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SOME MULTIVARIATE INFINITELY DIVISIBLE DISTRIBUTIONS AND THEIR PROJECTIONS

BY

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Abstract. Recently K. Sato constructed an infinitely divisible probability distribution μ on $\boldsymbol{R}^d$ such that μ is not selfdecomposable but every projection of μ to a lower dimensional space is selfdecomposable. Let $L_m(\boldsymbol{R}^d)$, $1 \leqslant m < \infty$, be the Urbanik–Sato type nested subclasses of the class $L_0(\boldsymbol{R}^d)$ of all selfdecomposable distributions on $\boldsymbol{R}^d$. In this paper, for each $1 \leqslant m < \infty$, a probability distribution μ with the following properties is constructed: μ belongs to $L_{m-1}(\boldsymbol{R}^d) \cap (L_m(\boldsymbol{R}^d))^c$, but every projection of μ to a lower k-dimensional space belongs to $L_m(\boldsymbol{R}^k)$. It is also shown that Sato's example is not only "non-selfdecomposable" but also "non-semi-selfdecomposable."

1. Introduction. Let $I(\boldsymbol{R}^d)$ and $S(\boldsymbol{R}^d)$ be the classes of all infinitely divisible distributions and all stable distributions on $\boldsymbol{R}^d$, respectively. Urbanik [9], [10] and Sato [4] studied the nested classes $L_m(\boldsymbol{R}^d)$, $m = 0, 1, 2, \ldots, \infty$, between $I(\boldsymbol{R}^d)$ and $S(\boldsymbol{R}^d)$, which are defined in the following way. For each $0 \leqslant m < \infty$, a distribution μ on $\boldsymbol{R}^d$ is said to *belong to the class* $L_m(\boldsymbol{R}^d)$ if $\mu \in I(\boldsymbol{R}^d)$ and for any $a \in (0, 1)$ there exists $\varrho_a \in L_{m-1}(\boldsymbol{R}^d)$ such that

$$(1.1) \qquad \hat{\mu}(z) = \hat{\mu}(az)\,\hat{\varrho}_a(z), \qquad \forall z \in \boldsymbol{R}^d,$$

with the convention $L_{-1}(\boldsymbol{R}^d) = I(\boldsymbol{R}^d)$, where $\hat{\mu}$ is the characteristic function of μ. The class $L_\infty(\boldsymbol{R}^d)$ is defined as $\bigcap_{m \geqslant 0} L_m(\boldsymbol{R}^d)$. (They actually defined $L_m(\boldsymbol{R}^d)$ as a class of limit distributions of independent random variables, and showed that (1.1) is a necessary and sufficient condition.) Then it was shown that

$$(1.2) \qquad I(\boldsymbol{R}^d) \supset L_0(\boldsymbol{R}^d) \supset L_1(\boldsymbol{R}^d) \supset \ldots \supset L_\infty(\boldsymbol{R}^d) \supset S(\boldsymbol{R}^d).$$

A distribution μ in $L_0(\boldsymbol{R}^d)$ is called *selfdecomposable*.

For a $k \times d$ real matrix A and a measure (or a signed measure) μ on $\boldsymbol{R}^d$, define $A\mu$ by $(A\mu)(B) = \mu(A^{-1}(B))$, $B \in \mathscr{B}(\boldsymbol{R}^k)$. If a $d \times d$ symmetric matrix A satisfies $A^2 = A$, and the dimension of the linear subspace $\{Ax\colon x \in \boldsymbol{R}^d\}$ is k ($\leqslant d-1$), A is called a *k-dimensional projector.*

It is well known that, for a distribution μ on $\boldsymbol{R}^d$, if $A\mu$ is Gaussian for any 1-dimensional projector A, then μ is Gaussian. For non-Gaussian stability, this fact does not necessarily remain true, but several conditions for its validity are known (see, e.g., [3]). Among those, if μ is infinitely divisible, then the stability of μ follows from the fact that $A\mu$ are stable for all 1-dimensional projectors A.

On the other hand, it is also known that even if $A\mu$ are infinitely divisible for all k-dimensional projectors A with $1 \leqslant k \leqslant d-1$, μ is not necessarily infinitely divisible. (As to the references on this fact, see [5].) An example by Shanbhag and Sreehari [7] gives us a multivariate distribution such that it is infinitely divisible and not selfdecomposable, but every linear combination of its components is selfdecomposable.

Recently Sato [5] has also given another example of $\mu \in I(\boldsymbol{R}^d)$ such that $\mu \notin L_0(\boldsymbol{R}^d)$, but $A\mu \in L_0(\boldsymbol{R}^k)$ for any $k \times d$ matrix A with $1 \leqslant k \leqslant d-1$, as follows.

$|x|$ denotes the Euclidean norm of $x \in \boldsymbol{R}^d$. Let $0 < \delta \leqslant 1, 0 < \varepsilon \leqslant 1$,

$$D_1 = \{x \in \boldsymbol{R}^d\colon 1 < |x| \leqslant 2\}, \qquad D_2 = \{x \in \boldsymbol{R}^d\colon |x| \leqslant \delta\},$$

$$\lambda_0(dx) = \big(1_{D_1}(x) - \varepsilon 1_{D_2}(x)\big)\, dx,$$

and define

$$\nu_0(B) = \int_{\boldsymbol{R}^d} \lambda_0(dx) \int_0^\infty 1_B(e^{-t}x)\, dx, \qquad B \in \mathscr{B}_0(\boldsymbol{R}^d), \tag{1.3}$$

where $\mathscr{B}_0(\boldsymbol{R}^d)$ is the class of all Borel sets B in $\boldsymbol{R}^d$ such that $B \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$, and $1_B(\cdot)$ is the indicator function of B. Then Sato [5] showed the following

THEOREM A. *The measure* ν_0 *in* (1.3) *is the Lévy measure of a distribution* $\mu_0 \in I(\boldsymbol{R}^d)$. *Further,* $\mu_0 \notin L_0(\boldsymbol{R}^d)$ *but* $A\mu_0 \in L_0(\boldsymbol{R}^k)$ *for any* $k \times d$ *matrix* A *with* $1 \leqslant k \leqslant d-1$.

The first purpose of this paper is to study the same problem for the nested classes $L_m(\boldsymbol{R}^d)$, $1 \leqslant m < \infty$, in (1.2). Namely, we show

THEOREM 1. *For each* $1 \leqslant m < \infty$, *there exists a distribution* μ_m *such that* $\mu_m \in L_{m-1}(\boldsymbol{R}^d)$, $\mu_m \notin L_m(\boldsymbol{R}^d)$, *but* $A\mu_m \in L_m(\boldsymbol{R}^k)$ *for any* $k \times d$ *matrix* A *with* $1 \leqslant k \leqslant d-1$.

In [2], the class of semi-selfdecomposable distributions $L_0(b, \boldsymbol{R}^d)$, $0 < b < 1$, has been introduced. We say that, for each $b \in (0, 1)$, μ *belongs to* $L_0(b, \boldsymbol{R}^d)$ if for some $\varrho \in I(\boldsymbol{R}^d)$, $\hat{\mu}(z) = \hat{\mu}(bz)\hat{\varrho}(z)$ for all $z \in \boldsymbol{R}^d$. It is easy to see that

$$L_0(b, \boldsymbol{R}^d) \subset I(\boldsymbol{R}^d) \quad \text{and} \quad L_0(\boldsymbol{R}^d) = \bigcap_{0<b<1} L_0(b, \boldsymbol{R}^d).$$

Therefore, for every $b \in (0, 1)$,

$$I(\boldsymbol{R}^d) \supset L_0(b, \boldsymbol{R}^d) \supset L_0(\boldsymbol{R}^d).$$

The second purpose of this paper is to show that the example constructed by Sato (μ_0 in Theorem A) is not only "non-selfdecomposable," but also "non-semi-selfdecomposable." Namely, we show

THEOREM 2. *Let μ_0 be the one in Theorem A. Then $\mu_0 \notin L_0(b, \mathbf{R}^d)$ for any $b \in (0, 1)$.*

Similarly to the nested classes $L_m(\mathbf{R}^d)$, $1 \leqslant m < \infty$, mentioned above, Maejima and Naito [2] have defined the nested classes $L_m(b, \mathbf{R}^d)$, $1 \leqslant m < \infty$, of $L_0(b, \mathbf{R}^d)$ as follows. Let $0 < b < 1$. For each $1 \leqslant m < \infty$, μ is said to *belong to the class* $L_m(b, \mathbf{R}^d)$ if $\mu \in I(\mathbf{R}^d)$ and there exists $\varrho \in L_{m-1}(b, \mathbf{R}^d)$ such that

$$\hat{\mu}(z) = \hat{\mu}(bz)\hat{\varrho}(z), \qquad \forall z \in \mathbf{R}^d.$$

It is easy to see that for each $b \in (0, 1)$, $L_m(b, \mathbf{R}^d) \supset L_m(\mathbf{R}^d)$ and $L_m(\mathbf{R}^d) = \bigcap_{0<b<1} L_m(b, \mathbf{R}^d)$. Related to Theorem 2 above, a natural question arises: For each $1 \leqslant m < \infty$, does μ_m in Theorem 1 belong to $L_m(b, \mathbf{R}^d)$ or not? The answer is the following

THEOREM 3. *Let $1 \leqslant m < \infty$, and let μ_m be the one in Theorem 1. Then $\mu_m \notin L_m(b, \mathbf{R}^d)$ for any $b \in (0, 1)$.*

2. Preliminary lemmas. To prove Theorem 1, the following characterization for $\mu \in L_0(\mathbf{R}^d)$ is our starting point. This is a reformulation by Sato and Yamazato [6] of a result of Urbanik [8].

THEOREM B. *$\mu \in L_0(\mathbf{R}^d)$ if and only if $\mu \in I(\mathbf{R}^d)$ and its Lévy measure ν is either zero or represented as*

$$\nu(B) = \int_{\mathbf{R}^d} \lambda(dx) \int_0^\infty 1_B(e^{-t}x)\,dt, \qquad B \in \mathscr{B}_0(\mathbf{R}^d), \tag{2.1}$$

where λ is a measure on $\mathbf{R}^d$ satisfying

$$\lambda(\{0\}) = 0, \tag{2.2}$$

$$\int_{|x| \leqslant 2} |x|^2 \lambda(dx) < \infty, \tag{2.3}$$

and

$$\int_{|x|>2} \log|x|\, \lambda(dx) < \infty. \tag{2.4}$$

This λ is uniquely determined by ν.

Since ν and λ are uniquely determined by $\mu \in I(\mathbf{R}^d)$, when we want to emphasize the correspondence between those, we may write $\nu = \nu_\mu$ and $\lambda = \lambda_\mu$.

In the following, we state two results by Jurek [1] on characterization for $\mu \in L_m(\mathbf{R}^d)$, $1 \leqslant m < \infty$, which will be used in the proof of Theorem 1. We say that an $\mathbf{R}^d$-valued stochastic process $\{Y(t), t \geqslant 0\}$ is a *Lévy process* if it has independent and stationary increments, it is right continuous, it has left limits and $Y(0) = 0$ a.s. The distribution of a random variable X is denoted by $\mathscr{L}(X)$.

For $c > 0$ and $B \subset \boldsymbol{R}^d$, write $cB = \{cx: x \in B\}$. For $a \in (0, 1)$ and a measure ξ on $\boldsymbol{R}^d$, define

$$\Delta_a \xi(B) = \xi(aB) - \xi(B),$$

when $\xi(B)$ and $\xi(aB)$ are finite, and for $n \geqslant 2$ and $a_1, \ldots, a_n \in (0, 1)$, define

$$(\Delta_{a_n \ldots a_1} \xi)(B) = \Delta_{a_n}(\Delta_{a_{n-1} \ldots a_1} \xi)(B)$$

successively.

Lemma 1 ([1], Corollary 2.6). *Let $0 \leqslant m < \infty$. μ belongs to $L_m(\boldsymbol{R}^d)$ if and only if $\mu \in I(\boldsymbol{R}^d)$ and its Lévy measure ν_μ satisfies*

(2.5) $$(\Delta_{a_l \ldots a_1} \nu_\mu)(B) \geqslant 0, \quad \forall a_1, \ldots, a_l \in (0, 1), \ \forall B \in \mathscr{B}_0(\boldsymbol{R}^d)$$

for any $l = 1, \ldots, m+1$.

Lemma 2 ([1], Theorem 2.3). *Let $1 \leqslant m < \infty$. μ belongs to $L_m(\boldsymbol{R}^d)$ if and only if there exists a Lévy process $\{Y(t)\}$ such that*

$$\mu = \mathscr{L}\Big(\int_0^\infty e^{-t}\, dY(t)\Big)$$

and $\mathscr{L}(Y(1)) \in L_{m-1}(\boldsymbol{R}^d) \cap I_{\log}(\boldsymbol{R}^d)$, where $I_{\log}(\boldsymbol{R}^d)$ is the set of all $\xi \in I(\boldsymbol{R}^d)$ satisfying $\int \log(1+|x|)\, \xi(dx) < \infty$.

For our purpose, we state Lemma 2 in terms of λ_μ as follows.

Lemma 3. *Let $1 \leqslant m < \infty$. μ belongs to $L_m(\boldsymbol{R}^d)$ if and only if $\mu \in L_0(\boldsymbol{R}^d)$ and $\lambda = \lambda_\mu$ in the representation* (2.1) *satisfies*

(2.6) $$(\Delta_{a_l \ldots a_1} \lambda_\mu)(B) \geqslant 0, \quad \forall a_1, \ldots, a_l \in (0, 1), \ \forall B \in \mathscr{B}_0(\boldsymbol{R}^d)$$

for any $l = 1, \ldots, m$.

Proof. Let $\mu \in L_0(\boldsymbol{R}^d)$. Note that the Lévy measure of $\mathscr{L}(Y(1))$ in Lemma 2 is λ_μ in our notation (see [6], p. 91). Then combining Lemmas 1 and 2, and noticing that $\lambda_\mu \in I_{\log}(\boldsymbol{R}^d)$ by (2.4), we conclude Lemma 3.

3. Proof of Theorem 1. For our construction of desired distributions in Theorem 1, we fully use the example by Sato [5] mentioned in Theorem A. We first show that the measure ν_0 in (1.3) satisfies (2.2), (2.3) and that

(3.1) $$\nu_0(|x| > 2) = 0.$$

Since ν_0 is the Lévy measure as shown in Theorem A, (2.2) and (2.3) are automatically satisfied. As to (3.1), we have

$$\nu_0(|x| > 2) = \int_{\boldsymbol{R}^d} \lambda_0(dy) \int_0^\infty 1(|e^{-t} y| > 2)\, dt = \int_{|y|>2} \lambda_0(dy) \int_0^\infty 1(|e^{-t} y| > 2)\, dt = 0,$$

because $\lambda_0(|y| > 2) = 0$.

Suppose for $0 \leqslant m < \infty$ we are given a measure ν_m on $\boldsymbol{R}^d$ satisfying (2.2), (2.3) and such that $\nu_m(|x| > 2) = 0$. ν_m also satisfies (2.4) trivially. Thus we can define the Lévy measure

$$\nu_{m+1}(B) = \int_{R^d} \nu_m(dx) \int_0^\infty 1_B(e^{-t}x)\,dt \tag{3.2}$$

by taking $\lambda = \nu_m$ in (2.1). If $\nu_m(|x|>2) = 0$, then $\nu_{m+1}(|x|>2) = 0$ as above. Thus ν_{m+1} also satisfies (2.2)–(2.4). Therefore starting with ν_0 in (1.3), we can construct a sequence of Lévy measures ν_m, $0 \leqslant m < \infty$, and denote by $\mu_m \in I(R^d)$ the distribution whose Lévy measure is ν_m. Note that

$$\nu_m = \lambda_{\mu_{m+1}} \tag{3.3}$$

in our notation. We will show that, for $1 \leqslant m < \infty$, μ_m is the desired distribution satisfying the requirements in Theorem 1.

By Theorem A, μ_0 is such that $\mu_0 \in I(R^d)$, $\mu_0 \notin L_0(R^d)$ and $A\mu_0 \in L_0(R^k)$ for any $k\times d$ matrix A with $1 \leqslant k \leqslant d-1$. We show the assertion of the theorem by induction on m.

Suppose, for some $m_0 \geqslant 0$, the distribution μ_{m_0} satisfies $\mu_{m_0} \in L_{m_0-1}(R^d)$, $\mu_{m_0} \notin L_{m_0}(R^d)$ and $A\mu_{m_0} \in L_{m_0}(R^k)$ for any $k\times d$ matrix A with $1 \leqslant k \leqslant d-1$. Since $\mu_{m_0} \notin L_{m_0}(R^d)$, we see from Lemma 1 that $\Delta_{a_l \ldots a_1} \nu_{m_0}(B) < 0$ for some $l = 1, \ldots, m_0+1$, $a_1, \ldots, a_l \in (0, 1)$, $B \in \mathscr{B}_0(R^d)$. Thus, by (3.3), $(\Delta_{a_l \ldots a_1} \lambda_{\mu_{m_0+1}})(B) < 0$ for such $l, a_1, \ldots, a_l$ and B, implying $\mu_{m_0+1} \notin L_{m_0+1}(R^d)$ by Lemma 2.

Next note that Lemma 1 remains true for $m = -1$, and that Lemma 2 also remains true for $m = 0$. Since $\mu_{m_0} \in L_{m_0-1}(R^d)$, we see from Lemma 1 (including the case for $m = -1$) that

$$(\Delta_{a_l \ldots a_1} \nu_{m_0})(B) \geqslant 0, \quad \forall a_1, \ldots, a_l \in (0, 1),\ \forall B \in \mathscr{B}_0(R^d)$$

for any $l = 1, \ldots, m_0$. Thus, by (3.3),

$$(\Delta_{a_l \ldots a_1} \lambda_{\mu_{m_0+1}})(B) \geqslant 0, \quad \forall a_1, \ldots, a_l \in (0, 1),\ \forall B \in \mathscr{B}_0(R^d)$$

for any $l = 1, \ldots, m_0$, implying $\mu_{m_0+1} \in L_{m_0}(R^d)$ by Lemma 2 (including the case for $m = 0$).

Finally, we suppose that A is any $k\times d$ matrix with $1 \leqslant k \leqslant d-1$. In general, if $\mu \in I(R^d)$, then $A\mu \in I(R^k)$ and its Lévy measure $\nu_{A\mu}$ is $[A\nu_\mu]_{R^k\setminus\{0\}}$. If

$$\nu_\mu(B) = \int_{R^d} \lambda_\mu(dx) \int_0^\infty 1_B(e^{-t}x)\,dt,$$

then for $B \in \mathscr{B}_0(R^k)$

$$\begin{aligned}\nu_{A\mu}(B) = \nu_\mu(A^{-1}(B)) &= \int_{R^d} \lambda_\mu(dx) \int_0^\infty 1_{A^{-1}(B)}(e^{-t}x)\,dt \\ &= \int_{R^d} (A\lambda_\mu)(dx) \int_0^\infty 1_B(e^{-t}x)\,dt.\end{aligned}$$

By induction hypothesis and Lemma 1, we see that

$$\left(\Delta_{a_l \dots a_1}(A\nu_{m_0})\right)(B) \geqslant 0, \quad \forall a_1, \dots, a_l \in (0, 1), \ \forall B \in \mathscr{B}_0(\boldsymbol{R}^k)$$

for any $l = 1, \dots, m_0+1$. On the other hand,

$$\nu_{A\mu_{m_0+1}}(B) = \int_{\mathbf{R}^d} (A\nu_{m_0})(dx) \int_0^\infty 1_B(e^{-t}x)\,dt.$$

Hence, by Lemma 2, $A\mu_{m_0+1} \in L_{m_0+1}(\boldsymbol{R}^k)$, which concludes that our μ_{m+1} having its Lévy measure ν_{m+1} in (3.2) is an example of the desired distribution. This completes the proof of Theorem 1.

4. Proof of Theorem 2. By Lemma 4.1 in [2], $\mu \in L_0(b, \boldsymbol{R}^d)$ if and only if $\nu_\mu(bB) \geqslant \nu_\mu(B)$ for any $B \in \mathscr{B}_0(\boldsymbol{R}^d)$. Thus, for a given $b \in (0, 1)$, if we could show

$$\nu_0(br_1 < |x| \leqslant br_2) < \nu_0(r_1 < |x| \leqslant r_2) \quad \text{for some } 0 < r_1 < r_2,$$

then Theorem 2 would be concluded. Here we use the calculation done by Sato [5]. He showed that if $0 < r_1 < r_2 < 1$, then

$$\begin{aligned} I(r_1, r_2) &= \frac{1}{c_d}\nu_0(r_1 < |x| \leqslant r_2) \\ &= -\int_{r_1}^{r_2} r^{d-1}\log\frac{r}{r_1}\,dr - \log\frac{r_2}{r_1}\int_{r_2}^{1} r^{d-1}\,dr + \log\frac{r_2}{r_1}\int_1^2 r^{d-1}\,dr, \end{aligned}$$

where c_d is the surface measure of the unit sphere in $\boldsymbol{R}^d$. Thus

$$I(br_1, br_2) = -\int_{br_1}^{br_2} r^{d-1}\log\frac{r}{br_1}\,dr - \log\frac{r_2}{r_1}\int_{br_2}^{1} r^{d-1}\,dr + \log\frac{r_2}{r_1}\int_1^2 r^{d-1}\,dr,$$

and we have

$$\begin{aligned} I &= I(r_1, r_2) - I(br_1, br_2) \\ &= -\int_{r_1}^{r_2} r^{d-1}\log\frac{r}{r_1}\,dr - \log\frac{r_2}{r_1}\int_{r_2}^{1} r^{d-1}\,dr \\ &\quad + \int_{br_1}^{br_2} r^{d-1}\log\frac{r}{br_1}\,dr + \log\frac{r_2}{r_1}\int_{br_2}^{1} r^{d-1}\,dr \\ &= (b^d-1)\int_{r_1}^{r_2} r^{d-1}\log\frac{r}{r_1}\,dr + \log\frac{r_2}{r_1}\int_{br_2}^{r_2} r^{d-1}\,dr \\ &\geqslant \frac{1}{d}\log\frac{r_2}{r_1}\{(b^d-1)(r_2^d - r_1^d) + (1-b^d)r_2^d\} = -\frac{1}{d}\log\frac{r_2}{r_1}(b^d-1)r_1^d > 0. \end{aligned}$$

This completes the proof.

5. Proof of Theorem 3. We need two lemmas corresponding to Lemmas 1 and 3.

LEMMA 4 [2]. *Let $0 < b < 1$ and $0 \leqslant m < \infty$. μ belongs to $L_m(b, \boldsymbol{R}^d)$ if and only if $\mu \in I(\boldsymbol{R}^d)$ and its Lévy measure ν_μ satisfies*

$$(\Delta_b^l \nu_\mu)(B) \geqslant 0, \quad \forall B \in \mathscr{B}_0(\boldsymbol{R}^d)$$

for any $l = 1, \ldots, m+1$, where $\Delta_b^l = \Delta_{b \ldots b}$.

LEMMA 5. *Let $0 < b < 1$ and $1 \leqslant m < \infty$. Suppose $\mu \in L_0(\boldsymbol{R}^d)$. Then μ belongs to $L_m(b, \boldsymbol{R}^d)$ if and only if $\lambda = \lambda_\mu$ in the representation (1.3) satisfies*

$$(\Delta_b^l \lambda_\mu)(B) \geqslant 0, \quad \forall B \in \mathscr{B}_0(\boldsymbol{R}^d)$$

for any $l = 1, \ldots, m$.

This lemma can be proved in exactly the same way as Lemma 3 with the replacement of Lemma 1 by Lemma 4.

Proof of Theorem 3. Since $\mu_0 \notin L_0(b, \boldsymbol{R}^d)$, by Lemma 4 we have $\Delta_b \nu_{\mu_0}(B) < 0$ for some $B \in \mathscr{B}_0(\boldsymbol{R}^d)$. As before

$$\Delta_b \lambda_{\mu_1}(B) = \Delta_b \nu_{\mu_0}(B) < 0.$$

Hence, by Lemma 4, $\mu_1 \notin L_1(b, \boldsymbol{R}^d)$. Repeating this argument, we conclude that $\mu_m \notin L_m(b, \boldsymbol{R}^d)$ for each $1 \leqslant m < \infty$.

6. Concluding remarks.

(i) We have the following two relations:

$$L_m(\boldsymbol{R}^d) \subset L_{m-1}(\boldsymbol{R}^d) \quad \text{and} \quad L_m(\boldsymbol{R}^d) \subset L_m(b, \boldsymbol{R}^d).$$

One might ask what the relationship between $L_{m-1}(\boldsymbol{R}^d)$ and $L_m(b, \boldsymbol{R}^d)$ is.

(I) $L_m(b, \boldsymbol{R}^d) \cap (L_{m-1}(\boldsymbol{R}^d))^c \neq \emptyset$. This can be shown by taking non-self-decomposable semi-stable distribution, the existence of which is well known.

(II) $L_{m-1}(\boldsymbol{R}^d) \cap (L_m(b, \boldsymbol{R}^d))^c \neq \emptyset$. Our μ_m constructed in Theorem 1 assures this non-emptiness.

(ii) It is known that if $A\mu \in S(\boldsymbol{R}^1)$ for any $1 \times d$ matrix A for some $\mu \in I(\boldsymbol{R}^d)$, then $\mu \in S(\boldsymbol{R}^d)$ (see, e.g., [3]). In Theorems A and 1, we have seen that this type of property does not hold for the classes $L_m(\boldsymbol{R}^d)$, $0 \leqslant m < \infty$. The same question about $L_\infty(\boldsymbol{R}^d)$ seems interesting, but it is still open.

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