $0 < a \leqslant s^\alpha$*

$$E^0(M_u(a, s)) \leqslant K s a^{-1/\alpha}. \tag{3.7}$$

If $\alpha > 1$, then for $a = s \leqslant 1$ we have

$$E^0(M_u(a, s)) \leqslant K. \tag{3.8}$$

Proof. For any $x \in \mathbf{R}^d$, by Fubini's theorem, (2.2) and (3.5), we have

$$(3.9)\qquad E^x\Big(\int_0^s 1_{B(x,a/3)}\big(X(t)\big)\,dt\Big) = \int_0^s P\big(t,\, x,\, B(x,\, a/3)\big)\,dt$$

$$= \int_0^s P\big(1,\, x/t^\alpha,\, B\big(x/t^\alpha,\, a/(3t^\alpha)\big)\big)\,dt \geqslant \int_0^{a^{1/\alpha}} P\big(1,\, x/t^\alpha,\, B(x/t^\alpha,\, r_0)\big)\,dt \geqslant K_2\, a^{1/\alpha}.$$

Hence (3.7) follows from (3.9) and Lemma 3.1 with $K = 2K_1 K_2^{-1}$. The proof of (3.8) is similar and we omit it.

Now we can calculate the Hausdorff dimension of the image and graph of certain self-similar Markov processes.

THEOREM 3.1. *Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ with transition function $P(t, x, A)$ satisfying* (3.5) *and* (3.6). *Then for every Borel set $F \subseteq [0, \infty)$ P^0-almost surely*

$$(3.10)\qquad \dim X(F) = \min\left\{d,\, \frac{1}{\alpha}\dim F\right\}.$$

If $0 < \alpha \leqslant 1$, then P^0-almost surely

$$(3.11)\qquad \dim \mathrm{Gr}\, X(F) = \min\left\{\dim F + (1-\alpha)d,\, \frac{1}{\alpha}\dim F\right\},$$

and if $\alpha > 1$, then P^0-almost surely

$$(3.12)\qquad \dim \mathrm{Gr}\, X(F) = \dim F.$$

Proof. We start by proving the upper bounds in (3.10). Clearly, $\dim X(F) \leqslant d$. For any fixed $\gamma > \dim F$ and every integer $m \geqslant 1$ there exists a sequence of intervals I_{im} $(i = 1, 2, \ldots)$ such that

$$(3.13)\qquad F \subseteq \bigcup_{i=1}^{\infty} I_{im} \quad \text{and} \quad \sum_{i=1}^{\infty} |I_{im}|^\gamma \leqslant \frac{1}{m}.$$

Take $a = |I_{im}|^\alpha$ in Lemma 3.2; then we have

$$(3.14)\qquad E^0(M_{im}) \leqslant K,$$

where M_{im} denotes the number of cubes in $\Lambda(|I_{im}|^\alpha)$ that intersect $X(I_{im})$. Now by (3.13) and (3.14) we have

$$X(F) \subseteq \bigcup_{i=1}^{\infty} X(I_{im})$$

and each $X(I_{im})$ can be covered by M_{im} cubes of side $|I_{im}|^\alpha$, and

$$E^0\Big(\sum_{i=1}^{\infty} M_{im}\big(|I_{im}|^\alpha\big)^{\gamma/\alpha}\Big) \leqslant K \sum_{i=1}^{\infty} |I_{im}|^\gamma \leqslant \frac{K}{m}.$$

It follows that

$$E^0\big(s^{\gamma/\alpha}\text{-}m\big(X(F)\big)\big) = 0,$$

which implies $s^{\gamma/\alpha}\text{-}m(X(F))=0$ P^0-almost surely. Hence $\dim X(F)\leqslant\gamma/\alpha$ P^0-almost surely. Finally, letting $\gamma\to\dim F$ along a sequence, we obtain the upper bound in (3.10).

The proof of the upper bound in (3.11) is similar. If we cover $\operatorname{Gr} X(F)$ by cubes in $\boldsymbol{R}_+\times\boldsymbol{R}^d$ of side $|I_{im}|^\alpha$, we will obtain

$$\dim\operatorname{Gr} X(F)\leqslant\frac{1}{\alpha}\dim F.$$

Since, for each i, $\operatorname{Gr} X(I_{im})$ can also be covered by $KM_{im}|I_{im}|^{(\alpha-1)d}$ cubes in $\boldsymbol{R}_+\times\boldsymbol{R}^d$ of side $|I_{im}|$, we have $\dim\operatorname{Gr} X(F)\leqslant\dim F+(1-\alpha)d$. The upper bound in (3.12) follows from (3.8) and by covering $\operatorname{Gr} X(F)$ by cubes of side $|I_{im}|$.

To prove the lower bounds in (3.10), we will use the usual capacity argument. See, e.g., Kahane [7]. If $\dim F=0$, there is nothing to prove. So we assume $\dim F>0$. For any fixed $0<\gamma<\min\{d,(\dim F)/\alpha\}$, there exists a positive measure σ on F with

$$\int_F\int_F\frac{\sigma(ds)\,\sigma(dt)}{|s-t|^{\alpha\gamma}}<\infty. \tag{3.15}$$

Standard capacity arguments imply that if

$$\int_F\int_F E^0\left(|X(s)-X(t)|^{-\gamma}\right)\sigma(ds)\,\sigma(dt)<\infty, \tag{3.16}$$

then $\dim X(F)\geqslant\gamma$ P^0-almost surely. Since $\gamma<\min\{d,(\dim F)/\alpha\}$ is arbitrary, this will prove the lower bound in (3.10). For every pair (s,t) with $s\leqslant t$, it follows from (2.2) and by change of variables that

$$\begin{aligned}E^0\left(|X(s)-X(t)|^{-\gamma}\right)&=\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}|x-y|^{-\gamma}P(s,0,dx)\,P(t-s,x,dy)\\&=|s-t|^{-\alpha\gamma}\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}|u-v|^{-\gamma}P(s,0,|s-t|^\alpha du)\,P(1,u,dv)\\&\leqslant|s-t|^{-\alpha\gamma}\sup_{u\in\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}|u-v|^{-\gamma}P(1,u,dv).\end{aligned} \tag{3.17}$$

Now it is clear that (3.16) follows from (3.15), (3.17) and the inequality

$$\sup_{u\in\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}|u-v|^{-\gamma}P(1,u,dv)\leqslant K_\gamma, \tag{3.18}$$

where K_γ is a positive finite constant. To prove (3.18), we fix $u\in\boldsymbol{R}^d$ and consider the image measure μ of $P(1,u,\cdot)$ under the mapping $T\colon v\to|v-u|$ from $\boldsymbol{R}^d$ to $\boldsymbol{R}_+$. Then by using a change of variable we have

$$\int_{\boldsymbol{R}^d}|u-v|^{-\gamma}P(1,u,dv)=\int_0^\infty\varrho^{-\gamma}\mu(d\varrho)=\gamma\int_0^\infty\varrho^{-\gamma-1}\mu(\varrho)\,d\varrho. \tag{3.19}$$

Since $\gamma<d$, (3.18) follows immediately from (3.6) and (3.19).

To prove the lower bound in (3.11), we notice that if $\dim F \leqslant \alpha d$, then $\dim \operatorname{Gr} X(F) = \dim X(F) = (\dim F)/\alpha$. So we only need to consider the case $\dim F > \alpha d$. Using an argument similar to the above, we can prove that, for every γ with $d < \gamma < \dim F + (1-\alpha) d$ and every pair $(s, t) \in F \times F$ with $s \leqslant t$,

$$E^0\left[(|s-t|^2 + |X(s)-X(t)|^2)^{-\gamma/2}\right]$$
$$= \int_{R^d}\int_{R^d} (|s-t|^2 + |x-y|^2)^{-\gamma/2} P(s, 0, dx) P(t-s, x, dy) \leqslant K|s-t|^{-\gamma+(1-\alpha)d}.$$

Therefore, there is a positive measure σ on F such that

$$\int_F\int_F E^0\left[(|s-t|^2 + |X(s)-X(t)|^2)^{-\gamma/2}\right] \sigma(ds)\sigma(dt) < \infty.$$

This proves the lower bound in (3.11). Finally, since we always have $\dim \operatorname{Gr} X(F) \geqslant \dim F$, the equality (3.12) holds.

Remark. It is clear that (3.1), (3.7) and (3.8) are still true if we replace the expectation E^0 by E^x for all $x \in S$. The proof of Theorem 3.1 can be slightly modified to show that (3.10)–(3.12) hold P^x-almost surely for all $x \in S$.

4. Hausdorff dimension of the zero set. Let $X(t)$ $(t \in \boldsymbol{R}_+)$ be an α-s.s. Markov process on S, where S is $\boldsymbol{R}^d$ or $\boldsymbol{R}^d_+$. In this section we consider the Hausdorff dimension of the zero set $X^{-1}(0) = \{t \geqslant 0 \colon X(t) = 0\}$. We notice that the size of the zero set can be totally different from the size of other level sets $X^{-1}(x)$ $(x \in S\backslash\{0\})$. This is clear from the following example. Let $X(t) = |Y(t)|$, where $Y(t)$ is a Brownian motion in $\boldsymbol{R}^2$; then $\dim X^{-1}(0) = 0$, but for every $x > 0$ almost surely $\dim X^{-1}(x) = 1/2$ (see, e.g., Testard [22]). We will only consider α-s.s. Markov processes on S with transition functions satisfying the following condition: there exist positive constants β, K_4 and K_5 such that for every $r \geqslant 0$ and $x \in S$ with $|x| \leqslant r$

$$K_4 \min\{1, r^\beta\} \leqslant P(1, x, B(0, r)) \leqslant K_5 \min\{1, r^\beta\}. \tag{4.1}$$

It is easy to verify that (4.1) is satisfied by strictly stable Lévy processes $X(t)$ with $\beta = d$; by $X(t)^{\langle\gamma\rangle}$, where $X(t)$ is a strictly stable Lévy process, with $\beta = d/\gamma$ and by a Bessel process of dimension δ (not necessarily an integer) with $\beta = \delta$.

We need the following lemma which is proved in Xiao [28].

LEMMA 4.1. *Let $X(t)$ $(t \geqslant 0)$ be a time homogeneous strong Markov process on S with transition function $P(t, x, A)$. Then for every $c > b > 0$ and $r > 0$ we have*

$$\frac{1}{2}\frac{\int_b^c P(t, 0, B(0, r))\,dt}{\sup_{|y|\leqslant r}\int_0^c P(t, y, B(0, r))\,dt} \leqslant P^0\left(|X(t)| \leqslant r \text{ for some } b \leqslant t \leqslant c\right) \tag{4.2}$$
$$\leqslant \frac{\int_b^{2c-b} P(t, 0, B(0, r))\,dt}{\inf_{|y|\leqslant r}\int_0^{c-b} P(t, y, B(0, r))\,dt}.$$

As a consequence, we have

LEMMA 4.2. *Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process on S with transition function $P(t, x, A)$ satisfying* (4.1). *Then for every $\varepsilon > 0$, $c > b \geqslant \varepsilon$ and for $r > 0$ small we have*

$$P^0\big(|X(t)| \leqslant r \text{ for some } b \leqslant t \leqslant c\big) \leqslant \begin{cases} Kr^{\beta-1/\alpha} & \text{if } \alpha\beta > 1, \\ K\log 1/r & \text{if } \alpha\beta = 1, \end{cases} \tag{4.3}$$

and if $\alpha\beta < 1$, then

$$K_6 \frac{c-b}{c} \leqslant P^0\big(|X(t)| \leqslant r \text{ for some } b \leqslant t \leqslant c\big) \leqslant K_7 (c-b)^{\alpha\beta}, \tag{4.4}$$

where K_6 and K_7 are positive and finite constants depending on α, β and ε only.

Proof. The two inequalities in (4.3) have been proved in Xiao [28]. To prove the upper bound in (4.4), we see that by (2.2) and (4.1), for any $y \in \boldsymbol{R}^d$ with $|y| \leqslant r$,

$$\begin{aligned} \int_0^{c-b} P\big(t, y, B(0, r)\big)\, dt &= \int_0^{c-b} P\big(1, y/t^\alpha, B(0, r/t^\alpha)\big)\, dt \\ &\geqslant K_4 \int_0^{c-b} \min\left(1, \frac{r^\beta}{t^{\alpha\beta}}\right) dt \geqslant Kr^\beta (c-b)^{1-\alpha\beta}, \end{aligned} \tag{4.5}$$

where $K > 0$ is a constant depending on α and β only. By the second inequality in (4.1) we have

$$\int_b^{2c-b} P\big(t, 0, B(0, r)\big)\, dt \leqslant \frac{K_5 r^\beta}{1-\alpha\beta}\big[(2c-b)^{1-\alpha\beta} - b^{1-\alpha\beta}\big] \leqslant Kr^\beta (c-b), \tag{4.6}$$

where $K > 0$ is a finite constant depending on α, β and ε only. The upper bound in (4.4) follows from Lemma 4.1, (4.5) and (4.6). The lower bound in (4.4) is proved similarly.

Now we are ready to prove the main theorem in this section.

THEOREM 4.1. *Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process on S with transition function $P(t, x, A)$ satisfying* (4.1). *Then P^0-almost surely*

$$\dim X^{-1}(0) \leqslant \max\{0, 1-\alpha\beta\}. \tag{4.7}$$

If $\alpha\beta < 1$, then for every interval $T = [0, b]$ with positive P^0-probability

$$\dim X^{-1}(0) \cap T = 1-\alpha\beta. \tag{4.8}$$

Proof. We prove (4.7) first. By the σ-stability of Hausdorff dimension, it is sufficient to prove that for every interval $[\varepsilon, M] \subseteq \boldsymbol{R}_+$ P^0-almost surely

$$\dim X^{-1}(0) \cap [\varepsilon, M] \leqslant \max\{0, 1-\alpha\beta\}. \tag{4.9}$$

We only prove (4.7) for the case $1 > \alpha\beta$ and the other cases can be proved similarly by using (4.3). For any integer $n \geqslant 1$, we divide $[\varepsilon, M]$ into n subintervals $I_{n,i}$ $(1 \leqslant i \leqslant n)$ of length $(\varepsilon - M)/n$. The collection of those subintervals $I_{n,i}$ satisfying $\inf_{t \in I_{n,i}} |X(t)| = 0$ constitutes a covering for $X^{-1}(0) \cap [\varepsilon, M]$. Since

$$E^0\left(s^{1-\alpha\beta}\text{-}m\left(X^{-1}(0) \cap [\varepsilon, M]\right)\right) \leqslant \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{M-\varepsilon}{n}\right)^{1-\alpha\beta} P^0\left(\inf_{t \in I_{n,i}} |X(t)| = 0\right) \leqslant K$$

by (4.4), this implies (4.9), and hence (4.7).

In order to prove the lower bound for $\dim X^{-1}(0) \cap T$, it is sufficient to show that for every $0 < \gamma < 1 - \alpha\beta$ we can construct a positive measure μ on $X^{-1}(0) \cap T$ such that

$$\int_T \int_T \frac{\mu(ds)\,\mu(dt)}{|s-t|^\gamma} < \infty, \tag{4.10}$$

which implies $\dim X^{-1}(0) \cap T \geqslant \gamma$ on $\{\mu > 0\}$. This capacity argument, which goes back to the early work of Kahane (see [7]), has been applied by many authors including Adler [1], Marcus [12], Shieh [16], Testard [22], Xiao [26], to cite a few, to study the existence and Hausdorff dimension of the level sets and multiple points.

Let $\mathcal{M}_\gamma^+$ be the space of all non-negative measures on $\boldsymbol{R}_+$ with finite γ-energy. It is known (see, e.g., Adler [1]) that $\mathcal{M}_\gamma^+$ is a complete metric space under the metric

$$\|\mu\|_\gamma = \int_{\boldsymbol{R}_+} \int_{\boldsymbol{R}_+} \frac{\mu(ds)\,\mu(dt)}{|s-t|^\gamma}.$$

For every $n \geqslant 1$, we define a random positive measure μ_n on $\mathcal{B}(\boldsymbol{R}_+)$ by

$$\mu_n(B, \omega) = \frac{1}{c_\beta n^{-\beta}} \int_{T \cap B} 1_{\{|X(t)| \leqslant 1/n\}}\, dt, \tag{4.11}$$

where c_β is a normalizing constant.

By a lemma of Testard [22], which simplifies the arguments of Kahane [7] and Marcus [12], if there are constants $K_8 > 0$ and $K_9 > 0$ such that

$$E^0(\|\mu_n\|) \geqslant K_8, \quad E(\|\mu_n\|^2) \leqslant K_9, \tag{4.12}$$

$$E^0(\|\mu_n\|_\gamma) < +\infty, \tag{4.13}$$

where $\|\mu_n\| = \mu_n(T)$, then there is a subsequence of $\{\mu_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu$ in $\mathcal{M}_\gamma^+$, and μ is strictly positive with probability at least $K_8^2/(2K_9)$. If $X(t)$ has continuous sample paths, then μ is supported on $X^{-1}(0) \cap T$ (cf. Marcus [12], p. 282). As noted by Shieh ([16], p. 557), this is also true if the sample path $X(\cdot, \omega)$ is right continuous and has left limit. This will imply (4.8).

Now we verify (4.12) and (4.13). It follows from (2.2) and (4.1) that

$$(4.14)\qquad E^0(\|\mu_n\|) = \frac{1}{c_\beta n^{-\beta}} \int_T P^0(|X(t)| \leqslant 1/n)\,dt$$

$$= \frac{1}{c_\beta n^{-\beta}} \int_T P(t, 0, B(0, 1/n))\,dt \geqslant Kn^\beta \int_T \min\left\{1, \left(\frac{1}{t^\alpha n}\right)^\beta\right\} dt \geqslant K_8$$

at least for $n \geqslant n_0$, where K_8 and n_0 depend on α, d and T only. We have

$$(4.15)\qquad E^0(\|\mu_n\|^2) = \frac{1}{c_\beta^2 n^{-2\beta}} \int_T \int_T P^0(|X(s)| \leqslant 1/n, |X(t)| \leqslant 1/n)\,ds\,dt.$$

For every fixed pair $(s, t) \in T \times T$ with $s \leqslant t$, it follows from (2.2) and the second inequality in (4.1) that

$$(4.16)\qquad P^0(|X(s)| \leqslant 1/n, |X(t)| \leqslant 1/n)$$

$$= \int_{B(0,1/n)} P(s, 0, du)\,P(t-s, u, B(0, 1/n))$$

$$\leqslant K_5^2 \min\left\{1, \left(\frac{1}{s^\alpha n}\right)^\beta\right\} \cdot \min\left\{1, \left(\frac{1}{|t-s|^\alpha n}\right)^\beta\right\}.$$

Putting (4.16) into (4.15) we see that

$$E^0(\|\mu_n\|^2) \leqslant K_9,$$

where K_9 depends on α, d and T only. Similarly, for every $0 < \gamma < 1-\alpha\beta$ we have

$$E^0(\|\mu_n\|_\gamma) \leqslant K \int_0^b \int_s^b \frac{1}{s^{\alpha\beta}} \cdot \frac{1}{|t-s|^{\alpha\beta+\gamma}}\,dt\,ds < \infty.$$

Hence we have proved that with P^0-probability at least $K_8^2/(2K_9)$ (independent of γ)

$$\dim X^{-1}(0) \cap T \geqslant 1-\alpha\beta.$$

This proves (4.8).

With one more assumption, which is satisfied by a large class of self-similar Markov processes including strictly stable Lévy processes and the self-similar Markov processes considered by Stone [18], we can prove the following probability 1 result.

THEOREM 4.2. *Let $X(t)$ $(t \geqslant 0)$ be an α-s.s. Markov process on S with transition function $P(t, x, A)$ satisfying (4.1) with $\alpha\beta < 1$. If we further assume that*

$$(4.17)\qquad P^0(\sup\{t: X(t) = 0\} = \infty) = 1,$$

then P^0-almost surely

$$(4.18)\qquad \dim X^{-1}(0) = 1-\alpha\beta.$$

Proof. We define a sequence of stopping times as follows. Let $\tau_0 = 0$ and for $n \geqslant 1$ define

$$\tau_n = \inf\{t \geqslant \tau_{n-1}+1:\ X(t) = 0\}.$$

Then, by (4.17), $\{\tau_n\}$ is well defined and, for every n, P^0-almost surely $X(\tau_n) = 0$. By the strong Markov property and (4.8), for every $n \geqslant 1$

$$\begin{aligned} P^0\big(\dim X^{-1}(0)\cap[0,\, \tau_{n+1}] \geqslant 1-\alpha\beta \mid \sigma\big(X(s),\, 0 \leqslant s \leqslant \tau_n\big)\big) \\ \geqslant P^{X(\tau_n)}\big(\dim X^{-1}(x)\cap[0,\, 1] \geqslant 1-\alpha\beta\big) \geqslant K_8^2/(2K_9). \end{aligned}$$

Therefore, by the conditional Borel–Cantelli lemma (see, e.g., Neveu [13]), we have

$$P^0\big(\dim X^{-1}(0)\cap[0,\, \tau_n] \geqslant 1-\alpha d \text{ for infinitely many } n\big) = 1.$$

This completes the proof of Theorem 4.2.

Remark. With a little more effort, we can apply the methods of this paper, combined with the methods in Pruitt and Taylor [14] and Xiao [26], to prove similar results for the Hausdorff dimension of the image, graph and zero set of the sample paths of the process $X(t)$ $(t \geqslant 0)$ on $\boldsymbol{R}^d$ defined by

$$X(t) = \big(X_1(t),\, \ldots,\, X_d(t)\big),$$

where $X_1, \ldots, X_d$ are independent α_i-s.s. Markov processes $(i = 1, \ldots, d)$.

5. Applications to elliptic diffusion processes. In this section, we apply the methods in Sections 2 and 3 and in Xiao [28] to study the Hausdorff dimension of the image, graph and level sets, and the escape rates for a class of elliptic diffusion processes which are not necessarily self-similar.

For any given number $\lambda \in (0, 1]$, let $\mathscr{A}(\lambda)$ denote the class of all measurable, symmetric matrix-valued functions a: $\boldsymbol{R}^d \to \boldsymbol{R}^d \otimes \boldsymbol{R}^d$ which satisfy the ellipticity condition

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leqslant \frac{1}{\lambda}|\xi|^2 \quad \text{for all } x,\, \xi \in \boldsymbol{R}^d.$$

For each $a \in \mathscr{A}(\lambda)$, let $L = \nabla\cdot(a\nabla)$ be the corresponding second order partial differential operator. By Theorem II.3.1 of Stroock [19], we know that L is the infinitesimal generator of a d-dimensional diffusion process $X = \big(X(t),\, t \geqslant 0\big)$, which is strongly Feller continuous. Moreover, its transition density function $p(t, x, y) \in C\big((0, \infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d\big)$ satisfies the following inequality:

$$\frac{1}{Mt^{d/2}}\exp\left(-\frac{M|y-x|^2}{t}\right) \leqslant p(t, x, y) \leqslant \frac{M}{t^{d/2}}\exp\left(-\frac{|y-x|^2}{Mt}\right) \tag{5.1}$$

for all $(t, x, y) \in (0, \infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $M = M(\alpha, d) \geqslant 1$ is a constant. The estimate in (5.1) is due to Aronson [2].

Even though $X(t)$ $(t \geqslant 0)$ is, in general, not self-similar, the estimate (5.1) makes it possible for us to apply the arguments in Sections 3 and 4, as well as those in Xiao [28] to prove results on the Hausdorff dimension of the image, graph, level sets and escape rates for such X. It turns out that these results are very similar to those of d-dimensional Brownian motion.

THEOREM 5.1. *Let $X(t)$ $(t \geqslant 0)$ be an elliptic diffusion process in $\boldsymbol{R}^d$ as given above. Then for every Borel set $F \subseteq [0, \infty)$ P^0-almost surely*

$$\dim X(F) = \min\{d, 2\dim F\}, \tag{5.2}$$

$$\dim \operatorname{Gr} X(F) = \min\{\dim F + d/2, 2\dim F\}. \tag{5.3}$$

If $d = 1$, then for every $x \in \boldsymbol{R}$ and $T = [a, b] \subseteq \boldsymbol{R}_+$, with positive P^0-probability

$$\dim X^{-1}(x) \cap T = \tfrac{1}{2}. \tag{5.4}$$

Proof. It is easy to verify that Lemma 3.2 and a variant of Lemma 4.2 (with $|X(t)|$ replaced by $|X(t)-x|$ in (4.3) and (4.4)) still hold with $\beta = d$ and $\alpha = 1/2$. Hence the right-hand sides of (5.2), (5.3) and (5.4) serve as the upper bounds for $\dim X(F)$, $\dim \operatorname{Gr} X(F)$ and $\dim X^{-1}(x)$, respectively. The proof of the lower bounds in (5.2) and (5.3) is almost the same as that of Theorem 3.1. The proof of the lower bound in (5.4) is similar to that of Theorem 4.1, we only need to modify the definition of the random measure $\mu_n(\cdot)$ in (4.11), and then apply (5.1) to prove inequalities in (4.12) and (4.13).

Using (5.1) and Lemma 3.1 in Xiao [28], and going through the proof of Theorem 3.1 in Xiao [28], we obtain the following results on the escape rates for X. Similar results for Brownian motion were proved by Dvoretsky and Erdös [4] and Spitzer [17]. See Xiao [28] for more information on escape rates for other processes.

THEOREM 5.2. *Let $X(t)$ $(t \geqslant 0)$ be a diffusion process in $\boldsymbol{R}^d$ as given above. If $d = 2$, then singletons are polar, but $(X(t), P^0)$ is neighborhood recurrent. If $d \geqslant 3$, then $X(t)$ is transient in the sense that for every $x \in \boldsymbol{R}^d$*

$$P^x\big(|X(t)| \to \infty \text{ as } t \to \infty\big) = 1.$$

For any positive non-increasing function $\phi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$, set

$$\mathscr{I}(\phi) = \begin{cases} \int_1^\infty t^{-1}\phi(t)^{d-2}\,dt & \text{if } d \geqslant 3, \\ \int_1^\infty \big(t\,|\log\phi(t)|\big)^{-1}\,dt & \text{if } d = 2. \end{cases}$$

Then P^0-almost surely

$$\liminf_{t\to\infty} \frac{|X(t)|}{t^{1/2}\phi(t)} = \begin{cases} \infty & \text{if } \mathscr{I}(\phi) < \infty, \\ 0 & \text{if } \mathscr{I}(\phi) = \infty. \end{cases}$$

We end this section with the following problem. By using the results in Sznitman [20], Chaleyat-Maurel and LeGall [3] have obtained some partial results for the Hausdorff measure of the image of elliptic diffusion processes. But the problem of finding exact Hausdorff measure of the image and graph remains open. We believe that the exact Hausdorff measure functions for the image and graph of $X(t)$ considered above are the same as those of Brownian motion. See Taylor [21] for the related results on Brownian motion. In order to prove this, some general Markov arguments have to be developed.

REFERENCES

[1] R. J. Adler, *The Geometry of Random Fields*, Wiley, New York 1981.

[2] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. 73 (1967), pp. 890–896.

[3] M. Chaleyat-Maurel and J.-F. LeGall, *Green functions, capacity and sample path properties for a class of hypoelliptic diffusion processes*, Probab. Theory Related Fields 83 (1989), pp. 219–264.

[4] A. Dvoretsky and P. Erdös, *Some problems on random walk in space*, Proceedings of the Second Berkeley Symposium, Univ. of California Press, Berkeley 1950, pp. 353–367.

[5] K. J. Falconer, *Fractal Geometry — Mathematical Foundations and Applications*, Wiley, 1990.

[6] S. E. Graversen and J. Vuolle-Apiala, *α-self-similar Markov processes*, Probab. Theory Related Fields 71 (1986), pp. 149–158.

[7] J.-P. Kahane, *Some Random Series of Functions*, 2nd edition, Cambridge University Press, 1985.

[8] S. W. Kiu, *Semi-stable Markov processes in $\boldsymbol{R}^n$*, Stochastic Process. Appl. 10 (1980), pp. 183–191.

[9] J. Lamperti, *Semi-stable Markov processes*, Z. Wahrsch. Verw. Gebiete 22 (1972), pp. 205–225.

[10] Bingzhang Li, Lihu Huang and Luqin Liu, *The exact measure functions of the images for a class of self-similar processes*, Preprint, 1997.

[11] Luqin Liu, *The Hausdorff dimensions of the image, graph and level sets of a self-similar process*, in: *Stochastic and Quantum Mechanics*, World Scientific Press, 1992, pp. 178–188.

[12] M. B. Marcus, *Capacity of level sets of certain stochastic processes*, Z. Wahrsch. Verw. Gebiete 34 (1976), pp. 279–284.

[13] J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco 1965.

[14] W. E. Pruitt and S. J. Taylor, *Sample path properties of processes with stable components*, Z. Wahrsch. Verw. Gebiete 12 (1969), pp. 267–289.

[15] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 2nd edition, Springer, Berlin 1994.

[16] N.-R. Shieh, *Multiple points of sample paths of Markov processes*, Ann. Probab. 20 (1992), pp. 553–562.

[17] F. Spitzer, *Some theorems concerning 2-dimensional Brownian motion*, Trans. Amer. Math. Soc. 87 (1958), pp. 187–197.

[18] C. Stone, *The set of zeros of a semistable process*, Illinois J. Math. 7 (1963), pp. 631–660.

[19] D. W. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, *Séminaire de Probabilité XXII*, Lecture Notes in Math. 1321, Berlin 1988, pp. 316–347.

[20] A. S. Sznitman, *Some bounds and limiting results for the measure of Wiener sausage of small radius associated with elliptic diffusions*, Stochastic Process. Appl. 25 (1987), pp. 1–25.

[21] S. J. Taylor, *The measure theory of random fractals*, Math. Proc. Cambridge Philos. Soc. 100 (1986), pp. 383–406.

[22] F. Testard, *Polarité, points multiples et géométrie de certains processus gaussiens*, Thèse doctorat, Orsay 1987.

[23] J. Vuolle-Apiala, *Time-change of self-similar Markov processes*, Ann. Inst. H. Poincaré Probab. Statist. 25 (1989), pp. 581–587.
[24] — *Itô excursion theory for self-similar Markov processes*, Ann. Probab. 22 (1994), pp. 546–565.
[25] — and S. E. Graversen, *Duality theory for self-similar Markov processes*, Ann. Inst. H. Poincaré Probab. Statist. 22 (1988), pp. 376–392.
[26] Yimin Xiao, *Dimension results for Gaussian vector fields and index α-stable fields*, Ann. Probab. 23 (1995), pp. 273–291.
[27] — *Fractal measures of the sets associated to Gaussian random fields*, to appear in: *Trends in Probability and Related Analysis: Proceedings of the Symposium on Analysis and Probability 1996*, N. Kôno and N.-R. Shieh (eds.), World Scientific.
[28] — *Asymptotic results for self-similar Markov processes*, to appear in: *Asymptotic Methods in Probability and Statistics (ICAMPS'97)*, B. Szyszkowicz (ed.), Elsevier Science.
[29] — and Luqin Liu, *Packing dimension results for certain Lévy processes and self-similar Markov processes*, Chinese Ann. Math. 17A (1996), pp. 392–400.

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