PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 19, Fasc. 1 (1999), pp. 123–132

ON A UNIQUENESS PROPERTY OF α-SPHERICAL DISTRIBUTIONS

BY

JACEK WESOŁOWSKI* (WARSZAWA)

Abstract. A distribution of an α-spherical random vector is shown to be uniquely determined by a distribution of quotients.

1. Introduction. Let X and Y be independent, zero-mean random variables. If they are normal, then the quotient X/Y is distributed according to the symmetric Cauchy distribution and the converse statement is false. This observation was given for the first time in 1958 by Laha [6]. Since that time many efforts have been devoted to explain relations between distributions of random vectors $X = (X_1, \ldots, X_n)$ and quotients $(X_1/X_n, \ldots, X_{n-1}/X_n)$; see, for instance, Kotlarski [5], Seshadri [9], Letac [7], Wesołowski [12], and Szabłowski et al. [11].

Extending Seshadri's [9] characterization of the normal distribution by a Cauchy quotient X/Y (of independent r.v.'s) being independent of X^2+Y^2, Wesołowski [13] proved that the bivariate central elliptically contoured distribution is identified by the same two conditions in the class of symmetric distributions. This result, in turn, has been recently generalized to any multivariate case and α-spherically invariant distributions by considering a special form of so-called *α-Cauchy distribution* for the vector of quotients $(X_1/X_n, \ldots, X_{n-1}/X_n)$ in Szabłowski [10]. An extensive study of the family of α-spherically invariant distributions, called also *L_α-norm spherical distributions*, has been given recently in Gupta and Song [3].

In the present paper we discover that the basic property allowing to determine the distribution of X by the distribution of quotients is its sign-symmetry:

DEFINITION 1. A (distribution of a) random vector $X = (X_1, \ldots, X_n)$ is said to be *sign-symmetric* iff the distributions of the random vectors

$$\big((-1)^{\varepsilon_1} X_1, \ldots, (-1)^{\varepsilon_n} X_n\big)$$

coincide for any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$.

* Mathematical Institute, Warsaw University of Technology.

If $X \stackrel{d}{=} -X$, then we simply say that (the distribution of) X is *symmetric*.

This notion allows us to consider a family of distributions which is much wider than α-spherically invariant distributions.

DEFINITION 2. A random vector $X = (X_1, \ldots, X_n)$ is said to have an *α-spherical distribution* ($\alpha > 0$) if there exist a positive random variable R and a random vector U_α with a sign-symmetric distribution concentrated on the unit α-sphere $S_\alpha = \{x \in R^n : \|x\|_\alpha = 1\}$ (where $\|x\|_\alpha = (|x_1|^\alpha + \ldots + |x_n|^\alpha)^{1/\alpha}$ for any $x = (x_1, \ldots, x_n) \in R^n$) such that R and U_α are independent and

$$X \stackrel{d}{=} RU_\alpha.$$

If the last formula holds, we say that X has the *U_α-spherical distribution.* Observe that a sign-symmetric random vector X is α-spherical iff $X/\|X\|_\alpha$ and $\|X\|_\alpha$ are independent. It is obvious that spherically invariant or, more generally, α-spherically invariant distributions (for the definition see for instance Szabłowski [10]) are included in the class of α-spherical distributions. Concrete examples include α-uniform distributions (see Gupta and Song [2]), α-generalized normal distributions (see Gupta and Song [1]) or generalized Liouville distributions (see Gupta et al. [4]). A recent application of ideas developed in this paper to study properties of the generalized Liouville distribution can be found in Matysiak [8].

Some basic properties of α-spherical distributions are given in Section 2. In Section 3 it is shown that sign-symmetry of the distribution of the vector of quotients is essential for a random vector X to be α-spherical. We assume throughout the paper that writing a quotient X/Y means that the assumption $P(Y = 0) = 0$ is additionally imposed.

2. Sign-symmetry and independence for α-spherical distributions. In this section we derive some properties of sign-symmetry and independence for any random vector X with an α-spherical distribution. These properties will be used for a unique determination of α-spherical distributions in Section 3.

THEOREM 1. *If a random vector X has an α-spherical distribution, then:*

(i) *X is sign-symmetric;*

(ii) *$X/\|X\|_\alpha$ and $\|X\|_\alpha$ are independent;*

(iii) *$(X_1/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i)$ is sign-symmetric for any $i = 1, \ldots, n$.*

Proof. (i) Observe that for any $(a_1, \ldots, a_n) \in R^n$ by the total probability rule

$$P(X_1 \leqslant a_1, \ldots, X_n \leqslant a_n) = \int_R P(U_1 \leqslant a_1/r, \ldots, U_n \leqslant a_n/r)\, dF_R(r),$$

where, by the definition, $X \stackrel{d}{=} RU_\alpha$, and $U_\alpha = (U_1, \ldots, U_n)$ and F_R denotes the distribution function of R. Since U_α is sign-symmetric, we have

$$P(U_1 \leqslant a_1/r, \ldots, U_n \leqslant a_n/r) = P\big((-1)^{\varepsilon_1} U_1 \leqslant a_1/r, \ldots, (-1)^{\varepsilon_n} U_n \leqslant a_n/r\big)$$

for any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$. Consequently, applying the total probability rule again but in the converse direction, we obtain (i).

(ii) Again using the representation $X \stackrel{d}{=} RU_\alpha$ it is easily seen that

$$\left(\frac{X}{\|X\|_\alpha}, \|X\|_\alpha\right) \stackrel{d}{=} (U_\alpha, R),$$

which are independent by the definition.

(iii) Without loosing generality we can consider $i = n$. Since, by (i), X is sign-symmetric, we obtain for any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$

$$\left(\frac{X_1}{X_n}, \ldots, \frac{X_{n-1}}{X_n}\right) = \left(\frac{(-1)^{\varepsilon_1} X_1}{(-1)^{\varepsilon_n} X_n}, \ldots, \frac{(-1)^{\varepsilon_{n-1}} X_{n-1}}{(-1)^{\varepsilon_n} X_n}\right)$$
$$= \left((-1)^{\varepsilon_1 - \varepsilon_n} \frac{X_1}{X_n}, \ldots, (-1)^{\varepsilon_{n-1} - \varepsilon_n} \frac{X_{n-1}}{X_n}\right).$$

Since for any $(\delta_1, \ldots, \delta_{n-1}) \in \{0, 1\}^{n-1}$ there exists $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$ such that

$$((-1)^{\delta_1}, \ldots, (-1)^{\delta_{n-1}}) = ((-1)^{\varepsilon_1 - \varepsilon_n}, \ldots, (-1)^{\varepsilon_{n-1} - \varepsilon_n}),$$

(iii) is proved. ∎

3. On determining the joint α-spherical distribution by the distribution of quotients. Now a kind of a converse question will be studied. It appears that even weaker conditions than (i), (ii) and (iii) of Theorem 1 suffice to characterize α-spherical distributions. Moreover, the distribution of quotients identifies U_α-spherical distributions.

THEOREM 2. *Assume that a random vector X has the following properties:*
(a) *X is symmetric;*
(b) *$(X_1/X_n, \ldots, X_{n-1}/X_n)$ and $|X_1|^\alpha + \ldots + |X_n|^\alpha$ are independent;*
(c) *$(X_1/X_n, \ldots, X_{n-1}/X_n)$ is sign-symmetric.*
Then X has an α-spherical distribution.

If X is sign-symmetric, then the vector of quotients is also sign-symmetric. The following auxiliary result, which will be used in the proof of Theorem 2, gives a kind of the converse statement.

LEMMA 1. *Let X be concentrated on a unit α-sphere, i.e. $\|X\|_\alpha = 1$. If X is symmetric and $(X_1/X_n, \ldots, X_{n-1}/X_n)$ is sign-symmetric, then X is also sign-symmetric.*

Proof. Observe that

$$(X_1, \ldots, X_n) = \left(\frac{X_1}{|X_n|}, \ldots, \frac{X_{n-1}}{|X_n|}, \frac{X_n}{|X_n|}\right) \frac{1}{\|(|X_1/X_n|, \ldots, |X_{n-1}/X_n|, 1)\|_\alpha}.$$

At first the above formula looks like being unnecessary complicated, but it allows to conclude immediately that showing sign-symmetry of the random vector

$$\left(\frac{X_1}{|X_n|}, \ldots, \frac{X_{n-1}}{|X_n|}, \frac{X_n}{|X_n|}\right)$$

is enough for proving the result.

Consider any $(a_1, \ldots, a_{n-1}) \in \boldsymbol{R}^{n-1}$. Then

$$P\left(\frac{X_1}{|X_n|} \leqslant a_1, \ldots, \frac{X_{n-1}}{|X_n|} \leqslant a_{n-1}, \frac{X_n}{|X_n|} = 1\right) = P\left(\frac{X_1}{X_n} \leqslant a_1, \ldots, \frac{X_{n-1}}{X_n} \leqslant a_{n-1}, X_n > 0\right).$$

But by the symmetry of X we get

$$P\left(\frac{X_1}{X_n} \leqslant a_1, \ldots, \frac{X_{n-1}}{X_n} \leqslant a_{n-1}, X_n > 0\right) = P\left(\frac{X_1}{X_n} \leqslant a_1, \ldots, \frac{X_{n-1}}{X_n} \leqslant a_{n-1}, X_n < 0\right) = \frac{1}{2} P\left(\frac{X_1}{X_n} \leqslant a_1, \ldots, \frac{X_{n-1}}{X_n} \leqslant a_{n-1}\right).$$

Consequently, since the vector of quotients is sign-symmetric, we have

$$P\left(\frac{X_1}{|X_n|} \leqslant a_1, \ldots, \frac{X_{n-1}}{|X_n|} \leqslant a_{n-1}, \frac{X_n}{|X_n|} = 1\right) = \frac{1}{2} P\left((-1)^{\varepsilon_1} \frac{X_1}{X_n} \leqslant a_1, \ldots, (-1)^{\varepsilon_{n-1}} \frac{X_{n-1}}{X_n} \leqslant a_{n-1}\right).$$

Similarly as above we have

$$\frac{1}{2} P\left((-1)^{\varepsilon_1} \frac{X_1}{X_n} \leqslant a_1, \ldots, (-1)^{\varepsilon_{n-1}} \frac{X_{n-1}}{X_n} \leqslant a_{n-1}\right) = P\left((-1)^{\varepsilon_1} \frac{X_1}{|X_n|} \leqslant a_1, \ldots, (-1)^{\varepsilon_{n-1}} \frac{X_{n-1}}{|X_n|} \leqslant a_{n-1}, \frac{X_n}{|X_n|} = (-1)^{\varepsilon_n}\right)$$

for any $(\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n) \in \{0, 1\}^n$, which proves the result. ∎

Proof of Theorem 2. Observe that for any $a_1, \ldots, a_n$ and any Borel set $B \subseteq \boldsymbol{R}$

$$P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n, \|X\|_\alpha \in B\right)$$

$$= P\left(C \frac{X_1}{|X_n|} \leqslant a_1, \ldots, C \frac{X_{n-1}}{|X_n|} \leqslant a_{n-1}, C \frac{X_n}{|X_n|} \leqslant a_n, \|X\|_\alpha \in B\right)$$

$$= P\left(C \frac{X_1}{X_n} \leqslant a_1, \ldots, C \frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, X_n > 0, \|X\|_\alpha \in B\right)$$

$$+ P\left(-C \frac{X_1}{X_n} \leqslant a_1, \ldots, -C \frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, X_n < 0, \|X\|_\alpha \in B\right),$$

where $C = 1/\|(|X_1/X_n|, \ldots, |X_{n-1}/X_n|, 1)\|_\alpha$ (again the formula, looking unnecessarily complicated, is kept in this form to stress that C is a function of the vector of quotients, which will be important in the final part of the proof).

Consider first the case $a_n > 0$. Then we have

$$P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n, \|X\|_\alpha \in B\right)$$
$$= P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, X_n > 0, \|X\|_\alpha \in B\right)$$
$$+ P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, X_n < 0, \|X\|_\alpha \in B\right).$$

Now by the symmetry of X it follows that

$$P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, X_n > 0, \|X\|_\alpha \in B\right)$$
$$= P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, X_n < 0, \|X\|_\alpha \in B\right)$$
$$= \frac{1}{2}P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, \|X\|_\alpha \in B\right),$$

and similarly

$$P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, X_n < 0, \|X\|_\alpha \in B\right)$$
$$= \frac{1}{2}P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, \|X\|_\alpha \in B\right).$$

Finally, the independence of $(X_1/X_n, \ldots, X_{n-1}/X_n)$ and $\|X\|_\alpha$ yields

$$P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_{n-1}}{\|X\|_\alpha} \leqslant a_n, \|X\|_\alpha \in B\right)$$
$$= \frac{1}{2}P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n\right)P(\|X\|_\alpha \in B)$$
$$+ \frac{1}{2}P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}\right)P(\|X\|_\alpha \in B)$$
$$= \left(P\left(C\frac{X_1}{X_n} \leqslant a_1, \ldots, C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, C \leqslant a_n, X_n > 0\right)\right.$$
$$\left. + P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, X_n < 0\right)\right)P(\|X\|_\alpha \in B)$$
$$= P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n\right)P(\|X\|_\alpha \in B).$$

In the case $a_n \leqslant 0$ the argument is quite similar. The first part is even simpler:

$$P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n, \|X\|_\alpha \in B\right)$$
$$= P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, X_n < 0, \|X\|_\alpha \in B\right).$$

Again using the independence property of the vector of quotients and the α-th norm of X we get

$$P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n, \|X\|_\alpha \in B\right)$$
$$= P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, X_n < 0, \|X\|_\alpha \in B\right)$$
$$= \frac{1}{2}P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, \|X\|_\alpha \in B\right)$$
$$= \frac{1}{2}P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n\right)P(\|X\|_\alpha \in B)$$
$$= P\left(-C\frac{X_1}{X_n} \leqslant a_1, \ldots, -C\frac{X_{n-1}}{X_n} \leqslant a_{n-1}, -C \leqslant a_n, X_n < 0\right)P(\|X\|_\alpha \in B)$$
$$= P\left(\frac{X_1}{\|X\|_\alpha} \leqslant a_1, \ldots, \frac{X_n}{\|X\|_\alpha} \leqslant a_n\right)P(\|X\|_\alpha \in B).$$

Consequently, $Y = X/\|X\|_\alpha$ and $\|X\|_\alpha$ are independent.

To show that X is α-spherical it suffices now to prove sign-symmetry for the random vector Y, which is concentrated on the unit α-sphere. But by the symmetry of X it follows that Y is symmetric and

$$(Y_1/Y_n, \ldots, Y_{n-1}/Y_n) = (X_1/X_n, \ldots, X_{n-1}/X_n).$$

Consequently, $(Y_1/Y_n, \ldots, Y_{n-1}/Y_n)$ is sign-symmetric. Now the final result follows from Lemma 1. ∎

Our next result says that for any U_α-spherical random vector X the distribution of U_α is uniquely determined by the distribution of quotients.

THEOREM 3. *If X is a U_α-spherical random vector, then the distribution of U_α is uniquely determined by the distribution of $(X_1/X_n, \ldots, X_{n-1}/X_n)$.*

The proof of the above theorem will be preceded by an auxiliary result on determination of the joint distribution by the distribution of absolute values for sign-symmetric random vectors, which seems to be also of independent interest.

LEMMA 2. *If X is sign-symmetric, then for any nonnegative $a_1, \ldots, a_n$*

$$P(X_1 \leqslant a_1, \ldots, X_n \leqslant a_n) = \frac{1}{2^n}\left(1+\sum_{r=1}^{n} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant n} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r})\right).$$

Thus the distribution of $X = (X_1, \ldots, X_n)$ is uniquely determined by the distribution of $(|X_1|, \ldots, |X_n|)$.

Proof. We use mathematical induction to show that

$$(1) \quad P(X_1 \leqslant a_1, \ldots, X_k \leqslant a_k, |X_{k+1}| \leqslant a_{k+1}, \ldots, |X_n| \leqslant a_n)$$
$$= \frac{1}{2^n}\Bigg[P(|X_{k+1}| \leqslant a_{k+1}, \ldots, |X_n| \leqslant a_n)$$
$$+\sum_{r=1}^{k} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant k} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}, |X_{k+1}| \leqslant a_{k+1}, \ldots, |X_n| \leqslant a_n)\Bigg]$$

for $k = 1, 2, \ldots, n-1$. Take first $k = 1$. Then

$$P(X_1 \leqslant a_1, |X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n)$$
$$= P(|X_1| \leqslant a_1, |X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n)$$
$$+P(|X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n) - P(X_1 \geqslant -a_1, |X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n).$$

Hence by the symmetry of X we have

$$P(X_1 \leqslant a_1, |X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n)$$
$$= \tfrac{1}{2}\big(P(|X_1| \leqslant a_1, |X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n) + P(|X_2| \leqslant a_2, \ldots, |X_n| \leqslant a_n)\big).$$

This proves the case $k = 1$.

Now assume that the condition (1) holds for $m-1 = k < n-1$. We will prove that it holds also for m. Observe that

$$P(X_1 \leqslant a_1, \ldots, X_m \leqslant a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$
$$= P(X_1 \leqslant a_1, \ldots, X_{m-1} \leqslant a_{m-1}, |X_m| \leqslant a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$
$$+P(X_1 \leqslant a_1, \ldots, X_{m-1} \leqslant a_{m-1}, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$
$$-P(X_1 \leqslant a_1, \ldots, X_m \geqslant -a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n).$$

And since $(X_1, \ldots, X_m, \ldots, X_n) \stackrel{d}{=} (X_1, \ldots, -X_m, \ldots, X_n)$, we get

$$P(X_1 \leqslant a_1, \ldots, X_m \leqslant a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$
$$= \tfrac{1}{2}\big(P(X_1 \leqslant a_1, \ldots, X_{m-1} \leqslant a_{m-1}, |X_m| \leqslant a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$
$$+P(X_1 \leqslant a_1, \ldots, X_{m-1} \leqslant a_{m-1}, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)\big).$$

Now by the induction assumption we have

$$P(X_1 \leqslant a_1, \ldots, X_m \leqslant a_m, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)$$

$$= \frac{1}{2}\Bigg\{\frac{1}{2^{m-1}}\Bigg[\sum_{r=1}^{m-1} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant m-1} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}, |X_m| \leqslant a_m, \ldots, |X_n| \leqslant a_n) + P(|X_m| \leqslant a_m, \ldots, |X_n| \leqslant a_n)\Bigg]$$

$$+ \frac{1}{2^{m-1}}\Bigg[\sum_{r=1}^{m-1} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant m-1} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n) + P(|X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)\Bigg]\Bigg\}$$

$$= \frac{1}{2^m}\Bigg[\sum_{r=1}^{m} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant m} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}, |X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n) + P(|X_{m+1}| \leqslant a_{m+1}, \ldots, |X_n| \leqslant a_n)\Bigg],$$

which ends the induction argument.

Consider now

$$P(X_1 \leqslant a_1, \ldots, X_n \leqslant a_n) = P(X_1 \leqslant a_1, \ldots, X_{n-1} \leqslant a_{n-1}, |X_n| \leqslant a_n)$$
$$+ P(X_1 \leqslant a_1, \ldots, X_{n-1} \leqslant a_{n-1}) - P(X_1 \leqslant a_1, \ldots, X_{n-1} \leqslant a_{n-1}, |X_n| \geqslant -a_n).$$

Hence again by the sign-symmetry of X we get

$$P(X_1 \leqslant a_1, \ldots, X_n \leqslant a_n) = \tfrac{1}{2}[P(X_1 \leqslant a_1, \ldots, X_{n-1} \leqslant a_{n-1}, |X_n| \leqslant a_n) + P(X_1 \leqslant a_1, \ldots, X_{n-1} \leqslant a_{n-1})].$$

Now we apply (1) with $k = n-1$ to the first summand above and with $k = n-1$ and $a_n \to \infty$ to the second to obtain finally

$$P(X_1 \leqslant a_1, \ldots, X_n \leqslant a_n)$$

$$= \frac{1}{2}\Bigg[\frac{1}{2^{n-1}}\Bigg(\sum_{r=1}^{n-1} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant n-1} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}, |X_n| \leqslant a_n) + P(|X_n| \leqslant a_n)\Bigg)$$

$$+ \frac{1}{2^{n-1}}\Bigg(\sum_{r=1}^{n-1} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant n-1} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}) + 1\Bigg)\Bigg]$$

$$= \frac{1}{2^n}\Bigg(\sum_{r=1}^{n} \sum_{1 \leqslant j_1 < \ldots < j_r \leqslant n} P(|X_{j_1}| \leqslant a_{j_1}, \ldots, |X_{j_r}| \leqslant a_{j_r}) + 1\Bigg). \blacksquare$$

Proof of Theorem 3. Observe that it suffices to consider the random vector $X/\|X\|_\alpha \stackrel{d}{=} U_\alpha$ which is sign-symmetric. Then by Lemma 2 its distribution is uniquely determined by the distribution of $(|X_1|, \ldots, |X_n|)/\|X\|_\alpha$. Now observe that

$$\left(\frac{|X_1|}{\|X\|_\alpha}, \ldots, \frac{|X_n|}{\|X\|_\alpha}\right)$$
$$= \left[\left(1+\left|\frac{X_2}{X_1}\right|^\alpha + \ldots + \left|\frac{X_n}{X_1}\right|^\alpha\right)^{-1/\alpha}, \ldots, \left(\left|\frac{X_1}{X_n}\right|^\alpha + \ldots + \left|\frac{X_{n-1}}{X_n}\right|^\alpha + 1\right)^{-1/\alpha}\right]$$
$$= [G_1^{-1/\alpha}(X_1/X_n, \ldots, X_{n-1}/X_n), \ldots, G_n^{-1/\alpha}(X_1/X_n, \ldots, X_{n-1}/X_n)],$$

where

$$G_i(X_1/X_n, \ldots, X_{n-1}/X_n)$$
$$= \left|\frac{X_1/X_n}{X_i/X_n}\right|^\alpha + \ldots + \left|\frac{X_{i-1}/X_n}{X_i/X_n}\right|^\alpha + 1 + \left|\frac{X_{i+1}/X_n}{X_i/X_n}\right|^\alpha + \ldots + \left|\frac{X_{n-1}/X_n}{X_i/X_n}\right|^\alpha + \left|\frac{X_n}{X_i}\right|^\alpha,$$

$i = 1, \ldots, n$.

Consequently, the distribution of $X/\|X\|_\alpha$ is uniquely determined by the distribution of $(X_1/X_n, \ldots, X_{n-1}/X_n)$. ■

As a final conclusion of Theorems 2 and 3 we have the following generalization of characterizations of two-dimensional spherically invariant distribution from Wesołowski [13] and of n-dimensional α-spherically invariant distribution from Szabłowski [10]:

THEOREM 4. *Let X be an n-dimensional random vector such that*:

1° *X is symmetric*;

2° *$(X_1/X_n, \ldots, X_{n-1}/X_n)$ has a sign-symmetric distribution μ*;

3° *$(X_1/X_n, \ldots, X_{n-1}/X_n)$ and $\|X\|_\alpha$ are independent.*

Then X has the U_α-spherical distribution, where the distribution of U_α (on a unit α-sphere) is uniquely determined by μ.

Acknowledgement. The author is greatly indebted to the referee for suggestions which helped to improve the text, and first of all for additional challenging questions related to the subject of the present paper.

REFERENCES

[1] A. K. Gupta and D. Song, *Characterization of p-generalized normality*, J. Multivariate Anal. 60 (1997), pp. 61–71.

[2] — *L_p-norm uniform distribution*, Proc. Amer. Math. Soc. 125 (1997), pp. 595–601.

[3] — *L_p-norm spherical distributions*, J. Statist. Plann. Inference 60 (1997), pp. 241–260.

[4] R. D. Gupta, J. K. Misiewicz and D. St. P. Richards, *Infinite sequences with sign-symmetric Liouville distributions*, Probab. Math. Statist. 16 (1996), pp. 29–44.

[5] I. Kotlarski, *On characterizing the gamma and the normal distribution*, Pacific J. Math. 20 (1967), pp. 69–76.
[6] R. G. Laha, *An example of a non-normal distribution where the quotient follows the Cauchy law*, Proc. Nat. Acad. Sci. U.S.A. 44 (1958), pp. 222–223.
[7] G. Letac, *Isotropy and sphericity: some characterisations of the normal distribution*, Ann. Statist. 9 (1981), pp. 408–417.
[8] W. Matysiak, *A characterization of sign-symmetric Liouville-type distributions*, Preprint 1–5 (1998).
[9] V. Seshadri, *A characterization of the normal and Weibull distributions*, Canad. Math. Bull. 12 (1969), pp. 257–260.
[10] P. J. Szabłowski, *Uniform distributions on spheres in finite dimensional L_α and their generalizations*, J. Multivariate Anal. 64 (1998), pp. 103–117.
[11] — J. Wesołowski and M. Ahsanullah, *Identification of probability measures via distribution of quotients*, J. Statist. Plann. Inference 63 (1997), pp. 377–385.
[12] J. Wesołowski, *Some characterizations connected with properties of the quotient of independent random variables*, Teor. Veroyatnost. i Primenen. 36 (1991), pp. 780–781.
[13] — *A characterization of the bivariate elliptically contoured distribution*, Statist. Papers 33 (1992), pp. 143–149.

Mathematical Institute
Warsaw University of Technology
plac Politechniki 1, 00-661 Warsaw, Poland
E-mail: wesolo@alpha.im.pw.edu.pl

Received on 22.9.1998