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APPROXIMATION BY PENULTIMATE STABLE LAWS

BY

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Abstract. In certain cases partial sums of i.i.d. random variables with finite variance are better approximated by a sequence of stable distributions with indices $\alpha_n \to 2$ than by a normal distribution. We discuss when this happens and how much the convergence rate can be improved by using penultimate approximations. Similar results are valid for other stable distributions.

1. Introduction. Let $X_1, X_2, \ldots$ be independent random variables with common distribution function F. We assume that F is either in the domain of attraction of a stable law with index less than 2, that is

$$\lim_{t\to\infty} \frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} = x^{-\alpha}, \quad x > 0,$$

(1.1)

$$\lim_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} = p,$$

for some parameters $\alpha \in (0, 2)$ and $p \in [0, 1]$, or in the domain of attraction of a normal law, i.e.

$$S(x) := \int_0^x \big(1-F(u)+F(-u)\big)u\,du \in RV_0.$$

Then there exist $a_n > 0$ and $b_n \in R$ such that

$$\lim_{n\to\infty} P\{\sum_{i=1}^{n} X_i/a_n - b_n \leqslant x\} = G_\alpha(x) \tag{1.2}$$

for all x, where G_α is a stable distribution function for $\alpha \in (0, 2)$ and

$$G_2(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\{-u^2/2\}\,du.$$

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Rate of convergence results in connection with (1.2) can be given under second order conditions. First let us concentrate on the case $\alpha < 2$.

Suppose there exists a function A with $\lim_{t\to\infty} A(t) = 0$ and not changing sign near infinity, such that

$$\lim_{t\to\infty} \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha}}{A(t)} = x^{-\alpha}\frac{x^{\varrho}-1}{\varrho}, \qquad x>0, \tag{1.3}$$

$$\lim_{t\to\infty} \frac{\frac{1-F(t)}{1-F(t)+F(-t)} - p}{A(t)} = q.$$

Here q is a real constant. The function $|A|$ is then regularly varying with non-positive index ϱ (notation: $|A(t)| \in RV_{\varrho}$).

De Haan and Peng [4] proved that under condition (1.3) for a suitable choice of the sequences a_n and b_n

$$\lim_{n\to\infty} \sup_{x\in R} \left|P\left\{\sum_{i=1}^{n} X_i/a_n - b_n \leqslant x\right\} - G_{\alpha}(x)\right| / |A(a_n)| \tag{1.4}$$

exists and is positive.

Now the question is: can we improve the convergence rate by using a sequence of stable distribution function G_{α_n} with $\alpha_n \to \alpha$ instead of G_{α} in relation (1.4)? In order to answer this question we note that an intermediate step in settling (1.4) is a second order relation for the characteristic function of F,

$$f(t) := \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

We take as an example the case $1 < \alpha < 2$ and $1-F(x) = F(-x)$ for $x > 0$. In this case the relation for f is the following (see Lemma 1 of de Haan and Peng [4]):

$$\lim_{n\to\infty} \frac{-n\log f(t/a_n) + \log g_{\alpha}(t)}{A(a_n)} = |t|^{\alpha}\left(s_{\alpha} + d_{\alpha-\varrho}\frac{|t|^{-\varrho}-1}{\varrho}\right), \tag{1.5}$$

where

$$g_{\alpha}(t) = \exp\{-|t|^{\alpha}\Gamma(1-\alpha)\cos(\pi\alpha/2)\}$$

is the characteristic function of $G_{\alpha}(x)$ and

$$d_{\alpha} := \int_0^{\infty} x^{-\alpha}\sin x\, dx = \Gamma(1-\alpha)\sin\frac{\pi(1-\alpha)}{2} \qquad (0<\alpha<2)$$

$$s_{\alpha} := \int_0^{\infty} x^{-\alpha}\log x\, \sin x\, dx$$

$$= \Gamma(1-\alpha)\sin\frac{\pi(1-\alpha)}{2}\left\{\frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} + \frac{\pi}{2}\operatorname{ctg}\frac{\pi(1-\alpha)}{2}\right\} \qquad (0<\alpha<2).$$

We want to replace α by a sequence α_n for which

(1.6) $$\lim_{n\to\infty} \frac{-n\log f(t/a_n)+\log g_{\alpha_n}(t)}{A(a_n)} = 0.$$

Note that for $n\to\infty$

$$\begin{aligned}
-\log g_{\alpha_n}(t)+\log g_\alpha(t) &= |t|^{\alpha_n}\Gamma(1-\alpha_n)\cos\frac{\pi\alpha_n}{2}-|t|^\alpha\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\\
&= (|t|^{\alpha_n}-|t|^\alpha)\Gamma(1-\alpha_n)\cos\frac{\pi\alpha_n}{2}\\
&\quad+|t|^\alpha\left(\Gamma(1-\alpha_n)\cos\frac{\pi\alpha_n}{2}-\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\right)\\
&\sim (\alpha_n-\alpha)|t|^\alpha\log|t|\,\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\\
&\quad-(\alpha_n-\alpha)|t|^\alpha\left[\Gamma'(1-\alpha)\cos\frac{\pi\alpha}{2}+\frac{\pi}{2}\Gamma(1-\alpha)\sin\frac{\pi\alpha}{2}\right]\\
&= (\alpha_n-\alpha)|t|^\alpha\log|t|\,d_\alpha-(\alpha_n-\alpha)|t|^\alpha s_\alpha.
\end{aligned}$$

This shows that if we take $\alpha_n:=\alpha-A(a_n)$,

$$\lim_{n\to\infty}\frac{-\log g_{\alpha_n}(t)+\log g_\alpha(t)}{A(a_n)} = \lim_{n\to\infty}\frac{-n\log f(t/a_n)+\log g_\alpha(t)}{A(a_n)}$$

when $\varrho=0$. So with that choice

$$\lim_{n\to\infty}\frac{-n\log f(t/a_n)+\log g_{\alpha_n}(t)}{A(a_n)} = 0,$$

i.e. the convergence rate can be improved.

If ϱ is less than zero, then for no choice of α_n cancellation is possible, so we cannot improve the convergence rate in this case.

Next let us consider the case $\alpha=2$. Since $x^{-2}S(x)\downarrow 0$ as $x\to\infty$, the function

$$a(x):=\sup\{a\colon 2a^{-2}S(a)\geqslant x^{-1}\}$$

is well defined for $x>1/2$. We have

(1.7) $$2x(a(x))^{-2}S(a(x)) = 1.$$

De Haan and Peng [5] proved that

(1.8) $$\lim_{n\to\infty}\sup_{x\in R}\frac{\left|P\left(\sum_{i=1}^n X_i/a(n)\leqslant x\right)-G_2(x)\right|}{n\left(1-F(a(n))+F(-a(n))\right)}$$

exists and is positive under the condition

$$\text{(1.9)}\qquad \begin{gathered} 1-F(x)+F(-x)\in RV_{\varrho-2} \quad (-1<\varrho\leqslant 0),\\ \lim_{x\to\infty}\frac{1-F(x)}{1-F(x)+F(-x)}=p^*\in[0,1]. \end{gathered}$$

Using the same arguments as in the case $\alpha<2$, we find that the rate of (1.8) can be improved only in the case $\varrho=0$ of (1.9).

The result in Section 2 shows that for $1<\alpha<2$ the convergence rate can be improved 'a little' if the condition (1.3) holds for $\varrho=0$, that is, if the convergence rate is slow. In that case the convergence rate $A(a_n)$ is replaced by $\{A(a_n)\}^2$. See also Remark 2.2 about the case $0<\alpha\leqslant 1$.

In Section 3 we consider the normal limit distribution. We shall show that if (1.9) holds for $\varrho=0$ the convergence rate can be improved 'a little' when one approximates by a sequence of stable distributions with $\alpha_n\to 2$ instead of by the normal distribution. In that case the rate $n\big(1-F(a(n))+F(-a(n))\big)$ is replaced by $\big[n\big(1-F(a(n))+F(-a(n))\big)\big]^2$. The phenomenon has been observed before in Iglesias Pereira et al. [10] and Oliveira [8].

2. Main result for $\alpha\in(1,2)$. Throughout this section we assume that $\alpha\in(1,2)$ (but see Remark 2.2 for $0<\alpha\leqslant 1$) and $EX_1=0$. We now need an even more stringent condition than the second order condition (1.3). In fact, we need a third order condition: suppose there exists a function $A_0(t)$ with $\lim_{t\to\infty}A_0(t)=0$ and not changing sign near infinity such that

$$\text{(2.1)}\qquad \lim_{t\to\infty}\frac{\dfrac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)}-x^{-\alpha}}{A(t)}-x^{-\alpha}\log x}{A_0(t)}=H(x),$$

where $H(x)$ is not a multiple of $x^{-\alpha}\log x$ and suppose that

$$\text{(2.2)}\qquad \lim_{t\to\infty}\frac{\dfrac{1-F(t)}{1-F(t)+F(-t)}-p}{A^2(t)}=q_0\in(-\infty,\infty).$$

Note that (2.1) is equivalent to

$$\text{(2.3)}\qquad \lim_{t\to\infty}\frac{(tx)^{\alpha}K(tx)-t^{\alpha}K(t)-A(t)\,t^{\alpha}K(t)\log x}{A(t)\,t^{\alpha}K(t)\,A_0(t)}=x^{\alpha}H(x),$$

where $K(x):=1-F(x)+F(-x)$. From Theorem 1 of de Haan and Stadtmüller [7] we can assume that

$$\text{(2.4)}\qquad H(x)=x^{-\alpha}\frac{1}{\varrho'}\left[\frac{x^{\varrho'}-1}{\varrho'}-\log x\right]\quad(\varrho'\leqslant 0).$$

Let us denote by $U(t)$ the generalized inverse of the function $1/(1-F(t)+F(-t))$. If (1.1) holds, $1<\alpha<2$ and $EX_1=0$, the sequence $\sum_{i=1}^{n} X_i/U(n)$ converges in distribution to G_α whose characteristic function is

$$g_\alpha(t) := \exp\left\{-|t|^\alpha \Gamma(1-\alpha)\left[\cos\frac{\pi\alpha}{2} - i\,\mathrm{sgn}(t)(2p-1)\sin\frac{\pi\alpha}{2}\right]\right\},$$

where

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{for } t \geqslant 0, \\ -1 & \text{for } t < 0. \end{cases}$$

Now we can state our main results.

THEOREM 2.1. *Let F be a non-lattice distribution function. Suppose* (2.1), (2.2) *and* (2.4) *hold for some $1<\alpha<2$ and $\varrho'<0$. Then (recall $EX_1=0$)*

$$\text{(2.5)} \quad \lim_{n\to\infty} \{A(U(n))\}^{-2}\left[P\left(\sum_{i=1}^{n} X_i/U(n) \leqslant x\right) - G_{\alpha-A(U(n))}(x)\right]$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}}{-it}\, g_\alpha(t)\big(C_1(t)+i\,\mathrm{sgn}(t)\,C_2(t)\big)\,dt$$

uniformly for all x, where

$$C_1(t) = \int_0^\infty \left[-(x/|t|)^{-\alpha}\big(\log(x/|t|)\big)^2/2\right]\sin x\,dx$$

and

$$C_2(t) = \int_0^\infty \left[-(2p-1)(x/|t|)^{-\alpha}\big((\log(x/|t|))^2/2\big) + 2q_0(x/|t|)^{-\alpha}\right](1-\cos x)\,dx.$$

Remark 2.1. Suppose (2.1), (2.2) and (2.4) hold for $\varrho'=0$. Assume

$$\lim_{t\to\infty} A_0(t)/A(t) = c \in (-\infty, \infty).$$

Then the left-hand side of (2.5) still exists, but the limit function is different.

Remark 2.2. For $\alpha \leqslant 1$ we also have a version of Theorem 2.1 when F is assumed to be symmetric (cf. Remark 5 of de Haan and Peng [5]).

3. Main result for $\alpha = 2$. We assume throughout this section that $EX_1 = 0$ and that G_α^* is the stable law with characteristic function

$$g_\alpha^* = \exp\left\{-|t|^\alpha(1-\alpha/2)\Gamma(1-\alpha)\left[\cos\frac{\pi\alpha}{2} - i\,\mathrm{sgn}(t)(2p^*-1)\sin\frac{\pi\alpha}{2}\right]\right\}.$$

Define $\alpha_n^* := 2-2n\big(1-F(a(n))+F(-a(n))\big)$. We now need a condition stronger than (1.9). Suppose there exists a function $A^*(t)$ with $\lim_{t\to\infty} A^*(t) = 0$

and not changing sign near infinity such that

(3.1)
$$\lim_{t\to\infty}\frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)}-x^{-2}}{A^*(t)}=x^{-2}\,\frac{x^{\varrho^*}-1}{\varrho^*},\qquad x>0,$$
$$\lim_{t\to\infty}\frac{\dfrac{1-F(t)}{1-F(t)+F(-t)}-p^*}{t\big(1-F(a(t))+F(a(t))\big)}=q^*,$$

where $\varrho^*\leqslant 0$ and q^* is a real constant.

Now we state our main theorem.

THEOREM 3.1. *Let F be a non-lattice distribution function. Suppose* (3.1) *holds for $\varrho^*<0$. Then*

(3.2)
$$\lim_{n\to\infty}\frac{P\left(\sum_{i=1}^{n}X_i/a(n)\leqslant x\right)-G^*_{\alpha_n^*}(x)}{\left[n\big(1-F(a(n))+F(-a(n))\big)\right]^2}$$
$$=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}}{-it}\exp\{-t^2/2\}\,[C_1^*(t)+i\,\mathrm{sgn}(t)\,C_2^*(t)]\,dt$$

uniformly for all x, where

$$C_1^*(t)=-2\int_{|t|}^{\infty}(x/|t|)^{-2}\log(x/|t|)\sin x\,dx-2\int_{0}^{|t|}(x/|t|)^{-2}\log(x/|t|)(\sin x-x)\,dx$$

and

$$C_2^*(t)=\int_{0}^{\infty}[-2(2p^*-1)(x/|t|)^{-2}\log(x/|t|)+2q^*(x/|t|)^{-2}](1-\cos x)\,dx.$$

Remark 3.1. Suppose (3.1) holds for $\varrho^*=0$. Assume that

$$\lim_{t\to\infty}A_0^*(t)/\left[n\big(1-F(a(n))+F(-a(n))\big)\right]=c_0\in(-\infty,\infty).$$

Then the left-hand of (3.2) still exists, but the limit function is different.

4. Proofs. The line of reasoning is as follows. The starting point is Lemma 4.1 which gives a reformulation of condition (2.1) for F suitable for our purposes. The next step is to give an equivalent relation for $1-f$, where f is the characteristic function of F. This involves application of Lebesgue's theorem on dominated convergence. The dominating function for this application is obtained in Lemmas 4.3–4.5. Next the limit relation for $1-f$ is translated into a relation for $-\log f$, hence for f^n, the characteristic function of the n-fold convolution of F (Lemma 4.6). The necessary inequalities for the last step, translating this relation into the promised limit relation for the n-fold convolution of F, are developed in Lemma 4.7. The proof of this last step is similar to the corresponding step in de Haan and Peng [4] and is omitted.

Lemmas 4.8–4.11 present a somewhat similar development for the case of the normal distribution.

LEMMA 4.1. *Suppose that* (2.1) *and* (2.4) *hold for* $\varrho' < 0$. *Then for* $x > 0$

$$\lim_{t\to\infty} \frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} = -x^{-\alpha}(\log x)^2/2.$$

Proof. Relation (2.1) implies that $A(t)$ is slowly varying and $A_0(t)$ is ϱ'-varying, and hence

$$\lim_{t\to\infty} A_0(t)/A(t) = 0. \tag{4.1}$$

Note that

$$\frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} \tag{4.2}$$

$$= \frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha} - A(t)\,x^{-\alpha}\log x}{A(t)\,A_0(t)} \cdot \frac{A_0(t)}{A(t)}$$

$$- \frac{x^{-\alpha+A(t)} - x^{-\alpha} - A(t)\,x^{-\alpha}\log x}{A^2(t)}.$$

We now use (2.1), (4.1), (4.2) and

$$x^y - x^{y_0} - (y-y_0)\,x^{y_0}\log x = \tfrac{1}{2}(y-y_0)^2\, x^{y_0+\theta(y-y_0)}(\log x)^2 \tag{4.3}$$

for all $x > 0$, where $\theta \in [0, 1]$. Lemma 4.1 follows easily since $\lim_{t\to 0} A(t) = 0$. ∎

LEMMA 4.2. *Suppose* (2.1), (2.2) *and* (2.4) *hold for* $\varrho' < 0$. *Then*

$$\frac{n\big(1-F(U(n)x)+F(-U(n)x)\big)-x^{-\alpha+A(U(n))}}{A^2(U(n))} \to -x^{-\alpha}(\log x)^2/2 \tag{4.4}$$

and

$$\frac{n\big(1-F(U(n)x)-F(-U(n)x)\big)-(2p-1)\,x^{-\alpha+A(U(n))}}{A^2(U(n))} \tag{4.5}$$

$$\to -(2p-1)\,x^{-\alpha}(\log x)^2/2 + 2q_0\,x^{-\alpha}.$$

The proof is similar to the proof of Proposition 2 of de Haan and Peng [4], using Lemma 4.1 and (2.2). ∎

The following lemma is an extension of a result of Drees [2].

LEMMA 4.3. *Let l be a measurable function. Suppose there exist a real parameter γ and functions $a_1(t) > 0$ and $a_2(t) \to 0$ with constant sign near infinity such* *that for all* $x>0$

$$\lim_{t\to\infty}\frac{[l(tx)-l(t)]/a_1(t)-(x^\gamma-1)/\gamma}{a_2(t)}=\tilde{h}(x)$$

exists as a finite limit and $\tilde{h}(x)$ *is not a multiple of* $(x^\gamma-1)/\gamma$. *The function* a_1 *is regularly varying of index* γ, *and* $|a_2(t)|$ *is regularly varying of index* $\beta\leqslant 0$.

Then there exist functions $a_3(t)>0$ *and* $a_4(t)$ *(where* $|a_4(t)|>0$*) with the property that for all* $\varepsilon,\varepsilon'>0$ *there exists* $t_0>0$ *such that for all* $t\geqslant t_0$, $tx\geqslant t_0$,

$$x^{-\gamma-\beta}\exp\{-\varepsilon'|\log x|\}\left|\frac{[l(tx)-l(t)]/a_3(t)-(x^\gamma-1)/\gamma}{a_4(t)}-h(x)\right|\leqslant\varepsilon,$$

where

$$h(x)=\begin{cases}(\log x)^2/2 & \text{for } \beta=0,\ \gamma=0,\\ x^\gamma\log x & \text{for } \beta=0,\ \gamma\neq 0,\\ (x^{\gamma+\beta}-1)/(\gamma+\beta) & \text{for } \beta<0.\end{cases}$$

Proof. Suppose $\beta=0$ and $\gamma=0$. We proceed as in Omey and Willekens [9] and Drees [2]. Write

$$l_1(t):=l(t)-\frac{1}{t}\int_0^t l(s)\,ds. \tag{4.6}$$

Then l_1 is in the class Π (for the definition of class Π, see Geluk and de Haan [3]). Hence by de Haan and Pereira [6], Appendix, there exists a slowly varying function L with the property that for all $\varepsilon^{(1)},\varepsilon^{(2)}>0$ there exists $t_0>0$ such that for $t\geqslant t_0$, $tx\geqslant t_0$,

$$\exp\{-\varepsilon^{(2)}|\log x|\}\left|\frac{l_1(tx)-l_1(t)}{L(t)}-\log x\right|\leqslant\varepsilon^{(1)}. \tag{4.7}$$

Next note that (4.6) implies

$$l(t)=l_1(t)+\int_0^t\frac{l_1(s)}{s}\,ds.$$

Hence

$$\frac{l(tx)-l(t)-l_1(t)\log x-L(t)\log x}{L(t)}-(\log x)^2/2$$
$$=\frac{l_1(tx)-l_1(t)}{L(t)}-\log x+\int_1^x\left(\frac{l_1(ts)-l_1(t)}{sL(t)}-\frac{\log s}{s}\right)ds.$$

Choose $\varepsilon>0$. By (4.7), for $t\geqslant t_0$, $tx\geqslant t_0$ we have

$$\left|\frac{l(tx)-l(t)-l_1(t)\log x-L(t)\log x}{L(t)}-(\log x)^2/2\right|$$
$$\leqslant \varepsilon^{(1)}\exp\{\varepsilon^{(2)}|\log x|\}+\varepsilon^{(1)}\left|\int_1^x \exp\{\varepsilon^{(2)}|\log s|\}\frac{ds}{s}\right|$$
$$=\varepsilon^{(1)}\exp\{\varepsilon^{(2)}|\log x|\}+\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}}\left|\exp\{\varepsilon^{(2)}|\log x|\}-1\right|.$$

Let $\varepsilon^{(1)}/\varepsilon^{(2)} \leqslant \varepsilon$, $\varepsilon^{(2)} \leqslant \varepsilon \wedge \varepsilon'$. Then the expression is at most $\varepsilon\exp\{\varepsilon'|\log x|\}$.

For $\beta = 0$ and $\gamma > 0$, by Theorem 2 of de Haan and Stadtmüller [7] the function $t^{-\gamma}l(t)$ is in the class Π. Hence, by (4.7), for each ε, $\varepsilon' > 0$ there exists $t_0 > 0$ such that for $t \geqslant t_0$, $tx \geqslant t_0$

$$x^{-\gamma}\left|\frac{l(tx)-l(t)-\gamma l(t)[(x^\gamma-1)\gamma^{-1}]}{t^\gamma L(t)}-x^\gamma\log x\right|$$
$$=\left|\frac{(tx)^{-\gamma}l(tx)-t^{-\gamma}l(t)}{L(t)}-\log x\right| \leqslant \varepsilon e^{\varepsilon|\log x|}.$$

Similarly for $\beta = 0$ and $\gamma < 0$.

For $\beta < 0$, from Theorem 2 of de Haan and Stadtmüller [7] we have for some positive $\tilde{a}_1$ and all $x > 0$

$$\lim_{t\to\infty}\frac{l_2(tx)-l_2(t)}{\tilde{a}_1(t)}=\frac{x^{\gamma+\beta}-1}{\gamma+\beta}$$

with

$$l_2(t)=l(t)-c\,\frac{t^\gamma-1}{\gamma}\qquad (c>0).$$

Hence by de Haan and Pereira [6], Appendix, there exists $a_5(t) > 0$ with the property that for all ε, $\varepsilon' > 0$ there exists $t_0 > 0$ such that for $t \geqslant t_0$, $tx \geqslant t_0$

$$x^{-\gamma-\beta}\left|\frac{l(tx)-l(t)-ct^\gamma[(x^\gamma-1)\gamma^{-1}]}{a_5(t)}-\frac{x^{\gamma+\beta}-1}{\gamma+\beta}\right|$$
$$=x^{-\gamma-\beta}\left|\frac{l_2(tx)-l_2(t)}{a_5(t)}-\frac{x^{\gamma+\beta}-1}{\gamma+\beta}\right| \leqslant \varepsilon\exp\{-\varepsilon'|\log x|\}.$$

This completes the proof of the lemma. ∎

LEMMA 4.4. *Suppose the conditions of Lemma* 4.1 *hold. Then for any* $\varepsilon > 0$ *there exists* $t_0 > 0$ *such that for all* $t \geqslant t_0$, $tx \geqslant t_0$

$$\left|\frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)}-x^{-\alpha+A(t)}}{A^2(t)}\right| \leqslant \varepsilon x^{-\alpha}(|\log x|+e^{\varepsilon|\log x|})+\tfrac{1}{2}x^{-\alpha}(\log x)^2 e^{\varepsilon|\log x|}.$$

Proof. Note that (2.3) implies

$$\lim_{t\to\infty} \frac{(tx)^\alpha K(tx)-t^\alpha K(t)-A(t)t^\alpha K(t)\left(1+A_0(t)/\varrho'\right)\log x}{A(t)t^\alpha K(t)A_0(t)} = \frac{1}{\varrho'}\frac{x^{\varrho'}-1}{\varrho'}.$$

By Lemma 4.3 for any $\varepsilon>0$ there exist functions $a_1(t)$, $a_2(t)$ and $t_0>0$ such that for all $t\geqslant t_0$, $tx\geqslant t_0$

$$\left|\frac{(tx)^\alpha K(tx)-t^\alpha K(t)-a_1(t)\log x}{a_2(t)}-\frac{1}{\varrho'}\frac{x^{\varrho'}-1}{\varrho'}\right| \leqslant \varepsilon x^{\varrho'} e^{\varepsilon|\log x|}. \tag{4.8}$$

It is easy to see that

$$\frac{a_1(t)}{A(t)t^\alpha K(t)}\to 1, \qquad \frac{a_2(t)}{A(t)t^\alpha K(t)A_0(t)}\to 1,$$

$$\frac{a_1(t)-A(t)t^\alpha K(t)}{A(t)t^\alpha K(t)A_0(t)}\to 1/\varrho'.$$

Note that

$$\frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)}-x^{-\alpha+A(t)}}{A^2(t)}$$

$$= x^{-\alpha}\frac{(tx)^\alpha K(tx)-t^\alpha K(t)-A(t)t^\alpha K(t)\log x}{A(t)t^\alpha K(t)A_0}\cdot\frac{A_0(t)}{A(t)}$$

$$-\frac{x^{-\alpha+A(t)}-x^{-\alpha}-A(t)x^{-\alpha}\log x}{A^2(t)}$$

$$= x^{-\alpha}\left[\frac{(tx)^\alpha K(tx)-t^\alpha K(t)-a_1(t)\log x}{a_2(t)}-\frac{1}{\varrho'}\frac{x^{\varrho'}-1}{\varrho'}\right]$$

$$\times\frac{a_2(t)}{A(t)t^\alpha K(t)A_0(t)}\cdot\frac{A_0(t)}{A_0(tx)}\cdot\frac{A_0(tx)}{A(tx)}\cdot\frac{A(tx)}{A(t)}$$

$$+x^{-\alpha}(\log x)\frac{a_1(t)-A(t)t^\alpha K(t)}{A(t)t^\alpha K(t)A_0(t)}\cdot\frac{A_0(t)}{A(t)}$$

$$+x^{-\alpha}\frac{x^{\varrho'}}{(\varrho')^2}\frac{a_2(t)}{A(t)t^\alpha K(t)A_0(t)}\cdot\frac{A_0(t)}{A_0(tx)}\cdot\frac{A_0(tx)}{A(tx)}\cdot\frac{A(tx)}{A(t)}$$

$$-x^{-\alpha}(\varrho')^{-2}\frac{a_2(t)}{A(t)t^\alpha K(t)A_0(t)}\cdot\frac{A_0(t)}{A(t)}$$

$$-\frac{x^{-\alpha+A(t)}-x^{-\alpha}-A(t)x^{-\alpha}\log x}{A^2(t)}$$

and $A_0(t)/A(t)\to 0$. Using (4.8), Potter bounds (see Bingham et al. [1]), and

Lemma 4.1, we obtain

$$|x^y - x^{y_0} - (y-y_0)\, x^{y_0} \log x| \leqslant \tfrac{1}{2}(y-y_0)^2\, x^{y_0} (\log x)^2 \exp\{|(y-y_0)\log x|\} \quad \text{for all } x>0$$

and

$$\left| \frac{\dfrac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} \right| \leqslant x^{-\alpha}\, \varepsilon\, (e^{\varepsilon|\log x|} + |\log x|) + \tfrac{1}{2} x^{-\alpha} (\log x)^2 e^{\varepsilon|\log x|}.$$

This completes the proof of the lemma. ∎

Lemma 4.5. *Suppose the conditions of Lemma 4.2 hold. Then for any $\varepsilon > 0$ there exists $N_0 > 0$ such that for all $n \geqslant N_0$, $U(n) \geqslant N_0$, $U(n)\, x \geqslant N_0$*

$$(4.9)\qquad \left| \frac{n\,[1-F(U(n)x)+F(-U(n)x)] - x^{-\alpha+A(U(n))}}{A^2(U(n))} \right| \leqslant \varepsilon x^{-\alpha} (|\log x| + e^{\varepsilon|\log x|}) + \tfrac{1}{2} x^{-\alpha} (\log x)^2 e^{\varepsilon|\log x|}$$

and

$$(4.10)\qquad \left| \frac{n\,[1-F(U(n)x)-F(-U(n)x)] - (2p-1)\, x^{-\alpha+A(U(n))}}{A^2(U(n))} \right| \leqslant \varepsilon x^{-\alpha} (|\log x| + e^{\varepsilon|\log x|}) + \frac{|2p-1|}{2} x^{-\alpha} (\log x)^2 e^{\varepsilon|\log x|} + 2q_0\, x^{-\alpha} e^{\varepsilon|\log x|}.$$

Proof. Note that

$$\frac{n\,[1-F(U(n)x)+F(-U(n)x)] - x^{-\alpha+A(U(n))}}{A^2(U(n))} = \frac{\dfrac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(-U(n))} - x^{-\alpha+A(U(n))}}{A^2(U(n))} + \frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(U(n))} \cdot \frac{n\,[1-F(U(n))+F(-U(n))]-1}{A^2(U(n))}$$

and

$$\frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(-U(n))} \in RV_{-\alpha}, \qquad \frac{n\,[1-F(U(n))+F(-U(n))]-1}{A^2(U(n))} \to 0.$$

Thus (4.9) follows from Lemma 4.4 and Potter bounds (see Bingham et al. [1]).

Note that

$$\frac{n\left[1-F\left(U(n)x\right)-F\left(-U(n)x\right)\right]-(2p-1)\,x^{-\alpha+A(U(n))}}{A^2\left(U(n)\right)}$$

$$=(2p-1)\,\frac{n\left[1-F\left(U(n)x\right)+F\left(-U(n)x\right)\right]-x^{-\alpha+A(U(n))}}{A^2\left(U(n)\right)}$$

$$+n\left[1-F\left(U(n)\right)+F\left(-U(n)\right)\right]\cdot\frac{1-F\left(U(n)x\right)+F\left(-U(n)x\right)}{1-F\left(U(n)\right)+F\left(-U(n)\right)}$$

$$\times\frac{\dfrac{1-F\left(U(n)x\right)-F\left(-U(n)x\right)}{1-F\left(U(n)x\right)+F\left(-U(n)x\right)}-(2p-1)}{A^2\left(U(n)x\right)}\cdot\frac{A^2\left(U(n)x\right)}{A^2\left(U(n)\right)}.$$

Hence (4.10) follows easily. This completes the proof of the lemma. ∎

LEMMA 4.6. *Suppose the conditions of Lemma 4.2 hold. Let f denote the characteristic function of F. Define $\alpha_n=\alpha-A\left(U(n)\right)$. Then*

$$\lim_{n\to\infty}\frac{-n\log f\left(t/U(n)\right)+\log g_{\alpha_n}(t)}{A^2\left(U(n)\right)}$$

$$=\int_0^\infty\left[-(x/|t|)^{-\alpha}\left(\log(x/|t|)\right)^2/2\right]\sin x\,dx$$

$$+i\,\mathrm{sgn}(t)\int_0^\infty\left[-(2p-1)(x/|t|)^{-\alpha}\left(\log(x/|t|)\right)^2/2+2q_0(x/|t|)^{-\alpha}\right](1-\cos x)\,dx$$

$$=:C_1(t)+i\,\mathrm{sgn}(t)\,C_2(t).$$

Proof. Note that for $|t|\neq 0$

$$n\left(1-f\left(t/U(n)\right)\right)-\log g_{\alpha_n}(t)$$

$$=n\int_0^\infty t\sin(tx)\left[1-F\left(U(n)x\right)+F\left(-U(n)x\right)\right]dx$$

$$+in\int_0^\infty t\left(1-\cos(tx)\right)\left[1-F\left(U(n)x\right)-F\left(-U(n)x\right)\right]dx$$

$$-|t|^{\alpha_n}\Gamma(1-\alpha_n)\cos\frac{\pi\alpha_n}{2}+i\,\mathrm{sgn}(t)\,|t|^{\alpha_n}\Gamma(1-\alpha_n)(2p-1)\sin\frac{\pi\alpha_n}{2}$$

$$=\int_0^\infty\left[n\left(1-F\left(U(n)x/|t|\right)+F\left(-U(n)x/|t|\right)\right)-(x/|t|)^{-\alpha_n}\right]\sin x\,dx$$

$$+i\,\mathrm{sgn}(t)\int_0^\infty\left[n\left(1-F\left(U(n)x/|t|\right)-F\left(-U(n)x/|t|\right)\right)\right.$$

$$\left.-(2p-1)(x/|t|)^{-\alpha_n}\right](1-\cos x)\,dx.$$

By Lemma 4.5, $\alpha > 1$ and Lebesgue's dominated convergence theorem we have

$$\frac{1}{A^2(U(n))}\int_1^\infty \left[n\left(1-F(U(n)x/|t|)+F(-U(n)x/|t|)\right)-(x/|t|)^{-\alpha_n}\right]\sin x\,dx$$
$$\to \int_1^\infty \left[-(x/|t|)^{-\alpha}(\log(x/|t|))^2/2\right]\sin x\,dx.$$

By Lemma 4.5, $|(\sin x)/x| \leqslant 1$ as $0 \leqslant x \leqslant 1$, $\alpha < 2$, and Lebesgue's dominated convergence theorem we have

$$\frac{1}{A^2(U(n))}\int_{|t|N_0/U(n)}^1 \left[n\left(1-F(U(n)x/|t|)+F(-U(n)x/|t|)\right)-(x/|t|)^{-\alpha_n}\right]\sin x\,dx$$
$$\to \int_0^1 \left[-(x/|t|)^{-\alpha}(\log(x/|t|))^2/2\right]\sin x\,dx.$$

Combining

$$\frac{1}{A^2(U(n))}\left|\int_0^{|t|N_0/U(n)} n\left[1-F(U(n)x/|t|)+F(-U(n)x/|t|)\right]\sin x\,dx\right|$$
$$=\frac{1}{A^2(U(n))}\left|\int_0^1 n\left(1-F(N_0 y)+F(-N_0 y)\right)\frac{|t|\,N_0}{U(n)}\sin\left(|t|\,N_0 y/U(n)\right)dy\right|$$
$$=O\left(\frac{n}{U^2(n)A^2(U(n))}\right)\to 0 \quad (\text{since } U\in RV_{1/\alpha})$$

and

$$\frac{1}{A^2(U(n))}\left|\int_0^{|t|N_0/U(n)} (x/|t|)^{-\alpha_n}\sin x\,dx\right| \to 0$$

(similarly to the proof of the above relation), we get

$$\frac{1}{A^2(U(n))}\int_0^\infty \left[n\left(1-F(U(n)x/|t|)+F(-U(n)x/|t|)\right)-(x/|t|)^{-\alpha_n}\right]\sin x\,dx \to C_1(t).$$

Similarly,

$$\frac{1}{A^2(U(n))}\int_0^\infty \left[n\left(1-F(U(n)x/|t|)-F(-U(n)x/|t|)\right)\right.$$
$$\left.-(2p-1)(x/|t|)^{-\alpha_n}\right](1-\cos x)\,dx \to C_2(t).$$

When expanding $-\log f = -\log(1-(1-f))$, we find that the second (and higher) order term is of lower order, hence the result of the lemma. ∎

LEMMA 4.7. *Suppose the conditions of Lemma 4.2 hold. Then for any* $\varepsilon > 0$ *there exists* $N_0 > 0$ *such that for all* $n \geqslant N_0$, $U(n) \geqslant N_0$, $U(n)/|t| \geqslant N_0$

$$\left|\frac{-n\log f(t/U(n))+\log g_{\alpha_n}(t)}{A^2(U(n))}\right| \leqslant C\left(|t|^{\alpha}(1+|\log|t||+(\log|t|)^2)(1+e^{\varepsilon|\log t|})\right),$$

where C *is a positive constant.*

The lemma follows by using the same arguments as in the proofs of Lemmas 4.4 and 4.5. ∎

Proof of Theorem 2.1. The proof is quite similar to the proof of Theorem 1 of de Haan and Peng [4] by using Lemmas 4.6 and 4.7.

For the proof of Theorem 3.1 we need also some lemmas.

LEMMA 4.8. *Suppose* (3.1) *holds for* $\varrho^* < 0$. *Then for* $x > 0$

$$\lim_{n\to\infty} \frac{\dfrac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}-x^{-\alpha_n^*}}{n\left(1-F(a(n))+F(-a(n))\right)} = -2x^{-2}\log x \tag{4.11}$$

and

$$\lim_{n\to\infty} \frac{\dfrac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))}-(2p^*-1)x^{-\alpha_n^*}}{n\left(1-F(a(n))+F(-a(n))\right)} = -2(2p^*-1)x^{-2}\log x+2q^*x^{-2}. \tag{4.12}$$

Proof. From the relations $S(x)\in RV_0$, (1.7) and $\varrho^* < 0$ we have

$$\lim_{n\to\infty} \frac{A^*(n)}{n\left(1-F(a(n))+F(-a(n))\right)} = 0. \tag{4.13}$$

Combining (4.13) with

$$\begin{aligned} &\frac{\dfrac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}-x^{-\alpha_n^*}}{n\left(1-F(a(n))+F(-a(n))\right)} \\ &\quad= \frac{\dfrac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}-x^{-2}}{A^*(n)}\cdot\frac{A^*(n)}{n\left(1-F(a(n))+F(-a(n))\right)} \\ &\quad\quad+\frac{x^{-2}-x^{-\alpha_n^*}}{n\left(1-F(a(n))+F(-a(n))\right)} \end{aligned} \tag{4.14}$$

and

$$x^y - x^{y_0} = (y-y_0)\,x^{y_0+\theta(y-y_0)}\log x, \qquad \theta\in[0,1], \tag{4.15}$$

we have (4.11). Note that

$$\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))}-(2p^*-1)x^{-\alpha_n^*}$$

$$=(2p^*-1)\left[\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}-x^{-\alpha_n^*}\right]$$

$$+\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}\cdot\left[\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n)x)+F(-a(n)x)}-(2p^*-1)\right]$$

and $n(1-F(a(n))+F(-a(n)))\in RV_0$. Then (4.12) follows easily. ∎

LEMMA 4.9. *Suppose* (3.1) *holds for* $\varrho^*<0$. *Then for any* $\varepsilon>0$ *there exists* $N_0>0$ *such that for all* $n\geqslant N_0$, $a(n)\geqslant N_0$, $a(n)x\geqslant N_0$

$$\frac{\left|\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))}-x^{-\alpha_n^*}\right|}{n(1-F(a(n))+F(-a(n)))}\leqslant \varepsilon x^{-2}e^{\varepsilon|\log x|}+2x^{-2}|\log x|\,e^{\varepsilon|\log x|}$$

and

$$\frac{\left|\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))}-(2p^*-1)x^{-\alpha_n^*}\right|}{n(1-F(a(n))+F(-a(n)))}$$

$$\leqslant \varepsilon x^{-2}e^{\varepsilon|\log x|}+2\,|2p^*-1|\,x^{-2}|\log x|\,e^{\varepsilon|\log x|}+2q^*x^{-2}e^{\varepsilon|\log x|}.$$

The proof is similar to the proofs of Lemmas 4.4 and 4.5. ∎

LEMMA 4.10. *Suppose* (3.1) *holds for* $\varrho^*<0$. *Then*

$$\lim_{n\to\infty}\frac{-n\log f(t/a(n))+\log g^*_{\alpha_n^*}(t)}{[n(1-F(a(n))+F(-a(n)))]^2}$$

$$=-2\int_{|t|}^{\infty}(x/|t|)^{-2}\log(x/|t|)\sin x\,dx-2\int_{0}^{|t|}(x/|t|)^{-2}\log(x/|t|)(\sin x-x)\,dx$$

$$+i\,\mathrm{sgn}(t)\int_0^{\infty}[-2(2p^*-1)(x/|t|)^{-2}\log(x/|t|)+2q^*(x/|t|)^{-2}](1-\cos x)\,dx.$$

Proof. Note that for $t\neq 0$

$$n(1-f(t/a(n)))-\log g^*_{\alpha_n^*}(t)$$

$$=\int_0^{\infty}\{n(1-F(a(n)x/|t|)+F(-a(n)x/|t|))-(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}\sin x\,dx$$

$$+i\,\mathrm{sgn}(t)\int_0^{\infty}\{n(1-F(a(n)x/|t|)-F(-a(n)x/|t|))$$

$$-(2p^*-1)(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}(1-\cos x)\,dx$$

$$= \int_{|t|}^{\infty} \{n(1-F(a(n)x/|t|)+F(-a(n)x/|t|))-(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\} \sin x\, dx$$

$$+ \int_{0}^{|t|} \{n(1-F(a(n)x/|t|)+F(-a(n)x/|t|))-(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\} (\sin x - x)\, dx$$

$$+ \int_{0}^{|t|} \{n(1-F(a(n)x/|t|)+F(-a(n)x/|t|))-(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\} x\, dx$$

$$+ i \operatorname{sgn}(t) \int_{0}^{\infty} \{n(1-F(a(n)x/|t|)-F(-a(n)x/|t|))$$

$$-(2p^* - 1)(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\} (1-\cos x)\, dx$$

and

$$\int_{0}^{|t|} \{n(1-F(a(n)x/|t|)+F(-a(n)x/|t|))-(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\} x\, dx$$

$$= |t|^2 \left\{ \int_0^1 n(1-F(a(n)x)+F(-a(n)x))x\, dx - (1-\alpha_n^*/2)\frac{1}{2-\alpha_n^*} \right\}$$

$$= 0 \quad \text{(by (1.7))}.$$

The rest of the proof is similar to that of Lemma 4.6. ∎

LEMMA 4.11. *Suppose* (3.1) *holds for* $\varrho^* < 0$. *Then for any* $\varepsilon > 0$ *there exists* $N_0 > 0$ *such that for all* $n \geqslant N_0$, $a(n) \geqslant N_0$, $a(n)/|t| \geqslant N_0$

$$\frac{|-n \log f(t/a(n)) + \log g_{\alpha_n^*}^*(t)|}{[n(1-F(a(n))+F(-a(n)))]^2} \leqslant C^* |t|^2 (1+|\log|t||)(1+e^{\varepsilon|\log|t||}),$$

where C^* *is a positive constant.*

The proof is similar to the proof of Lemma 4.7 by using Lemma 4.10. ∎

Proof of Theorem 3.1. The proof is quite similar to the proof of Theorem 1 of de Haan and Peng [4] by using Lemmas 4.10 and 4.11. ∎

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