PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 20, Fasc. 2 (2000), pp. 373–381

ON SOME CONNECTIONS BETWEEN RANDOM PARTITIONS OF THE UNIT SEGMENT AND THE POISSON PROCESS

BY

KRZYSZTOF KANIOWSKI (ŁÓDŹ)

Abstract. Let D_t be the diameter of a partition of the interval $[0, t]$ by renewal moments of a standard Poisson process. Then $D_t/\ln t \to 1$ for $t \to \infty$, in probability. Other theorems on diameters are obtained. Jajte's theorem on random partitions of the unit segment is used.

1991 Mathematics Subject Classification: 60F10, 60G55.

1. Introduction. The aim of this paper is to describe some connections between random partitions of the unit segment and the Poisson process. In the first section we shall present several limit theorems concerning the random partitions of the segment $[0, 1]$ by the sequence of independent random variables uniformly distributed on $[0, 1]$. Next we will show that it is reasonable to search for analogous theorems for sequences of independent random variables with exponential distribution (which can be interpreted as time distances between successive renewal moments in the Poisson process). In the last section we prove some limit theorems for the Poisson process using analogous theorems for random partitions of the unit segment. In particular, we obtain some stochastic limits and limits in law for normed sequences d_t, D_t, where

$$(0)\qquad \begin{aligned} d_t &= \min\{\sigma_1, \sigma_2-\sigma_1, \ldots, t-\sigma_{N_t}\} \quad \text{for } t>0,\\ D_t &= \max\{\sigma_1, \sigma_2-\sigma_1, \ldots, t-\sigma_{N_t}\} \quad \text{for } t>0, \end{aligned}$$

for $\sigma_1, \sigma_2, \ldots$ being successive renewal moments for standard Poisson process $\{N_t, t \geqslant 0\}$. In the theory of queues, let $\sigma_1, \sigma_2, \ldots$ be moments of arrivals of successive customers. Let us assume also that customers are served at once. Then d_t and D_t are the shortest and the longest time intervals in which no customer is served, when we observe the queue to the moment t.

2. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables uniformly distributed on the interval $[0, 1]$. Denote by δ_n and Δ_n lengths of the shortest and the longest interval, respectively, which we obtain by partitioning the segment $[0, 1]$ by the random points $\xi_1, \ldots, \xi_{n-1}$. It can be easily shown that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \Delta_n = 0 \text{ with probability one.}$$

The following theorem has been proved by Jajte [3]:

THEOREM 2.1. *The sequence $\{(n\Delta_n)/\ln n\}$ converges in probability to unity.*

One can also prove the following

PROPOSITION 2.2. *The joint distribution of the random vector (δ_n, Δ_n), $n \geqslant 2$, is given by*

$$(1)\quad P(x < \delta_n, \Delta_n < y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-(n-k)x-ky)_+^{n-1} \quad \textit{for } 0 < x < y < 1,$$

where $z_+ = \max(z, 0)$.

Now we can easily prove the following

PROPOSITION 2.3. *The sequence $\{n^2 \delta_n\}$ converges weakly to the exponential distribution.*

Proof. From (1) we get

$$P(\delta_n < x) = 1-(1-nx)_+^{n-1} \quad \text{for } x > 0.$$

Hence for $x > 0$ we obtain

$$(2)\quad \lim_{n\to\infty} P(n^2 \delta_n < x) = 1-e^{-x}.$$

We shall find now the distribution of the random variable δ_n/Δ_n, $n \geqslant 3$.

PROPOSITION 2.4. *The distribution function of the random variable δ_n/Δ_n, $n \geqslant 3$, is given by*

$$P\left(\frac{\delta_n}{\Delta_n} < s\right) = 1 - \frac{1}{1-s} \frac{n!}{\left(\frac{s}{1-s}n+1\right)\left(\frac{s}{1-s}n+2\right)\ldots\left(\frac{s}{1-s}n+n\right)} \quad \textit{for } s \in (0, 1).$$

Proof. By (1) for $0 < x < y < 1$ we get

$$P(\delta_n < x, \Delta_n < y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-ky)_+^{n-1} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-(n-k)x-ky)_+^{n-1}.$$

Differentiating the above distribution function we obtain density of the random vector (δ_n, Δ_n):

$$f(x, y) = (n-1)(n-2)\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}k(n-k)\left(1-(n-k)x-ky\right)_{+}^{n-3},$$

where $n \geqslant 3$, $0 < x < y < 1$. After routine calculations we get for $s > 1$

$$P\left(\frac{\Delta_n}{\delta_n} > s\right) = \int_0^{1/s}\int_{sx}^{1} f(x, y)\,dx\,dy = \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}\frac{n-k}{n-k+ks}. \tag{3}$$

Using the following formulas:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0,$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k} = \frac{n!}{x(x+1)\ldots(x+n)} \quad \text{for } x \notin \{-n, \ldots, -1, 0\},$$

we obtain for $s \in (0, 1)$

$$\begin{aligned} P\left(\frac{\delta_n}{\Delta_n} < s\right) &= P\left(\frac{\Delta_n}{\delta_n} > \frac{1}{s}\right) = \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}\frac{s(n-k)}{s(n-k)+k} \\ &= 1 + \frac{s}{1-s}\sum_{k=0}^{n}(-1)^k\binom{n}{k} - \frac{sn}{1-s}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{sn+k(1-s)} \\ &= 1 - \frac{sn}{(1-s)^2}\,\frac{n!}{\frac{s}{1-s}n\left(\frac{s}{1-s}n+1\right)\left(\frac{s}{1-s}n+2\right)\ldots\left(\frac{s}{1-s}n+n\right)}. \end{aligned}$$

This completes the proof of the proposition. ∎

In order to investigate asymptotic behaviour of the sequence δ_n/Δ_n, $n = 2, 3, \ldots$, we shall use the following proposition:

PROPOSITION 2.5. *If $\xi_n \Rightarrow \xi$ (by the symbol $\Rightarrow$ we denote weak convergence of distributions) and $\eta_n \to c$ in probability, where $c \in \mathbf{R}$, then*

(i) $\xi_n \eta_n \Rightarrow c\xi$,

(ii) $\xi_n/\eta_n \Rightarrow \xi/c$ *if* $c \neq 0$.

Now we can easily prove the following

PROPOSITION 2.6. *The following formulas hold:*

$$\lim_{n\to\infty} P\left(n \ln n\,(\delta_n/\Delta_n) < x\right) = 1 - e^{-x} \quad \textit{for } x > 0, \tag{i}$$

$$\lim_{n\to\infty} (\delta_n/\Delta_n) = 0 \ \textit{in probability}. \tag{ii}$$

Proof. (i) can be obtained immediately from Proposition 2.5, Theorem 2.1 and (2). Formula (ii) follows from (i) and the equality

$$\frac{\delta_n}{\Delta_n} = \frac{1}{n\ln n} n\ln n \frac{\delta_n}{\Delta_n}. \blacksquare$$

3. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables uniformly distributed on the interval $[0, 1]$. By $\xi_{1:n}, \xi_{2:n}, \ldots, \xi_{n:n}$ we denote the sequence $\xi_1, \xi_2, \ldots, \xi_n$ arranged in increasing order. One can show (see [2]) that for $k \leqslant n$ and $t_1, \ldots, t_k > 0$ we have

$$P(\xi_{1:n} > t_1, \xi_{2:n} - \xi_{1:n} > t_2, \ldots, \xi_{k:n} - \xi_{k-1:n} > t_k) = (1-t_1-\ldots-t_k)_+^n.$$

Therefore we obtain

$$\lim_{n\to\infty} P\left(n\xi_{1:n} > t_1, n(\xi_{2:n}-\xi_{1:n}) > t_2, \ldots, n(\xi_{k:n}-\xi_{k-1:n}) > t_k\right) = e^{-t_1} \ldots e^{-t_k},$$

which means that random variables $n\xi_{1:n}, n(\xi_{2:n}-\xi_{1:n}), \ldots, n(\xi_{k:n}-\xi_{k-1:n})$ converge weakly to the exponential distribution and are asymptotically independent. This fact may suggest some analogies between random partitions of the unit segment and the Poisson process.

Let $\eta_1, \eta_2, \ldots$ be a sequence of independent random variables with exponential distributions. We can interpret $\eta_1, \eta_2, \ldots$ as time distances between successive renewal moments for the Poisson process. We shall consider now the following two random variables:

$$d_n = \min\{\eta_1, \ldots, \eta_n\}, \quad D_n = \max\{\eta_1, \ldots, \eta_n\}.$$

The joint distribution of the random variables d_n and D_n is given by

$$P(x < d_n, D_n < y) = (e^{-x} - e^{-y})^n \quad \text{for } 0 < x < y. \tag{4}$$

In particular, we have

$$P(d_n < x) = 1 - e^{-nx} \quad \text{for } x > 0 \tag{5}$$

and

$$P(D_n < y) = (1 - e^{-y})^n \quad \text{for } y > 0. \tag{6}$$

We prove the following

PROPOSITION 3.1. *The sequence $\{D_n/n\}$ converges with probability one to zero.*

Proof. Let us compute the second moment of D_n. It is well known that for any nonnegative random variable ξ we have

$$E\xi^p = p\int_0^\infty y^{p-1} P(\xi \geqslant y)\, dy \quad \text{for } p > 0.$$

From (6) we have

$$P(D_n \geqslant y) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} e^{-ky}$$

and, consequently,

$$ED_n^2 = 2 \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^{\infty} y e^{-ky}\, dy = 2 \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k^2}. \tag{7}$$

We put

$$\gamma_n^2 = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k^2}, \qquad n = 1, 2, \ldots$$

It can be shown (see [3]) that

$$\gamma_n^2 = \sum_{1 \leqslant i \leqslant j \leqslant n} 1/ij$$

and

$$\lim_{n\to\infty} 2\gamma_n^2/\ln^2 n = 1. \tag{8}$$

Thus from (7) and (8) we have

$$E\left(\frac{D_n}{n}\right)^2 = 2\frac{\gamma_n^2}{n^2}$$

and

$$\lim_{n\to\infty} E\left(\frac{D_n}{n}\right)^2 \Big/ \left(\frac{\ln n}{n}\right)^2 = 1. \tag{9}$$

Since $\sum_{n=1}^{\infty} ((\ln n)/n)^2 < \infty$, (9) implies that

$$\sum_{n=1}^{\infty} E(D_n/n)^2 < \infty.$$

Therefore we can deduce that

$$\lim_{n\to\infty} (D_n/n) = 0 \text{ with probability } 1.$$

This completes the proof of the proposition. ∎

It is easily seen from (5) that for $x > 0$ we have

$$P(nd_n < x) = 1 - e^{-x}.$$

Let us consider now the random variable $D_n/\ln n$. From (6) we get

$$P(D_n/\ln n < y) = (1 - n^{-y})^n \quad \text{for } y > 0.$$

Now it is easy to verify that

$$\lim_{n\to\infty} P\left(\frac{D_n}{\ln n} < y\right) = \begin{cases} 0, & y \in (0, 1), \\ 1, & y \in (1, \infty). \end{cases}$$

Thus, we have proved the following

PROPOSITION 3.2. *The sequence $\{D_n/\ln n\}$ converges in probability to unity.*

We shall prove now the following

PROPOSITION 3.3. *Distributions of the random variables δ_n/Δ_n and d_n/D_n for $n \geqslant 3$ are identical.*

Proof. Let us put

$$\zeta_i = \eta_i / \sum_{k=1}^{n} \eta_k \quad \text{for } i = 1, 2, \ldots, n,$$

where $\eta_1, \ldots, \eta_n$ are independent and have exponential distributions.

It is known (see Feller [2]) that distributions of the random vectors $(\xi_{1:n-1}, \xi_{2:n-1} - \xi_{1:n-1}, \ldots, 1 - \xi_{n-1:n-1})$ and $(\zeta_1, \ldots, \zeta_n)$ are identical. From this the conclusion follows almost immediately.

From Propositions 2.6 and 3.3 we get

PROPOSITION 3.4. *The following equalities hold:*

(i) $$\lim_{n\to\infty} (d_n/D_n) = 0 \ \textit{in probability},$$

(ii) $$\lim_{n\to\infty} P\big(n \ln n (d_n/D_n) < x\big) = 1 - e^{-x} \quad \textit{for } x > 0.$$

4. Let $\{N_t, t \geqslant 0\}$ be a standard Poisson process. Let us remind that d_t and D_t are minimal and maximal waiting times given by (0).

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables uniformly distributed on the interval $[0, t]$ for $t > 0$. Denote by $\delta_n^{(t)}$ and $\Delta_n^{(t)}$ lengths of the shortest and the longest interval, which we obtain by partitioning the segment

$[0, t]$ by the random points $\xi_1, \ldots, \xi_{n-1}$. By Δ_n we shall denote a random variable $\Delta_n^{(1)}$. One can easily see that random vectors $(\Delta_n^{(t)}/t, \delta_n^{(t)}/t)$ and (Δ_n, δ_n) have the same distributions. It is known (see [2]) that if $N_t = n-1$, then the random vector $(\sigma_1, \sigma_2-\sigma_1, \ldots, \sigma_{N_t}-\sigma_{N_t-1})$ has a conditional uniform distribution on the set $\{(s_1, \ldots, s_{n-1}): 0 < s_1+\ldots+s_{n-1} < t\}$. Since the distribution of the random vector $(\xi_{1:n-1}, \xi_{2:n-1}-\xi_{1:n-1}, \ldots, \xi_{n-1:n-1}-\xi_{n-2:n-1})$ is the same, we conclude that if $N_t = n-1$, then the conditional distribution of the random vector (D_t, d_t) is the same as the distribution of $(\Delta_n^{(t)}, \delta_n^{(t)})$.

Let us prove now the following lemma:

LEMMA 4.1. *Let F_n, $n = 0, 1, \ldots,$ be a sequence of distribution functions weakly convergent to the distribution function F. If for every $t > 0$ a given sequence $p_n(t)$, $n = 0, 1, \ldots,$ satisfies the following two conditions:*

(i) $\sum_{n=0}^{\infty} p_n(t) = 1$, $p_n(t) \geqslant 0$, $n = 0, 1, \ldots$

(ii) *for every $\varepsilon > 0$ and $k = 0, 1, \ldots$ there exists $T > 0$ such that*

$$\sum_{n=k}^{\infty} p_n(t) > 1-\varepsilon \quad \textit{for every } t > T, \tag{10}$$

then distribution functions $F_t = \sum_{n=0}^{\infty} F_n p_n(t)$ are weakly convergent to F.

Remark. Condition (ii) is equivalent to the following:

$$\lim_{t\to\infty} p_n(t) = 0 \quad \text{for } n \geqslant 0.$$

Proof of Lemma 4.1. Let $x \in \boldsymbol{R}$ be a continuity point of F. Take $\varepsilon > 0$. We can choose $k \in N$ such that

$$|F_n(x)-F(x)| < \varepsilon \quad \text{for every } n \geqslant k.$$

Let $T > 0$ satisfy (10); then for every $t > T$ we have

$$F_t(x) = \sum_{n=0}^{k-1} F_n(x)p_n(t)+\sum_{n=k}^{\infty} F_n(x)p_n(t) \leqslant \varepsilon+F(x)+\varepsilon = F(x)+2\varepsilon$$

and

$$F_t(x) = \sum_{n=0}^{k-1} F_n(x)p_n(t)+\sum_{n=k}^{\infty} F_n(x)p_n(t) \geqslant \big(F(x)-\varepsilon\big)(1-\varepsilon).$$

Thus

$$\big(F(x)-\varepsilon\big)(1-\varepsilon) \leqslant \liminf_{t\to\infty} F_t(x) \leqslant \limsup_{t\to\infty} F_t(x) \leqslant F(x)+2\varepsilon.$$

Letting $\varepsilon \to 0$ we get

$$\lim_{t\to\infty} F_t(x) = F(x).$$

This completes the proof of the lemma. ∎

We shall use the above lemma to investigate asymptotic behaviour of the random variables d_t and D_t. Let us prove the following theorem:

THEOREM 4.2. *The sequence $\{D_t/\ln t\}_{t>0}$ converges in probability to unity.*

Proof. Let us consider first the random variable

$$\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}}.$$

For $x > 0$ we have

$$P\left(\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}} < x\right) = \sum_{n=0}^{\infty} P\left(\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}} < x \mid N_t = n\right)\frac{e^{-t}t^n}{n!}$$
$$= e^{-t}+te^{-t}+\sum_{n=2}^{\infty} P\left(\frac{n\Delta_{n+1}}{\ln n} < x\right)\frac{e^{-t}t^n}{n!}.$$

It is easy to see that the sequence $\{e^{-t}t^{n-1}/(n-1)!\}$, $t > 0$, satisfies the assumptions of Lemma 4.1. By this, Lemma 4.1 and Theorem 2.1 we conclude that

$$\lim_{t\to\infty} P\left(\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}} < x\right) = \begin{cases} 0, & x\in(0, 1),\\ 1, & x\in(1, \infty),\end{cases}$$

that is,

$$\lim_{t\to\infty}\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}} = 1 \text{ in probability.}$$

It is known that $\lim_{t\to\infty}(N_t/t) = 1$ with probability one (and, in consequence, also $\lim_{t\to\infty}(\ln N_t/\ln t) = 1$ with probability one). From this, Proposition 2.5 and the equality

$$\frac{D_t}{\ln t}\mathbf{1}_{\{N_t>1\}} = \left(\frac{t}{N_t}\frac{\ln N_t}{\ln t}\right)\frac{N_t}{\ln N_t}\frac{D_t}{t}\mathbf{1}_{\{N_t>1\}}$$

we deduce also that

$$\lim_{t\to\infty}\frac{D_t}{\ln t}\mathbf{1}_{\{N_t>1\}} = 1 \text{ in probability,}$$

but as the sequence $\{N_t > 1\}_{t>0}$ increases and $\lim_{t\to\infty} P(N_t > 1) = 1$, we finally get

$$\lim_{t\to\infty} (D_t/\ln t) = 1 \text{ in probability.}$$

This completes the proof of the theorem. ∎

In the same way, using analogous formulas for random partitions of the unit segment, we can prove the following

PROPOSITION 4.3. *The following formulas hold*:

(i) $$\lim_{t\to\infty} P(td_t < x) = 1 - e^{-x} \quad \textit{for } x > 0,$$

(ii) $$\lim_{t\to\infty} (d_t/D_t) = 0 \textit{ in probability},$$

(iii) $$\lim_{t\to\infty} P\big(t \ln t\,(d_t/D_t) < x\big) = 1 - e^{-x} \quad \textit{for } x > 0.$$

REFERENCES

[1] D. A. Darling, *On a class of problems related to the random division of an interval*, Ann. Math. Statist. 24 (1953), pp. 239–253.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, New York 1966.

[3] R. Jajte, *On random partitions of the segment*, Bull. Acad. Polon. Sci. 19 (1971), pp. 231–233.

Faculty of Mathematics, University of Łódź
90-238 Łódź, Banacha 22
Poland

Received on 11.5.2000