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MOMENTS AND GENERALIZED CONVOLUTIONS. III

BY

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Abstract. The paper deals with some uniqueness and characterization theorems for probability measures and generalized convolutions in terms of negative moments.

1. Preliminaries and notation. This paper is a continuation of the author's earlier works [8], [9] and is organized as follows. Section 1 collects together some basic facts and notation concerning generalized convolutions needed in the sequel. In Sections 2 and 3 some properties of negative moments are discussed. Section 4 contains auxiliary results on mappings of measures and differentiable characteristic functions. In the last section these results are applied to some uniqueness and characterization theorems for probability measures and generalized convolutions in terms of negative moments. For the terminology and notation used here, see [6]. In particular, P will stand for the set of all probability measures defined on Borel subsets of the half-line $[0, \infty)$. The set P is endowed with the metrizable topology of weak convergence. As usual, we let δ_c stand for the probability measure concentrated at the point c. Given a random variable X we shall denote by $\operatorname{distr} X$ its probability distribution. Further, for any pair X and Y of independent random variables with $\operatorname{distr} X = \mu$ and $\operatorname{distr} Y = \nu$, respectively, we put $\mu\nu = \operatorname{distr} XY$. Two measures μ and ν from P are said to be *similar* if $\mu = \delta_c \nu$ for a certain positive number c.

A continuous commutative and associative P-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to the convex combinations of measures and the operations $\mu \to \delta_c \mu$ $(c \geqslant 0)$, δ_0 is its unit element and an analogue of the law of large numbers is fulfilled:

$$\delta_{c_n} \delta_1^{\circ n} \to \gamma \neq \delta_0 \tag{1.1}$$

for a choice of a norming sequence c_n of positive numbers. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$.

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Let m_0 be the sum of δ_0 and the Lebesgue measure on $[0, \infty)$. It has been proved in [6], Theorem 4.1 and Corollary 4.4, that each generalized convolution admits a characteristic function, i.e. a one-to-one correspondence $\mu \to \hat{\mu}$ between measures μ from P and real-valued Borel functions $\hat{\mu}$ from $L_\infty(m_0)$ such that

$$(c\mu+(1-c)\nu)^\wedge = c\hat{\mu}+(1-c)\hat{\nu}, \quad (\delta_a \mu)^\wedge(t) = \hat{\mu}(at), \quad (\mu \circ \nu)^\wedge = \hat{\mu}\hat{\nu}$$

for all $c \in [0, 1]$, $a \in (0, \infty)$ and $\mu, \nu \in P$. The weak convergence $\mu_n \to \mu$ is equivalent to the convergence $\hat{\mu}_n \to \hat{\mu}$ in $L_1(m_0)$-topology of $L_\infty(m_0)$. Moreover, if μ is absolutely continuous with respect to the measure m_0, then the function $\hat{\mu}$ is continuous. The characteristic function is uniquely determined up to a scale change and is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\mu(dx).$$

The kernel Ω is a Borel function with $\Omega(0) = 1$ and

$$|\Omega(t)| \leqslant 1 \quad \text{for } t \in [0, \infty). \tag{1.2}$$

It is easy to check the formula

$$\mu\nu(t)^\wedge = \int_0^\infty \hat{\mu}(tx)\nu(dx) \tag{1.3}$$

for $\mu, \nu \in P$. By $P_+(\circ)$ we shall denote the subset of P consisting of measures μ with nonnegative characteristic function $\hat{\mu}$. It is clear that the set $P_+(\circ)$ does not depend upon the choice of a characteristic function.

A measure μ from P other than δ_0 is said to be $\circ$-*stable* if the measures $\mu_1 \circ \mu_2$ and μ are similar provided the measures μ_1, μ_2 and μ are similar. In the sequel the set of all $\circ$-stable measures will be denoted by $S(\circ)$. It was shown in [6], Theorem 4.2, that there exists a constant $\varkappa(\circ)$ $(0 < \varkappa(\circ) \leqslant \infty)$ with the following property: $\mu \in S(\circ)$ if and only if either

$$\hat{\mu}(t) = \exp(-ct^p) \tag{1.4}$$

with $c \in (0, \infty)$ and $p \in (0, \varkappa(\circ)] \cap (0, \infty)$ or $\varkappa(\circ) = \infty$ and $\mu = \delta_a$ for some $a \in (0, \infty)$. The constant p does not depend upon the choice of the characteristic function and is called the *exponent* of μ. Obviously, all $\circ$-stable measures with the same exponent are similar. The limit measure γ from (1.1) is $\circ$-stable with the exponent $\varkappa(\circ)$.

Given $p, q > 0$ we denote by $\varrho(p, q)$ the probability measure on the half-line $[0, \infty)$ with the density function $p\Gamma(q/p)^{-1}x^{q-1}\exp(-x^p)$.

Many examples of generalized convolutions are to be found in various branches of probability theory (see [10]). We shall quote some of them. In all examples the random variables X and Y are assumed to be independent.

EXAMPLE 1.1. *The convolutions* $*_p$ $(0 < p < \infty)$. These convolutions are defined by the formula

$$(\operatorname{distr} X) *_p (\operatorname{distr} Y) = \operatorname{distr}(X^p + Y^p)^{1/p}.$$

Here we have $\varkappa(*_p) = p$ and $\Omega(t) = \exp(-t^p)$. For $p = 1$ we obtain the ordinary convolution $*$.

EXAMPLE 1.2. *Max-convolution* $*_\infty$. This convolution is defined by the formula

$$(\operatorname{distr} X) *_\infty (\operatorname{distr} Y) = \operatorname{distr}\big(\max(X, Y)\big).$$

Here we have $\varkappa(*_\infty) = \infty$ and $\Omega(t)$ is equal to the indicator of the unit interval $[0, 1]$. By Lemma 2.1 in [6], $\varkappa(\circ) = \infty$ if and only if $\circ = *_\infty$.

EXAMPLE 1.3. *Kingman convolutions* $*_{p,q}$ $(p > 0,\ q \geqslant 1)$. The generalized convolution $*_{p,q}$ is defined by the formula

$$(\operatorname{distr} X) *_{p,q} (\operatorname{distr} Y) = \operatorname{distr}(X^{2p} + Y^{2p} + 2X^p Y^p U_q)^{1/2p},$$

where the random variable U_q is independent of X and Y, $\operatorname{distr} U_1 = \frac{1}{2}(\delta_{-1} + \delta_1)$ and for $q > 1$ the probability distribution of U_q is concentrated on the interval $[-1, 1]$ and has the density function

$$B(1/2, q/2)^{-1}(1-x^2)^{(q-3)/2},$$

where B is the beta function ([5], Example 1.2). Here we have $\varkappa(*_{p,q}) = 2p$ and

$$\Omega_{p,q}(t) = \Gamma(q/2)(2/t^p)^{(q-2)/2} J_{(q-2)/2}(t^p), \tag{1.5}$$

where J_r is the Bessel function. The probability measure $\varrho(2p, 2pq-p)$ is $*_{p,q}$-stable with exponent $2p$.

2. Negative moments. Given $s > 0$ and $\mu \in P$ we put

$$\pi_s(\mu) = \int_0^\infty x^{-s}\mu(dx).$$

The set of all measures μ from P with finite moment $\pi_s(\mu)$ will be denoted by P_s. It is clear that

$$\pi_s(\mu) > 0, \tag{2.1}$$

$$\pi_s(\mu\nu) = \pi_s(\mu)\pi_s(\nu) \tag{2.2}$$

for all $\mu, \nu \in P$ and

$$P_s \subset P_u \tag{2.3}$$

whenever $0 < u < s$.

Given $s > 0$ and a generalized convolution $\circ$ with the characteristic function $\mu \to \hat{\mu}$ we denote by $F_s(\circ)$ the set of all probability measures μ from P for which the limit

$$f_s(\hat{\mu}) = \lim_{c\to\infty} \int_0^c \hat{\mu}(t)\, t^{s-1}\, dt$$

exists. It is evident that the set $F_s(\circ)$ does not depend upon the choice of the characteristic function. As an immediate consequence of formula (1.4) and Example 1.2 we get the following inclusion:

$$S(\circ) \subset F_s(\circ) \tag{2.4}$$

for all generalized convolutions $\circ$ and $s > 0$.

LEMMA 2.1. *If $\mu \in F_s(\circ)$ and $\nu \in P_s$, then $\mu\nu \in F_s(\circ)$ and*

$$f_s(\mu\nu)^\wedge = f_s(\hat{\mu})\,\pi_s(\nu).$$

Proof. For $c > 0$ we have, by (1.3),

$$\int\limits_0^c \mu\nu(t)^\wedge t^{s-1}\,dt = \nu(\{\circ\})\frac{c^s}{s} + \int\limits_{0+}^{\infty}\int\limits_0^{cx} \hat{\mu}(y)\,y^{s-1}\,dy\,x^{-s}\,\nu(dx). \tag{2.5}$$

Since $\nu \in P_s$, the measure ν has no mass at the origin. Consequently, by the bounded convergence theorem, the right-hand side of (2.5) has the limit $f_s(\hat{\mu})\,\pi_s(\nu)$, which completes the proof.

LEMMA 2.2. *If $\mu \in P_+(\circ)$ and $\mu\nu \in F_s(\circ)$, then $\mu \in F_s(\circ)$ and $\nu \in P_s$.*

Proof. The condition $\mu \in P_+(\circ)$ and formula (2.5) yield the inequality

$$\int\limits_0^c \mu\nu(t)^\wedge t^{s-1}\,dt \geqslant \nu(\{0\})\frac{c^s}{s} \qquad (c > 0).$$

Since the left-hand side of this inequality has a finite limit as $c \to \infty$, we conclude that $\nu(\{\circ\}) = 0$. Now, by (2.5) and the Fatou lemma, we get the inequality

$$f_s(\mu\nu)^\wedge \geqslant \pi_s(\nu)\int\limits_0^\infty \hat{\mu}(t)\,t^{s-1}\,dt,$$

which, by (2.1), yields

$$\pi_s(\nu) < \infty \quad \text{and} \quad f_s(\hat{\mu}) = \int\limits_0^\infty \hat{\mu}(t)\,t^{s-1}\,dt < \infty.$$

The lemma is thus proved.

LEMMA 2.3. *If $P_s \cap P_+(\circ) \neq \emptyset$, then*

$$P_s \cap P_+(\circ) \subset F_s(\circ) \subset P_s.$$

Proof. We note that, by (2.4), $F_s(\circ) \neq \emptyset$. Taking $\mu \in F_s(\circ)$ and $\nu \in P_s \cap P_+(\circ)$ we get, by Lemma 2.1, $\mu\nu \in F_s(\circ)$. Applying now Lemma 2.2 we have $\mu \in P_s$ and $\nu \in F_s(\circ)$, which completes the proof.

As an immediate consequence of the above lemma we get the following result:

COROLLARY 2.1. *If $P_s \cap P_+(\circ) \neq \emptyset$, then $P_s \cap P_+(\circ) = F_s(\circ) \cap P_+(\circ)$.*

LEMMA 2.4. *If $\mu \in P_+(\circ) \cap F_s(\circ)$ and $\nu \in P$, then $\mu \circ \nu \in F_s(\circ)$.*

Proof. It is clear that the function $\hat{\mu}(t)t^{s-1}$ is integrable on the half-line $[0, \infty)$. By the inequality $|(\mu \circ \nu)^\wedge| = |\hat{\mu}||\hat{\nu}| \leqslant \hat{\mu}$, also the function $(\mu \circ \nu)^\wedge(t)t^{s-1}$ is integrable. This yields the assertion of the lemma.

THEOREM 2.1. *The following conditions are equivalent*:

(i) $$S(\circ) \subset P_s,$$

(ii) $$S(\circ) \cap P_s \neq \emptyset,$$

(iii) $$P_s \cap P_+(\circ) \neq \emptyset,$$

(iv) $$F_s(\circ) \subset P_s.$$

Proof. Since $S(\circ) \neq \emptyset$, the implication (i) $\Rightarrow$ (ii) is evident. The inclusion $S(\circ) \subset P_+(\circ)$ yields the implication (ii) $\Rightarrow$ (iii). By Lemma 2.3 we have the implication (iii) $\Rightarrow$ (iv). Finally, inclusion (2.4) yields the implication (iv) $\Rightarrow$ (i), which completes the proof.

In the sequel, $I(\circ)$ will denote the set of all positive real numbers s fulfilling the condition $S(\circ) \subset P_s$. The problem whether the set $I(\circ)$ is non-void for all generalized convolutions $\circ$ is still open.

EXAMPLE 2.1. From the implication (iii) $\Rightarrow$ (i) of Theorem 2.1 we get the equality $I(\circ) = (0, \infty)$ whenever $P_+(\circ) = P$. In particular, $I(*_p) = (0, \infty)$ $(0 < p \leqslant \infty)$.

EXAMPLE 2.2. Given $p > 0$ and $q \geqslant 1$ we conclude, by Example 1.3, that $\varrho(2p, 2pq-p) \in S(*_{p,q})$. It is easy to check that $\varrho(2p, 2pq-p) \in P_s$ if and only if $s < 2pq-p$. Consequently, applying Theorem 2.1 (parts (i) and (ii)), we get the formula $I(*_{p,q}) = (0, 2pq-p)$.

THEOREM 2.2. *Given $s \in I(\circ)$ and a characteristic function $\mu \to \hat{\mu}$ there exists a positive constant c_s such that*

$$\pi_s(\mu) = c_s f_s(\hat{\mu}) \quad \textit{for every } \mu \in F_s(\circ).$$

Proof. Let $\mu \in F_s(\circ)$. Taking a $\circ$-stable measure λ we have, by Theorem 2.1, $\lambda \in P_s$. Applying Lemma 2.1 we get the formula

$$f_s(\mu\lambda)^\wedge = f_s(\hat{\mu})\pi_s(\lambda). \tag{2.6}$$

Moreover, by Theorem 2.1 (iv), $\mu \in P_s$ and, by (2.4), $\lambda \in F_s(\circ)$. Now, applying Lemma 2.1, we get the formula

$$f_s(\mu\lambda)^\wedge = f_s(\hat{\lambda})\pi_s(\mu),$$

which together with (2.6) yields the assertion of the theorem with the constant

$$c_s = \pi_s(\lambda)/f_s(\hat{\lambda}).$$

Changing the scale $\tilde{\mu}(t) = \hat{\mu}(c_s^{1/s}t)$ we get a new characteristic function $\mu \to \tilde{\mu}$ for which the constant appearing in Theorem 2.2 is equal to 1. This characteristic function will be called *s-normed.*

3. Semigroups associated with generalized convolutions. Given $s > 0$ and a generalized convolution $\circ$ we denote by $Q_s(\circ)$ the set of all probability measures μ fulfilling the condition $\mu^{\circ n} \in P_s$ for all positive integers n.

THEOREM 3.1. *$Q_s(\circ) \neq \emptyset$ if and only if $s \in I(\circ)$.*

Proof. Suppose that $\mu \in Q_s(\circ)$. Then $\mu \circ \mu \in P_s \cap P_+(\circ)$, which, by Theorem 2.1 (iii), yields $s \in I(\circ)$. Conversely, suppose that $s \in I(\circ)$. Let $\lambda \in S(\circ)$. The power $\lambda^{\circ n}$ being similar to λ also belongs to $S(\circ)$. Consequently, $\lambda^{\circ n} \in P_s$ $(n = 1, 2, \ldots)$ or, in other words, $\lambda \in Q_s(\circ)$, which completes the proof.

The following result will play a crucial role in our considerations.

THEOREM 3.2. *A measure μ belongs to $Q_s(\circ)$ if and only if both measures μ and $\mu \circ \mu$ belong to P_s.*

Proof. The necessity of our conditions is evident. To prove the sufficiency let us assume that μ and $\mu \circ \mu$ belong to P_s. Since $\mu \circ \mu \in P_+(\circ)$, we have, by Theorem 2.1 (implication (iii)$\Rightarrow$(iv)), the inclusion

$$F_s(\circ) \subset P_s. \tag{3.1}$$

Moreover, by Corollary 2.1, $\mu \circ \mu \in F_s(\circ) \cap P_+(\circ)$. Applying Lemma 2.4, we infer that $\mu^{\circ n} \in F_s(\circ)$ for $n \geqslant 2$, which, by (3.1), shows that $\mu \in Q_s(\circ)$. The theorem is thus proved.

THEOREM 3.3. *For every $s > 0$ and every generalized convolution $\circ$ the inclusion $P_s \cap P_+(\circ) \subset Q_s(\circ)$ is true.*

Proof. The case $P_s \cap P_+(\circ) = \emptyset$ is evident. In the opposite case we have, by Theorem 2.1 (iii), $s \in I(\circ)$. Let $\mu \in P_s \cap P_+(\circ)$. Then, by Corollary 2.1, $\mu \in F_s(\circ)$, which, by Lemma 2.4, yields $\mu \circ \mu \in F_s(\circ)$. Consequently, by Theorem 2.1 (iv), $\mu \circ \mu \in P_s$, which, by Theorem 3.2, shows that $\mu \in Q_s(\circ)$.

THEOREM 3.4. *Let $s \in I(\circ)$. The set $Q_s(\circ)$ is a semigroup under operation $\circ$.*

Proof. Let $\mu, \nu \in Q_s(\circ)$. Since $\mu^{\circ 2n}, \nu^{\circ 2n} \in P_s \cap P_+(\circ)$, we have, by Corollary 2.1, the relation $\mu^{\circ 2n}, \nu^{\circ 2n} \in F_s(\circ)$ $(n = 1, 2, \ldots)$. Consequently, the functions $\hat{\mu}^{2n}(t)t^{s-1}$, $\hat{\nu}^{2n}(t)t^{s-1}$ are integrable on the half-line $[0, \infty)$. By the inequality

$$\left|\left(\mu \circ \nu(t)^\wedge\right)^n\right| = |\hat{\mu}(t)|^n |\hat{\nu}(t)|^n \leqslant \tfrac{1}{2}\left(\hat{\mu}^{2n}(t) + \hat{\nu}^{2n}(t)\right)$$

the functions $\left(\mu \circ \nu(t)^\wedge\right)^n t^{s-1}$ $(n = 1, 2, \ldots)$ are also integrable. Thus $(\mu \circ \nu)^{\circ n} \in F_s(\circ)$ $(n = 1, 2, \ldots)$. Now, applying Theorem 2.1 (iv), we get $\mu \circ \nu \in Q_s(\circ)$, which completes the proof.

THEOREM 3.5. *Let $s \in I(\circ)$. If $\mu \in Q_s(\circ)$ and $\nu \in P_s$, then $\mu\nu \in Q_s(\circ)$.*

Proof. By (2.2), $\mu\nu \in P_s$ and, by (1.3),

$$\left((\mu\nu) \circ (\mu\nu)\right)^\wedge(t) = \left(\mu\nu(t)^\wedge\right)^2 \leqslant \int_0^\infty \hat{\mu}^2(tx)\,\nu(dx) = \left((\mu \circ \mu)\nu\right)^\wedge(t). \tag{3.2}$$

Since $(\mu \circ \mu)\nu \in P_s \cap P_+(\circ)$, we infer, by Corollary 2.1, that $(\mu \circ \mu)\nu \in F_s(\circ)$ and, consequently, the function $((\mu \circ \mu)\nu)^\wedge(t)t^{s-1}$ is integrable on the half-line $[0, \infty)$. Inequality (3.2) shows that also the function $((\mu\nu)\circ(\mu\nu))^\wedge(t)t^{s-1}$ is integrable. Thus $(\mu\nu)\circ(\mu\nu) \in F_s(\circ)$, which, by Theorem 2.1 (iv), yields the relation $(\mu\nu)\circ(\mu\nu) \in P_s$. Applying Theorem 3.2 we get $\mu\nu \in Q_s(\circ)$, which completes the proof.

THEOREM 3.6. *$Q_s(\circ) = P_s$ if and only if $\delta_1 \circ \delta_1 \in P_s$.*

Proof. The necessity of the condition $\delta_1 \circ \delta_1 \in P_s$ follows from the relation $\delta_1 \in P_s$ and Theorem 3.4. Conversely, by Theorem 3.2, the relations $\delta_1 \in P_s$ and $\delta_1 \circ \delta_1 \in P_s$ yield $\delta_1 \in Q_s(\circ)$. Now, applying Theorem 3.5, we conclude that $\mu = \delta_1 \mu \in Q_s(\circ)$ for every $\mu \in P_s$, which completes the proof.

EXAMPLE 3.1. As an immediate consequence of Theorem 3.3 we get the formula $Q_s(\circ) = P_s$ for all $s > 0$ provided $P_+(\circ) = P$. In particular, we have $Q_s(*_p) = P_s$ for all $s > 0$ and $0 < p \leqslant \infty$.

EXAMPLE 3.2. *Kingman convolutions.* According to Example 1.3 we have the formula

$$\delta_1 *_{p,q} \delta_1 = \operatorname{distr}(2+2U_q)^{1/2p} \qquad (p > 0,\ q \geqslant 1).$$

Consequently,

$$\pi_s(\delta_1 *_{p,q} \delta_1) = B(1/2, q/2)^{-1} \int_{-1}^{1} (2+2x)^{-s/2p}(1-x^2)^{(q-3)/2}\,dx.$$

Hence it follows that $\delta_1 *_{p,q} \delta_1 \in P_s$ if and only if $0 < s < pq-p$. Taking into account Theorems 2.1 and 3.5 and Example 2.2 we get

$$\begin{aligned} Q_s(*_{p,q}) &= P_s && \text{if } 0 < s < pq-p, \\ \varnothing \neq Q_s(*_{p,q}) &\neq P_s && \text{if } pq-p \leqslant s < 2pq-p, \\ Q_s(*_{p,q}) &= \varnothing && \text{if } s \geqslant 2pq-p. \end{aligned}$$

4. Auxiliary results. In this section we gather some auxiliary results on mappings of measures and differentiable characteristic functions needed in the sequel. Given a positive number s we denote by m_s the measure on the half-line $[0, \infty)$ with the density function x^{s-1}. As usual, the set of all real-valued Borel square m_s-integrable functions on $[0, \infty)$ will be denoted by $L^2(m_s)$. Given $f \in L^2(m_s)$ we put

$$(fm_s)(E) = m_s(f^{-1}(E))$$

for Borel subsets E of $[0, \infty)$.

The proof of the following lemma is based on an idea due to Braverman et al. ([3], Theorem 2.1).

LEMMA 4.1. *Let $g \in L^2(m_s)$ and $x_0 \in (0, \infty)$. Suppose that $g(x_0) > 0$ and the derivative of g at x_0 exists and is equal to 0. Then for every positive number c and* *sufficiently small positive number h the inequality*

$$(gm_s)([g(x_0)-h,\ g(x_0)+h]) \geqslant ch$$

is true.

Proof. Setting for $y \in (0, \infty)$

$$a(y) = \sup\{|g(x_0)-g(x)|/|x_0-x|:\ x \geqslant 0,\ 0 < |x_0-x| \leqslant y\}$$

we have $a(y) \to 0$ as $y \to 0$. Consequently, for every $b > 0$ there exists a positive number $h(b)$ such that the inequalities $bh < x_0$ and $a(bh)b \leqslant 1$ are fulfilled for $0 < h < h(b)$. Observe that the inequalities $0 < h < \min(g(x_0), h(b))$ and $|x_0-x| \leqslant bh$ imply

$$|g(x_0)-g(x)| \leqslant a(bh)|x_0-x| \leqslant a(bh)bh \leqslant h.$$

Hence we get the inclusion

$$[x_0-bh,\ x_0+bh] \subset \{x:\ |g(x_0)-g(x)| \leqslant h\} \subset g^{-1}([g(x_0)-h,\ g(x_0)+h]).$$

Consequently,

$$\begin{aligned}(gm_s)([g(x_0)-h,\ g(x_0)+h]) &\geqslant m_s([x_0-bh,\ x_0+bh])\\ &= s^{-1}(x_0+bh)^s - s^{-1}(x_0-bh)^s \geqslant ch,\end{aligned}$$

where $c = x_0^{s-1}b$ and $0 < h < \min(g(x_0), h(b))$, which completes the proof.

THEOREM 4.1. *Let $f, g \in L^2(m_s)$. Suppose that both functions are continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Further, suppose that the derivative of g is negative on $(0, \infty)$ and*

$$f(0) = g(0) > g(x) \qquad (x > 0). \tag{4.1}$$

Then the equality

$$fm_s = gm_s \tag{4.2}$$

on $(-\infty, 0) \cup (0, \infty)$ yields $f = g$.

Proof. By the assumption the function f is decreasing on $[0, \infty)$. Since $f \in L^2(m_s)$, it converges to 0 at ∞. Thus f is positive on $[0, \infty)$, which shows, by (4.1) and (4.2), that both measures fm_s and gm_s are concentrated on the interval $[0, f(0)]$. Hence it follows that the function g is nonnegative. Denote by φ the inverse function for f mapping the interval $(0, f(0)]$ into $[0, \infty)$. Of course, φ is decreasing and differentiable on $(0, f(0))$. Suppose that $x_0 > 0$ and $g(x_0) > 0$. Then, by (4.1), the function φ is differentiable at the point $g(x_0)$. Moreover, by (4.2),

$$\begin{aligned}(gm_s)([g(x_0)-h,\ g(x_0)+h]) &= (fm_s)([g(x_0)-h,\ g(x_0)+h])\\ &= s^{-1}\varphi^s(g(x_0)-h) - s^{-1}\varphi^s(g(x_0)+h)\end{aligned}$$

whenever $0 < h < g(x_0)$. This shows that the left-hand side of the above equality is less than ah for some constant a. Applying Lemma 4.1 we conclude that the derivative of the function g at x_0 does not vanish. In particular, the function g has no local extremum at the points $x_0 \in (0, \infty)$ with $g(x_0) > 0$. Hence and from the assumption $g \in L^2(m_s)$ it follows that the equality $g(x_1) = 0$ for $x_1 > 0$ yields $g(x) = 0$ for $x \geqslant x_1$. In this case we have, by (4.2), the contradiction

$$\infty = m_s((0, \infty)) = (fm_s)\big((0, f(0)]\big) = (gm_s)\big((0, f(0)]\big)$$
$$= m_s([0, x_1)) < \infty,$$

which shows that $g(x) > 0$ for all $x \in (0, \infty)$ and, consequently, the derivative of g does not vanish on $(0, \infty)$. By inequality (4.1) this derivative is negative, which shows that the function g is decreasing on $(0, \infty)$. Thus

$$(gm_s)\big([g(x), g(0)]\big) = s^{-1}x^s \quad (x > 0). \tag{4.3}$$

By (4.1) we have the formula $\varphi(g(0)) = \varphi(f(0)) = 0$. Consequently,

$$(fm_s)\big([g(x), g(0)]\big) = s^{-1}\varphi\big(g(x)\big)^s,$$

which, by (4.2) and (4.3), yields $\varphi(g(x)) = x$ for $x > 0$. Thus $f = g$, which completes the proof.

To state the next result we introduce some notation. Given a generalized convolution $\circ$ we denote by $D(\circ)$ the set of all probability measures μ from P whose characteristic functions $\hat{\mu}$ are continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and fulfil the inequality $\hat{\mu}(t) < 1$ for $t \in (0, \infty)$. It is clear that this definition does not depend upon the choice of a characteristic function.

The subset of $D(\circ)$ consisting of measures μ for which the derivative of $\hat{\mu}$ is negative on $(0, \infty)$ will be denoted by $\Delta(\circ)$.

THEOREM 4.2. *For every generalized convolution* $\circ$ *the inclusion* $\Delta(\circ) \subset P_+(\circ)$ *is true.*

Proof. It is clear that for $\mu \in \Delta(\circ)$ the limit

$$\lim_{t\to\infty} \hat{\mu}(t) = c \tag{4.4}$$

exists and

$$\hat{\mu}(t) > c \quad \text{for } t \geqslant 0. \tag{4.5}$$

Let λ be a $\circ$-stable measure with finite exponent p and the characteristic function $\hat{\lambda}(t) = \exp(-t^p)$. Put $\nu = \mu\lambda$. By (1.3),

$$\hat{\nu}(t) = \int_0^\infty \exp(-t^p x^p)\,\mu(dx),$$

which yields

$$\lim_{t\to\infty} \hat{\nu}(t) = \mu(\{0\}). \tag{4.6}$$

On the other hand, $\hat{\nu}(t) = \int_0^\infty \hat{\mu}(tx)\,\lambda(dx)$. Since, by Lemma 2.2 in [6], the measure λ has no atom at the origin, the last equality and (4.4) yield

$$\lim_{t\to\infty} \hat{\nu}(t) = c.$$

Comparing this with (4.6) we get the inequality $c \geqslant 0$, which, by (4.5), shows that $\mu \in P_+(\mathrm{o})$. The theorem is thus proved.

Given an absolutely continuous measure μ from P with the probability density function g we denote by $T(\mu)$ the total variation of the function $xg(x)$ on the half-line $[0, \infty)$. Denote by P^* the set of probability measures μ with finite total variation $T(\mu)$.

LEMMA 4.2. *If* $\mu \in P^*$, $\nu \in P$ *and* $\nu(\{\mathrm{o}\}) = 0$, *then* $\mu\nu \in P^*$.

Proof. Observe that the probability density function of $\mu\nu$ is given by the expression $\int_0^\infty g(x/y)\,y^{-1}\,\nu(dy)$, where g is the density function of μ. It is easy to check the inequality $T(\mu\nu) \leqslant T(\mu)$, which yields the assertion of the lemma.

We define the family of transformations V_p $(p > 0)$ of P by setting $V_p(\mathrm{distr}\,X) = \mathrm{distr}\,X^{1/p}$.

LEMMA 4.3. *The set* P^* *is invariant under transformations* V_p $(p > 0)$.

Proof. Let $\mu \in P^*$. Observe that the transformed measure $V_p\mu$ has the density function $pg(x^p)\,x^{p-1}$, where g denotes the density function of μ. Hence the equality $T(V_p\mu) = pT(\mu)$ follows, which completes the proof.

A probability measure λ on the real-line $(-\infty, \infty)$ is said to be *unimodal* if for some c the function $x \to \lambda\big((-\infty, x)\big)$ is convex on $(-\infty, c)$ and concave on (c, ∞). The point c is called a *mode* of λ. The mode of a unimodal probability measure is not necessarily unique, but the mode of a probability measure from P concentrated on $[0, \infty)$ is nonnegative.

LEMMA 4.4. *Let* μ *be a unimodal measure from* P *with the probability density function* g *and the mode* c. *If the function* cg *is bounded on* $[0, \infty)$, *then* $\mu \in P^*$.

Proof. First observe that the measure $\mu * \delta_{-c}$ regarded on the whole line $(-\infty, \infty)$ is unimodal with the mode 0 and the density function $g_c(x) = g(x+c)$ for $x \in (-c, \infty)$ and $g_c(x) = 0$ otherwise. By the Khintchine theorem ([4], Chapter 6) the function

$$\int_{-\infty}^{x} g_c(u)\,du - xg_c(x)$$

is non-decreasing and bounded on $(-\infty, \infty)$. Hence it follows that the function

$$\int_{0}^{x} g(u)\,du - (x-c)\,g(x)$$

is non-decreasing and bounded on $[0, \infty)$. Now it is easy to verify that the total variation $T(\mu)$ is finite provided the function cg is bounded on $[0, \infty)$. The lemma is thus proved.

Denote by $S^*(\circ)$ the subset of $S(\circ)$ consisting of $\circ$-stable measures with exponent less than $\varkappa(\circ)$.

THEOREM 4.3. *For every generalized convolution* $\circ$ *the inclusion* $S^*(\circ) \subset P^*$ *is true.*

Proof. First consider an ordinary convolution $*$. It is well known that the measures from $S^*(*)$ are absolutely continuous with bounded density function. Moreover, by the Yamazato theorem [11], they are unimodal. Consequently, by Lemma 4.4, we have the inclusion

$$S^*(*) \subset P^*. \tag{4.7}$$

By Lemma 2.5 and Proposition 4.4 in [6], each measure μ from $S^*(\circ)$ is of the form $\mu = (V_p \nu)\lambda$, where $p > 0$, $\nu \in S^*(*)$ and $\lambda \in S(\circ)$. By Lemma 2.2 in [6], the measure λ has no atom at the origin. The relation $\mu \in P^*$ is a consequence of (4.7) and Lemmas 4.2 and 4.3, which completes the proof.

THEOREM 4.4. *For every generalized convolution* $\circ$ *other than the max-convolution the inclusion* $P^* \subset D(\circ)$ *is true.*

Proof. Let $\mu \in P$. The continuity of $\hat{\mu}$ on $[0, \infty)$ is an immediate consequence of the absolute continuity of μ. Further, by Lemma 2.2 in [7], we have the inequality $\hat{\mu}(t) < 1$ for $t > 0$. To prove the differentiability of $\hat{\mu}$ on $(0, \infty)$ we introduce a signed bounded measure m by setting

$$m(dx) = \mu(dx) - d\big(xg(x)\big) = -x\,dg(x),$$

where g is the probability density function of μ. By (1.2) the integral transform

$$\hat{m}(t) = \int_0^\infty \Omega(tx)\, m(dx)$$

is bounded and for $t > 0$

$$\frac{1}{t}\int_0^t \hat{m}(u)\,du = -\frac{1}{t}\int_0^t\int_0^\infty \Omega(ux)\,x\,dg(x)\,du = -\int_0^\infty\int_0^x \Omega(ty)\,dy\,dg(x). \tag{4.8}$$

Observe that, by (1.2),

$$\Big|\int_0^x \Omega(ty)\,dy\Big| \leqslant x \qquad (x \geqslant 0).$$

Since the total variation $T(\mu)$ is finite, the function $xg(x)$ has the limits as $x \to 0$ and $x \to \infty$. These limits are equal to 0 because of the integrability of g. Using this fact and integrating by parts the right-hand side of (4.8) we get the formula

$$\hat{\mu}(t) = \frac{1}{t}\int_0^t \hat{m}(u)\,du \qquad (t > 0),$$

which shows that the function $\hat{\mu}$ is differentiable on $(0, \infty)$. The theorem is thus proved.

As an application of Lemma 4.2 and Theorem 4.4 we deduce the following result:

COROLLARY 4.1. *Let* $\circ$ *be a generalized convolution other than the max-convolution. If* $\mu \in P^* \cap \Delta(\circ)$ *and* $\nu \in P$ *with* $\nu(\{0\}) = 0$, *then* $\mu\nu \in \Delta(\circ)$.

We conclude this section by some examples.

EXAMPLE 4.1. Taking into account Example 1.1 we get the equality

$$\Delta(*_p) = P \tag{4.9}$$

whenever $0 < p < \infty$. Moreover, by Example 1.2, we have $\Delta(*_\infty) \neq P$.

EXAMPLE 4.2. It is easy to verify, by Example 1.2, that $\varrho(p, q) \in \Delta(*_\infty)$ and $\varrho(p, q) \in P^*$ for all $p, q > 0$. Applying Theorem 4.4 we get the relation

$$\varrho(p, q) \in D(\circ) \tag{4.10}$$

for all $p, q > 0$ and all generalized convolutions $\circ$.

Before we go further we establish some results for Kingman convolutions.

LEMMA 4.5. *If* $1 \leqslant q < r$ *and* $p > 0$, *then the inclusions*

$$P_+(*_{p,q}) \subset P_+(*_{p,r}) \tag{4.11}$$

and

$$P^* \cap \Delta(*_{p,q}) \subset \Delta(*_{p,r}) \tag{4.12}$$

are true.

Proof. Given $a, b, c > 0$ we denote by $\lambda(a, b, c)$ the probability measure concentrated on the interval $[0, 1]$ with the density function

$$2aB(b, c)^{-1} x^{2ab-1}(1-x^{2a})^{c-1}.$$

From the Sonine integral (see [2], 7.7 (5)) and (1.5) we get the formula

$$\Omega_{p,r}(t) = \int_0^1 \Omega_{p,q}(tx)\lambda(p, q-1/2, r-q)(dx).$$

Consequently, denoting by $\mu \to \hat{\mu}$ and $\mu \to \tilde{\mu}$ the characteristic functions for the Kingman convolutions $*_{p,r}$ and $*_{p,q}$, respectively, and taking into account (1.3), we get the equality

$$\hat{\mu}(t) = \int_0^1 \tilde{\mu}(tx)\lambda(p, q-1/2, r-q)(dx) = \big(\mu\lambda(p, q-1/2, r-q)\big)^{\sim}(t), \tag{4.13}$$

which yields inclusion (4.11). Suppose now that $\mu \in P^* \cap \Delta(*_{p,q})$. Then, by Corollary 4.1, $\mu\lambda(p, q-1/2, r-q) \in \Delta(*_{p,q})$. Comparing this with (4.13) we get inclusion (4.12), which completes the proof.

EXAMPLE 4.3. *Let* $p, q > 0$. *Then*

$$\varrho(p, q) \in \Delta(*_{p,r}) \quad \textit{if } r \geqslant \max(1, q/p) \tag{4.14}$$

and

(4.15) $\varrho(p,q)\notin P_+(*_{p,r})$ *if* $q>p$ *and* $1\leqslant r<q/p$.

Proof. Given $p>0$ and $r\geqslant 1$ we denote by $\mu\to\hat{\mu}$ the characteristic function for the Kingman convolution $*_{p,r}$. From the Hankel integral (see [2], 7.3 (16)) for every $q>0$ we get the formula

$$(4.16)\quad \hat{\varrho}(p,q)(t)=(1+t^{2p})^{-q/2p}\,{}_2F_1\big(q/2p,(rp-p-q)/2p;\ r/2;\ t^{2p}/(1+t^{2p})\big),$$

where ${}_2F_1$ denotes the hypergeometric function.

First consider the case $q\geqslant p$. Then setting $r=q/p$ we have, by (4.16) and formula 2.8 (4) in [1],

$$\hat{\varrho}(p,q)(t)=(1+t^{2p})^{-(p+q)/2p},$$

which shows that $\varrho(p,q)\in\Delta(*_{p,q/p})$. Since $\varrho(p,q)\in P^*$, using Lemma 4.5 (inclusion (4.12)) we get relation (4.14) for $q\geqslant p$.

Suppose now that $q<p$ and put $r=1$ into (4.16). Then, by [1], 2.8 (11),

$$\hat{\varrho}(p,q)(t)=(1+t^{2p})^{-q/2p}\cos\frac{q}{p}\left(\arcsin\left(t^p/\sqrt{1+t^{2p}}\right)\right),$$

which, by a standard calculation, shows that $\varrho(p,q)\in\Delta(*_{p,1})$. Arguing as before we get relation (4.14) for $q<p$.

Now assume that $q>p$ and $\max(1,q/p-2)<s<q/p$. Then

$$\Gamma\big((sp-q)/2p\big)<0.$$

Setting $r=s$ into (4.16) we have, by formula 2.8 (46) in [1],

$$\lim_{t\to\infty}(1+t^{2p})^{q/2p}\hat{\varrho}(p,q)(t)={}_2F_1\big(q/2p,(sp-p-q)/2p;\ s/2;\ 1\big)$$
$$=\Gamma(s/2)\Gamma(1/2)\Gamma\big((sp-q)/2p\big)^{-1}\Gamma\big((p+q)/2p\big)^{-1}<0.$$

Consequently, $\varrho(p,q)\notin P_+(*_{p,s})$, which, by Lemma 4.5 (inclusion (4.11)), yields relation (4.15). This completes the proof.

5. Characterization theorems. We start by the definition of an equivalence relation. Given $s>0$ by $[\circ,\mu]_s$ we denote the pair consisting of a generalized convolution $\circ$ and a probability measure μ from $Q_s(\circ)$. Two pairs $[\circ,\mu]_s$ and $[\Box,\nu]_s$ are said to be *equivalent*, in symbols $[\circ,\mu]_s\sim[\Box,\nu]_s$, if $\pi_s(\mu^{\circ n})=\pi_s(\nu^{\Box n})$ $(n=1,2,\ldots)$.

THEOREM 5.1. *Let* $\mu\to\hat{\mu}$ *and* $\nu\to\hat{\nu}$ *be characteristic functions for generalized convolutions* $\circ$ *and* $\Box$, *respectively. Suppose that*

$$\mu\in F_s(\circ)\cap Q_s(\circ),\qquad \nu\in F_s(\Box)\cap Q_s(\Box)$$

and

$$(5.1)\qquad f_s(\hat{\mu}^n)=bf_s(\hat{\nu}^n)\quad (n=1,2,\ldots)$$

for a certain constant $b>0$. *Then* $[\circ, \mu]_s \sim [\square, \delta_a \nu]_s$, *where*

$$a = \left(\pi_s(\nu)/\pi_s(\mu)\right)^{1/s}. \tag{5.2}$$

Proof. From (5.1) and Theorem 2.2 we obtain the equalities

$$\pi_s(\mu^{\circ n}) = c\pi_s(\nu^{\square n}) \qquad (n = 1, 2, \ldots)$$

for a certain constant c. Setting $n=1$ and taking the constant a defined by (5.2) we have $c = a^{-s}$. Consequently, by (2.2),

$$\pi_s(\mu^{\circ n}) = a^{-s}\pi_s(\nu^{\square n}) = \pi_s\left((\delta_a \nu)^{\square n}\right) \qquad (n = 1, 2, \ldots),$$

which completes the proof.

As an application of Theorem 5.1 we get the following result:

EXAMPLE 5.1. *Suppose that* $0 < p \leqslant q$ *and* $0 < s < q$. *Then*

$$[*_{p,q/p}, \varrho(p, q)]_s \sim [*_{p,2+q/p}, \delta_a \varrho(p, p+q)]_s \sim [*_{2p}, \delta_b \varrho(2p, p+q)]_s, \tag{5.3}$$

where $a = (1-s/q)^{1/s}$ *and*

$$b = \Gamma\left((p+q-s)/2p\right)^{1/s} \Gamma(q/p)^{1/s} \Gamma\left((p+q)/2p\right)^{-1/s} \Gamma\left((q-s)/p\right)^{-1/s}.$$

Proof. By a standard calculation we get the formula

$$\pi_s\left(\varrho(p, q)\right) = \Gamma\left((q-s)/p\right)/\Gamma(q/p) \quad \text{for } 0 < s < q. \tag{5.4}$$

By (4.14) and Theorem 4.2 we have the relation

$$\varrho(p, q) \in P_+(*_{p,r}) \qquad \text{for } r \geqslant q/p,$$

which, by Corollary 2.1 and Theorem 3.3, yields

$$\varrho(p, q) \in F_s(*_{p,r}) \cap Q_s(*_{p,r}) \quad \text{for } r \geqslant q/p \text{ and } 0 < s < q. \tag{5.5}$$

A similar reasoning leads to the relation

$$\varrho(p, q) \in F_s(*_r) \cap Q_s(*_r) \quad \text{for } r > 0 \text{ and } 0 < s < q. \tag{5.6}$$

Denote by $\mu \to \hat{\mu}$, $\mu \to \tilde{\mu}$ and $\mu \to \check{\mu}$ the characteristic functions for the generalized convolutions $*_{p,q/p}$, $*_{p,2+q/p}$ and $*_{2p}$, respectively. Setting $r = q/p$ into (4.16) we obtain $\hat{\varrho}(p, q)$. Similarly, setting $r = 1+q/p$ into (4.16) and replacing q by $p+q$ we get $\tilde{\varrho}(p, p+q)$. Further, by a standard calculation, we get $\check{\varrho}(2p, p+q)$. This yields the equality

$$\hat{\varrho}(p, q)(t) = \tilde{\varrho}(p, p+q)(t) = \check{\varrho}(2p, p+q)(t) = (1+t^{2p})^{-(p+q)/2p},$$

which, by (5.5), (5.6) and Theorem 5.1, proves relations (5.3). The constants a and b can be calculated by means of formulae (5.2) and (5.4).

The following result plays a crucial role in our considerations:

THEOREM 5.2. *Let* $\mu \to \hat{\mu}$ *and* $\nu \to \tilde{\nu}$ *be s-normed characteristic functions for generalized convolutions* $\circ$ *and* $\square$, *respectively. If* $[\circ, \mu]_s \sim [\square, \nu]_s$, *then*

$$\hat{\mu} m_s = \tilde{\nu} m_s \tag{5.7}$$

on the set $[-1, 0) \cup (0, 1]$. *If in addition* $\mu \in \Delta(\circ)$ *and* $\nu \in D(\Box)$, *then*

$$\hat{\mu}(t) = \tilde{\nu}(t) \quad \text{for } t \in [0, \infty). \tag{5.8}$$

Proof. Since $\mu \in Q_s(\circ)$, we have, by Corollary 2.1 and Lemma 2.4, $\mu^{\circ n} \in F_s(\circ)$ for $n \geqslant 2$. Consequently, by Theorem 2.2,

$$\pi_s(\mu^{\circ n}) = f_s(\hat{\mu}^n) = \int_0^\infty \hat{\mu}^n(t)\, t^{s-1}\, dt = \int_{-1}^{1} x^n (\hat{\mu} m_s)(dx) \quad \text{for } n \geqslant 2.$$

Thus

$$\int_{-1}^{1} x^2 e^{zx} (\hat{\mu} m_s)(dx) = \sum_{n=0}^{\infty} \pi_s(\mu^{\circ(n+2)})\, z^n/n!,$$

which shows that the equivalence relation $[\circ, \mu]_s \sim [\Box, \nu]_s$ yields the equality of the Laplace transforms of the measures $x^2(\hat{\mu} m_s)(dx)$ and $x^2(\tilde{\nu} m_s)(dx)$. This implies equality (5.7).

Finally, suppose that $\mu \in \Delta(\circ)$ and $\nu \in D(\Box)$. Then equality (5.8) is an immediate consequence of Theorem 4.1. This completes the proof.

We are now in a position to prove the following characterization theorem:

THEOREM 5.3. *Let* $\mu \in \Delta(\circ) \cap Q_s(\circ)$ *and* $\nu \in D(\circ)$. *Then the equalities* $\pi_s(\mu^{\circ n}) = \pi_s(\nu^{\circ n})$ $(n = 1, 2, \ldots)$ *yield* $\mu = \nu$.

Proof. The conditions of the theorem can be written in the form $[\circ, \mu]_s \sim [\circ, \nu]_s$. Applying Theorem 5.2 we get $\hat{\mu} = \hat{\nu}$ and, consequently, $\mu = \nu$, which completes the proof.

In the sequel the following lemma will be used:

LEMMA 5.1. *Let* $\mu \to \hat{\mu}$ *and* $\nu \to \tilde{\nu}$ *be characteristic functions for generalized convolutions* $\circ$ *and* $\Box$, *respectively. If* $\hat{\mu}(t) = \tilde{\mu}(t)$ *for all* $t \in [0, \infty)$ *and a certain* $\mu \in P_s$ *with* $s \in I(\circ) \cap I(\Box)$, *then* $\circ = \Box$.

Proof. We can find two probability measures η and σ fulfilling the conditions $\eta \in S(\circ)$, $\sigma \in S(\Box)$ and $\hat{\eta}(t) = \tilde{\sigma}(t) = \exp(-t^p)$ for some $p > 0$. Of course, $\eta, \sigma \in P_s$ and, by (1.3),

$$\int_0^\infty \exp(-t^p x^p)(\mu\sigma)(dx) = (\mu\sigma\eta)^\wedge(t) = \int_0^\infty \hat{\mu}(tx)(\sigma\eta)(dx)$$
$$= \int_0^\infty \tilde{\mu}(tx)(\sigma\eta)(dx) = (\mu\sigma\eta)^\sim(t) = \int_0^\infty \exp(-t^p x^p)(\mu\eta)(dx),$$

which, by the uniqueness theorem for the Laplace transformation, yields the equality $\mu\eta = \mu\sigma$. Consequently, by (2.2), $\pi_r(\mu)\pi_r(\eta) = \pi_r(\mu)\pi_r(\sigma)$ for $0 < r \leqslant s$. Taking into account (2.1) we conclude that $\pi_r(\eta) = \pi_r(\sigma)$ for $0 < r \leqslant s$, which, by the uniqueness theorem for the Mellin transformation, yields the equality $\eta = \sigma$. Thus $\hat{\eta}(t) = \tilde{\eta}(t) = \exp(-t^p)$. Given an arbitrary measure λ from P, we have

$$\int_0^\infty \hat{\lambda}(tx)\,\eta(dx) = \int_0^\infty \hat{\eta}(tx)\,\lambda(dx) = \int_0^\infty \tilde{\eta}(tx)\,\lambda(dx) = \int_0^\infty \tilde{\lambda}(tx)\,\eta(dx).$$

Applying Lemma 2.2 of [5] we conclude that $\hat{\lambda} = \tilde{\lambda}$. Thus for every pair $\lambda, \nu \in P$ we have $(\lambda \circ \nu)^\wedge = \hat{\lambda}\hat{\nu} = \tilde{\lambda}\tilde{\nu} = (\lambda \square \nu)^\sim = (\lambda \square \nu)^\wedge$, which yields the equality $\lambda \circ \nu = \lambda \square \nu$. This completes the proof.

THEOREM 5.4. *Let* $\circ$ *and* $\square$ *be a pair of generalized convolutions. Suppose that* $\mu \in \Delta(\circ) \cap Q_s(\circ)$, $\nu \in D(\square)$ *and the measures* μ *and* ν *are similar. Then the equalities* $\pi_s(\mu^{\circ n}) = \pi_s(\nu^{\square n})$ $(n = 1, 2, \ldots)$ *yield* $\circ = \square$.

Proof. Denoting by $\lambda \to \hat{\lambda}$ and $\lambda \to \tilde{\lambda}$ s-normed characteristic functions for the convolutions $\circ$ and $\square$, respectively, we have, by Theorem 5.2, the equality $\hat{\mu}(t) = \tilde{\nu}(t)$ for all $t \in [0, \infty)$. By the assumption we obtain $\nu = \delta_c \mu$ for some $c > 0$. We define a new characteristic function $\lambda \to \check{\lambda}$ for the convolution $\square$ by setting $\check{\lambda} = (\delta_c \lambda)^\sim$. Then $\hat{\mu}(t) = \check{\mu}(t)$ for $t \in [0, \infty)$, which, by Lemma 5.1, yields the assertion of the theorem.

We shall now illustrate the above theorems by some examples.

EXAMPLE 5.2. *Let* $\circ$ *be a generalized convolution,* $s \in I(\circ)$ *and* $0 < p < \varkappa(\circ)$. *If* $\mu \in D(\circ)$ *and* $\pi_s(\mu^{\circ n}) = bn^{-s/p}$ $(n = 1, 2, \ldots)$ *for a certain constant* b, *then* $\mu \in S^*(\circ)$.

Proof. Let ν be a $\circ$-stable measure with exponent p and the characteristic function of the form (1.4). Then, by Theorem 2.2,

$$\pi_s(\nu^{\circ n}) = c_s \int_0^\infty \exp(-cnt^p) t^{s-1}\, dt = c^{-s/p} c_s \Gamma(s) n^{-s/p} \qquad (n = 1, 2, \ldots).$$

Taking the constant c satisfying the condition $c^{-s/p} c_s \Gamma(s) = b$ we have $\pi_s(\mu^{\circ n})$ $(n = 1, 2, \ldots)$, which, by Theorem 5.3, yields $\mu = \nu$. This completes the proof.

EXAMPLE 5.3. *Let* $\circ$ *and* $\square$ *be a pair of generalized convolutions. Suppose that* $s \in I(\circ)$, μ *is a* $\circ$-*stable measure with exponent* $p < \varkappa(\circ)$, *and* $\pi_s(\mu^{\square n}) = bn^{-s/p}$ $(n = 1, 2, \ldots)$ *for a certain constant* b. *Then* $\circ = \square$.

Proof. Arguing as before we can find a measure ν similar to μ such that $\pi_s(\nu^{\circ n}) = bn^{-s/p}$ $(n = 1, 2, \ldots)$. Now our assertion is an immediate consequence of Theorem 5.4.

EXAMPLE 5.4. *Let* $p > 0$ *and* $0 < s < q$. *If a generalized convolution* $\circ$ *fulfils the condition* $\pi_s(\varrho(p, q)^{\circ n}) = b\Gamma((nq-s)/p)/\Gamma(nq/p)$ $(n = 1, 2, \ldots)$ *for a certain constant* b, *then* $\circ = *_p$.

Proof. It is easy to check the formula $\varrho(p, q) *_r \varrho(p, r) = \varrho(p, q+r)$. Consequently, by (5.4),

$$\pi_s(\varrho(p, q)^{*_p n}) = \Gamma((nq-s)/p)\Gamma(nq/p) \qquad (n = 1, 2, \ldots).$$

Observe that, by (4.9) and (4.10), $\varrho(p, q) \in \Delta(*_p) \cap D(\circ)$. Now our assertion is an immediate consequence of Theorem 5.4.

EXAMPLE 5.5. *Let $p > 0$ and $0 < s < q$. If a generalized convolution $\circ$ fulfils the condition $\pi_s(\varrho(p, q)^{\circ n}) = b\Gamma((nq-s)/2p)/\Gamma(nq/2p)$ $(n = 1, 2, \ldots)$ for a certain constant b, then $\circ = *_{p,1+q/p}$.*

Proof. Denoting by $\mu \to \hat{\mu}$ the characteristic function for the Kingman convolution $*_{p,1+q/p}$ and setting $r = 1+q/p$ into (4.16) we get the formula

$$\hat{\varrho}(p, q)(t) = (1+t^{2p})^{-q/2p}.$$

Thus, by an integral representation of the beta function (see [1], 1.5 (2)), we get

$$f_s(\hat{\varrho}(p, q)^n) = \int_0^\infty t^{s-1}(1+t^{2p})^{-nq/2p}\,dt = \frac{1}{2p}B(s/2p, (nq-s)/2p),$$

which yields, by Theorem 2.2,

$$\pi_s(\varrho(p, q)^{*_{p,1+q/p}n}) = \pi_s((\delta_c\varrho(p, q))^{\circ n}) \quad (n = 1, 2, \ldots)$$

for a certain constant c. Observe that, by (4.10), (4.14) and (5.5),

$$\varrho(p, q) \in D(\circ) \cap \Delta(*_{p,1+q/p}) \cap Q_s(*_{p,1+q/p}).$$

Now our assertion is an immediate consequence of Theorem 5.4.

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