PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 20, Fasc. 2 (2000), pp. 399–423

ON FIRST-PASSAGE TIMES FOR ONE-DIMENSIONAL JUMP-DIFFUSION PROCESSES

BY

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Abstract. Some problems of first-crossing times over two time-dependent boundaries for one-dimensional jump-diffusion processes are considered. The moments of the first-crossing times over each boundary are shown to be the solutions of certain partial differential-difference equations with suitable outer conditions. An approach based on the Laplace transform allows us to compare the moments of the first-crossing times of the jump-diffusion process with those of the corresponding simple-diffusion without jumps. For some examples where the boundaries are constant, the results are illustrated graphically.

AMS 1991 Subject Classification: Primary 60J60; Secondary 60J99; 60H10.

Key words and phrases: Jump-diffusion process, exit probability, first-crossing time.

1. INTRODUCTION

Many stochastic phenomena in applied sciences are very well described in terms of a *jump-diffusion process* $X(t)$, i.e. a diffusion process to which jumps at Poisson-distributed instants are superimposed. As in the case of diffusion processes, it is relevant to study the first-passage time of $X(t)$ over certain curves. Even in the simplest case of constant boundaries, however, few analytical results are known. In [9] and [15] the authors found some recursive differential-difference equations for the moments of the first exit time of the process $X(t)$ from a set A in the phase space, in the case when $X(t)$ is a one-dimensional jump-diffusion process which is temporally homogeneous. These equations are the generalization to the actual case of Darling and Siegert's equations [5], holding for simple-diffusions.

In [12] new integral equations for those moments have been obtained, in the case of constant amplitude Poisson time distributed jumps. On the other hand, also in the case of simple-diffusion processes, the available analytical solutions to first-passage-time problems through time-dependent boundaries appear to be fragmentary (see, e.g., [6]–[8], [3], [4], [10], [11], [2], and references therein). In the case of time-homogeneous diffusion processes and constant boundaries, some closed forms have been obtained in [1] for moments of the first-passage time through any of the two boundaries of an interval (α, β).

In this paper, going back to the analogous case of simple-diffusions, treated in [2], we consider a one-dimensional jump-diffusion process $X(t)$ which is temporally homogeneous, and two time-dependent boundaries $\alpha(t)$ and $\beta(t)$ such that $\alpha(t) < \beta(t)$ for all $t \geqslant s$. We suppose that the process starts at the initial time s from a point x such that $\alpha(s) < x < \beta(s)$ and we consider the first-passage time of $X(t)$ through either the curve $x = \alpha(t)$ or $x = \beta(t)$. More precisely, by using the generalized Itô's formula for jump-diffusion processes (see e.g. [9]), we derive some partial differential-difference equations (PDDE's) which are analogous to the corresponding PDE's found in [14] for the moments of first-exit time of a simple-diffusion from the domain $\Omega = \{(t, y):\ t \geqslant s,\ \alpha(t) < y < \beta(t)\}$, and to those found in [2] for the moments of the first-arrival time at the boundary of Ω with the condition that the exit takes place at $x = \alpha(t)$ or $x = \beta(t)$.

In Section 2, the main results on first-passage times are shown.

Section 3 is devoted to the case of constant boundaries; for some examples, explicit computations are carried out. Notice that the differential-difference equations for the moments of the first-passage time are equations with outer conditions, thus, also in simple cases, the analytical solutions are hard to obtain. Moreover, the uniqueness of the solution has to be proved. Then we discuss this issue, and also we point out how to find approximate solutions to the above equations, by solving simpler differential-difference equations with boundary conditions.

In Section 4, using an approach based on the Laplace transform, we compare, in the case of constant boundaries, the moments of first-passage times of the jump-diffusion process with those of the simple-diffusion obtained by disregarding the jumps.

2. NOTATION AND MAIN RESULTS

We consider a one-dimensional jump-diffusion process $X(t)$ which is the solution of the stochastic differential equation (SDE):

$$dX(t) = b\big(X(t)\big)\,dt + \sigma\big(X(t)\big)\,dW_t + \int_{-\infty}^{+\infty} \gamma\big(X(t), u\big)\,\nu(dt, du) \tag{2.1}$$

with the assigned initial condition; here, W_t is a standard Brownian motion and $\nu(t, \cdot)$ is a temporally homogeneous Poisson random measure. If the functions $b(\cdot)$, $\sigma(\cdot)$ and $\gamma(\cdot,\cdot)$ are sufficiently regular, then there exists a unique solution to (2.1) which is a temporally homogeneous Markov process:

$$X(t) = X(s)+\int_s^t b\big(X(r)\big)\,dr+\int_s^t \sigma\big(X(r)\big)\,dW_r+\int_s^t\int_{-\infty}^{+\infty} \gamma\big(X(r),u\big)\,\nu(dr,du), \tag{2.2}$$

where $X(s)$ is the initial value of $X(t)$ at the instant s. For the definitions of the integrals on the right-hand side of (2.2) and the Poisson measure, see e.g. [9].

We denote by $\Pi(\cdot)$ the positive measure defined on $\mathscr{B}(R)$ such that

$$E[\nu(t,B)] = t\Pi(B), \qquad B\in\mathscr{B}(R), \tag{2.3}$$

and we suppose that the jump intensity

$$\Lambda = \int_{-\infty}^{+\infty} \Pi(du) \geqslant 0 \tag{2.4}$$

is finite.

Notice that if $\gamma = 0$ in (2.1) or $\nu = 0$, then the equation (2.1) becomes the usual Itô's stochastic differential equation (SDE) for a simple-diffusion.

In the special case when the measure Π is concentrated e.g. over the set $\{u_1, u_2\} = \{-1, 1\}$ with $\Pi(u_i) = \lambda_i$ and $\gamma(u_i) = \varepsilon_i$, we can rewrite the equation (2.1) as

$$dX(t) = b\big(X(t)\big)\,dt+\sigma\big(X(t)\big)\,dW_t+\varepsilon_2\,dN_2(t)+\varepsilon_1\,dN_1(t), \tag{2.5}$$

where $N_i(t)$, $t\geqslant 0$, are independent homogeneous Poisson processes of amplitudes $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ and rates λ_1 and λ_2, respectively, governing downward (N_1) and upward (N_2) jumps.

If conditions hold in order that the transition probability of $X(t)$ has a density $p(y,t\,|\,x,s)$, $t>s$, this density satisfies the generalized backward Kolmogorov equation (see [9]):

$$-\frac{\partial p}{\partial s} = -\Lambda p+b(x)\frac{\partial p}{\partial x}+\frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2}+\int_{-\infty}^{+\infty} p\big(y,t\,|\,x+\gamma(x,u),s\big)\,\Pi(du). \tag{2.6}$$

Of course, when $\Lambda = 0$, (2.6) becomes the usual backward Kolmogorov equation.

Let D be the class of functions $f(t,x)$ defined in $R\times R$, differentiable with respect to t and twice differentiable with respect to x, for which the function $f\big(t,x+\gamma(x,u)\big)-f(t,x)$ is Π-integrable for any (t,x). We recall the generalized Itô's formula for jump-diffusion processes (see e.g. [9]):

$$\begin{aligned} df\big(t,X(t)\big) = \Big[&\frac{\partial f}{\partial t}\big(t,X(t)\big)+b\big(X(t)\big)\frac{\partial f}{\partial x}\big(t,X(t)\big) \\ &+\frac{1}{2}\sigma^2\big(X(t)\big)\frac{\partial^2 f}{\partial x^2}\big(t,X(t)\big)\Big]dt+\frac{\partial f}{\partial x}\big(t,X(t)\big)\sigma\big(X(t)\big)\,dW_t \\ &+\int\big[f\big(t,X(t)+\gamma\big(X(t),u\big)\big)-f\big(t,X(t)\big)\big]\,\nu(dt,du). \end{aligned} \tag{2.7}$$

The differential operator associated with the process $X(t)$ which is a solution of (2.1) is defined, for any function $f \in D$, by

$$Lf(t, x) = L_d f(t, x) + L_j f(t, x), \tag{2.8}$$

where the "diffusion part" is

$$L_d f(t, x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 f(t, x)}{\partial x^2} + b(x)\frac{\partial f(t, x)}{\partial x} \tag{2.9}$$

and the "jump part" is

$$L_j f(t, x) = \int_{-\infty}^{+\infty} \left[f\big(t, x+\gamma(x, u)\big) - f(t, x)\right] \Pi(du). \tag{2.10}$$

Then, from (2.7), taking expectation, we obtain

$$E\left[f\big(t, X(t)\big)\right] = f(s, x) + \int_s^t \left[\frac{\partial f}{\partial r}\big(r, X(r)\big) + Lf\big(r, X(r)\big)\right] dr.$$

Now, let $P\big(X(s) = x\big) = 1$, and let $\alpha(t)$ and $\beta(t)$ be C^1-functions of time such that $\alpha(t) < \beta(t)$ for all $t \geqslant s$ and $\alpha(s) < x < \beta(s)$. Assume that

$$\tau(s, x) = \tau_{\alpha\beta}(s, x) = \inf_{t \geqslant s}\{t\colon\ X(t) \notin \big(\alpha(t), \beta(t)\big) \mid X(s) = x\} \tag{2.11}$$

is the first-crossing time, that is the first time at which the process $X(t)$, starting from x at the initial instant s, crosses one of the two boundaries: $\alpha(t)$ or $\beta(t)$.

Now, let us suppose that τ is an honest random variable, i.e. $P\big(\tau(s, x) < \infty\big) = 1$, so that $X(t)$ crosses one of the boundaries $\alpha(t)$ or $\beta(t)$ in a finite time with probability one. We denote by $\pi_\alpha(s, x)$ and $\pi_\beta(s, x) = 1 - \pi_\alpha(s, x)$, respectively, the probability that the process, starting from x at the initial time s, exits for the first time from the domain

$$\Omega = \{(t, y) \in R^2\colon\ t \geqslant s,\ \alpha(t) < y < \beta(t)\} \tag{2.12}$$

through the boundary $\alpha(t)$ and $\beta(t)$. Then

$$\begin{aligned} \pi_\alpha(s, x) &= P\big(X(\tau) \leqslant \alpha(\tau) \mid X(s) = x\big), \\ \pi_\beta(s, x) &= P\big(X(\tau) \geqslant \beta(\tau) \mid X(s) = x\big). \end{aligned} \tag{2.13}$$

Let us suppose that the moments of n-th order $M_n(s, x) \doteq E\left[\big(\tau(s, x) - s\big)^n\right]$, $n = 0, 1, \ldots,$ exist. We have the following

THEOREM 2.1. *The probability $M_0(s, x)$ that $X(t)$, starting from x at time s, ever leaves the domain Ω satisfies the PDDE:*

(2.14)

$$\frac{\partial M_0}{\partial s} - \Lambda M_0 + b(x)\frac{\partial M_0}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 M_0}{\partial x^2} + \int_{-\infty}^{+\infty} M_0\big(s, x+\gamma(x, u)\big)\Pi(du) = 0$$

with the boundary condition

$$M_0(s, x) = 1 \quad \text{if } (s, x) \notin \Omega.$$

Moreover, let us suppose that the solution of (2.14) *is* $M_0 \equiv 1$ *for all* $(s, x) \in \Omega$, *then the moments of n-th order* $M_n(s, x)$ *satisfy the recursive PDDE*:

$$\frac{\partial M_n}{\partial s} - \Lambda M_n + b(x)\frac{\partial M_n}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 M_n}{\partial x^2} + \int_{-\infty}^{+\infty} M_n\big(s, x+\gamma(x, u)\big)\Pi(du) = -nM_{n-1}(s, x) \tag{2.15}$$

with the conditions

$$M_n(s, x) = 0 \quad \text{if } (s, x) \notin \Omega. \tag{2.16}$$

Proof. Let $F(x, s, t) = P\big(\tau(s, x) \leqslant t\big)$ be the distribution function of $\tau(s, x)$. Then $F(x, s, t)$ satisfies the equation

$$\frac{\partial F(x, s, t)}{\partial s} + LF(x, s, t) = 0, \quad (s, x) \in \Omega, \tag{2.17}$$

with the initial condition

$$F(x, s, s) = \begin{cases} 0, & (s, x) \in \Omega, \\ 1, & (s, x) \notin \Omega, \end{cases} \tag{2.18}$$

and the boundary condition

$$F(x, s, t) = 1, \quad (s, x) \notin \Omega. \tag{2.18'}$$

Indeed, let us use the argument of the proof of Theorem 2 in [15], and choose the function denoted there by Φ as follows:

$$\Phi\big(X(t), t\big) = \begin{cases} 0, & \big(t, X(t)\big) \in \Omega, \\ 1, & \big(t, X(t)\big) \notin \Omega. \end{cases}$$

Then we infer that the probability $P(x, s, t)$ that $\big(t, X(t)\big) \in \Omega$ throughout the entire time interval $[s, t]$, given $X(s) = x$, satisfies

$$\frac{\partial P(x, s, t)}{\partial s} + LP(x, s, t) = 0, \quad (s, x) \in \Omega, \tag{2.19}$$

with the initial condition

$$P(x, s, s) = \begin{cases} 1, & (s, x) \in \Omega, \\ 0, & (s, x) \notin \Omega, \end{cases} \tag{2.20}$$

and the boundary condition

$$P(x, s, t) = 0, \quad (s, x) \notin \Omega. \tag{2.20'}$$

Thus, since $P(x, s, t) = P(\tau(s, x) > t) = 1 - P(\tau(s, x) \leqslant t) = 1 - F(x, s, t)$, (2.17), (2.18) and (2.18′) easily follow by (2.19), (2.20) and (2.20′).

Now, the moments M_n are given by

$$M_n(s, x) = \int_s^{+\infty} (t-s)^n F(x, s, dt), \tag{2.21}$$

where the integral is understood in the Stieltjes sense. Then, if we multiply both members of (2.17), (2.18), (2.18′) by $(t-s)^n$ and we integrate between s and $+\infty$, using the fact that

$$(t-s)^n \frac{\partial F}{\partial s}$$

can be written as

$$\frac{\partial}{\partial s}[(t-s)^n F] + n(t-s)^{n-1} F,$$

we easily infer that M_n satisfies (2.15) and (2.16). In particular, for $n = 0$ we obtain (2.14). ∎

Remark 2.1. The equations (2.15) for $n = 1, 2$ can be also obtained by means of the generalized Itô's formula, following the approach used in the proof of the next Theorems 2.2 and 2.3 (see also [9]).

Remark 2.2. If $\Lambda = 0$, (2.15) become the well-known equations (see [14]) for the moments of the first-crossing time of a simple-diffusion process.

If $\alpha(t) = \alpha = \text{const}$ and $\beta(t) = \beta = \text{const}$, then the moments M_n are independent of the initial instant s, and (2.15) become the well-known equations obtained by Tuckwell [15].

Finally, if $\Lambda = 0$ and the boundaries are constant, then (2.15) become the well-known Darling and Siegert's equations (see [5]) for the moments of the first-exit time of a diffusion process from an interval (α, β).

Notice that, while for simple-diffusions we have to do with a PDE, when $\Lambda \neq 0$, the equations (2.15) are PDDE's.

Now, we go to consider the distribution of the first-exit time of $X(t)$ from the domain Ω, with the condition that the exit has occurred through the particular boundary $\alpha(t)$ or $\beta(t)$.

THEOREM 2.2. *Let us suppose that, for any continuous function $h(s, x)$, there exists $w(s, x) \in D$, bounded in (s, x), which satisfies the problem:*

$$\begin{aligned} &\frac{\partial w}{\partial s}(s, x) + Lw(s, x) = 0, \qquad s \geqslant 0,\ \alpha(s) < x < \beta(s), \\ &w(s, y) = h(s, y), \qquad y \leqslant \alpha(s) \text{ or } y \geqslant \beta(s). \end{aligned} \tag{2.22}$$

Then $\pi_\alpha(s, x)$ satisfies the following PDDE:

$$\begin{aligned} &\frac{\partial u}{\partial s}(s, x) + Lu(s, x) = 0, \qquad s \geqslant 0,\ \alpha(s) < x < \beta(s); \\ &u(s, y) = 1,\ y \leqslant \alpha(s); \qquad u(s, y) = 0,\ y \geqslant \beta(s). \end{aligned} \tag{2.23}$$

Analogously, $\pi_\beta(s, x)$ *satisfies*

$$\frac{\partial u}{\partial s}(s, x)+Lu(s, x) = 0, \quad s \geqslant 0,\ \alpha(s) < x < \beta(s);$$
$$u(s, y) = 0,\ y \leqslant \alpha(s); \quad u(s, y) = 1,\ y \geqslant \beta(s). \tag{2.23'}$$

Proof. It follows easily by a straightforward modification of the analogous result holding for simple-diffusion processes (see [2]). Indeed, let h be a continuous function, w the solution of (2.22) and $X(t)$ the solution of (2.1) starting from x at the initial instant s. By using the generalized Itô's formula for functions of the solution of the jump-diffusion process on the bounded Markov time-interval (see [9]), if $\tau(t) = \min(t, \tau(s, x))$, we get

$$w\big(\tau(t), X(\tau(t))\big) = w(s, x)+\int_s^{\tau(t)} (\partial w/\partial r+L_d w)\,dr+\int_s^{\tau(t)} \sigma\big(X(r)\big)\frac{\partial w}{\partial x}\,dW_r$$
$$+\int_s^{\tau(t)}\int\big[w\big(r, X(r)+\gamma(r, u)\big)-w\big(r, X(r)\big)\big]\,\nu(dr, du).$$

Taking expectation, we obtain

$$E\big[w\big(\tau(t), X(\tau(t))\big)\big] = w(s, x)+E\Big[\int_s^{\tau(t)} (\partial w/\partial r+Lw)\,dr\Big],$$

and therefore

$$E\big[w\big(\tau(t), X(\tau(t))\big)\big] = w(s, x).$$

Now, letting $t \to \infty$, we get $w(s, x) = E\big[h\big(\tau(s, x), X(\tau(s, x))\big)\big]$. Then, considering in place of h a sequence $\{h_n(t, y)\}$ of continuous approximations of the indicator function of the set $U = \{(t, y)\colon y \leqslant \alpha(t)\}$, the result for π_α easily follows. The second part can be proved analogously. ■

Now, if $X(s) = x$, let $\tau_\alpha(s, x)$ and $\tau_\beta(s, x)$ be, respectively, the first-crossing time of $X(t)$ over the boundary of Ω, with the condition that the exit takes place through $\alpha(t)$ and $\beta(t)$. We are interested in the quantities:

$$E\big(\tau_\alpha^n(s, x)\big) \doteq E\big[\tau^n(s, x) \mid \{X(\tau) \leqslant \alpha(\tau)\}\big], \tag{2.24}$$

$$E\big(\tau_\beta^n(s, x)\big) \doteq E\big[\tau^n(s, x) \mid \{X(\tau) \geqslant \beta(\tau)\}\big]. \tag{2.25}$$

Then we have

THEOREM 2.3. *Assume that* $T_\alpha(s, x) = \pi_\alpha(s, x)E[\tau_\alpha(s, x)-s]$. *Then* $T_\alpha(s, x)$ *satisfies the PDDE*:

$$\begin{aligned} \partial w/\partial s+Lw &= -\pi_\alpha, \quad (s, x)\in\Omega,\\ w(s, x) &= 0, \quad (s, x)\notin\Omega. \end{aligned} \tag{2.26}$$

Analogously, if $T_\beta(s, x) = \pi_\beta(s, x) E[\tau_\beta(s, x) - s]$, *then* $T_\beta(s, x)$ *satisfies:*

$$\begin{aligned} \partial w/\partial s + Lw &= -\pi_\beta, \quad (s, x) \in \Omega, \\ w(s, x) &= 0, \quad (s, x) \notin \Omega. \end{aligned} \tag{2.27}$$

Proof. Let $X(t)$ be the solution of (2.1) starting from x at the initial time s. Once again, by the generalized Itô's formula, if $\bar{\tau}(t) = \min(t, \tau_\alpha(s, x))$, we get

$$E\left[T_\alpha\left(\bar{\tau}(t), X(\bar{\tau}(t))\right)\right] = T_\alpha(s, x) + \int_s^{\bar{\tau}(t)} (\partial T_\alpha/\partial r + L T_\alpha)\, dr. \tag{2.28}$$

Then, when $t \to \infty$, we obtain

$$E\left[T_\alpha(\tau_\alpha, X(\tau_\alpha))\right] = T_\alpha(s, x) + E\left[\int_s^{\tau_\alpha} -\pi_\alpha(r, X(r))\, dr\right]. \tag{2.29}$$

Therefore, by using the boundary condition $T_\alpha(\tau_\alpha, X(\tau_\alpha)) = 0$ (in fact, $(\tau_\alpha, X(\tau_\alpha)) \notin \Omega$), we have

$$T_\alpha(s, x) = E\left[\int_s^{\tau_\alpha} \pi_\alpha(r, X(r))\, dr\right]. \tag{2.30}$$

But the last quantity is equal to $\pi_\alpha(s, x) E[\tau_\alpha(s, x) - s]$. Indeed, by Itô's formula, we have

$$\pi_\alpha(r, X(r)) = \pi_\alpha(s, x) + \int_s^r (\partial \pi_\alpha/\partial t + L\pi_\alpha)\, dt + \int_s^r \sigma(X(t)) \frac{\partial \pi_\alpha}{\partial x} dW_t + \omega(r),$$

where

$$\omega(r) = \int_s^r \int \left[\pi_\alpha\left(t, X(t) + \gamma(X(t), u)\right) - \pi_\alpha(t, X(t))\right] \tilde{\nu}(dt, du),$$

$\tilde{\nu}$ being defined by $\tilde{\nu}(dt, du) = \nu(dt, du) - dt\Pi(du)$ and $E(\omega(r)) = 0$, due to (2.3). Then, by (2.30) and (2.23), we obtain

$$T_\alpha(s, x) = \pi_\alpha(s, x) E[\tau_\alpha(s, x) - s] + E\left[\int_s^{\tau_\alpha} dr \int_s^r \sigma(X(t)) \frac{\partial \pi_\alpha}{\partial x} dW_t\right] + E\left[\int_s^{\tau_\alpha} \omega(r) dr\right],$$

where the last two expectations are zero; indeed, by changing the order of integration, the first integral is equal to

$$\int_s^{\tau_\alpha} (\tau_\alpha - t)\, \sigma(X(t)) \frac{\partial \pi_\alpha}{\partial x} dW_t.$$

Similarly, as easily seen, the expectation of the second integral is zero.

The second part of the theorem can be proved analogously. ■

Remark 2.3. By summing the equations in (2.26) and (2.27), since π_α and π_β are supposed to sum up to unity, the functions T_α and T_β satisfy (2.15) with

$n=1$. Then (if the uniqueness holds), we get $T_\alpha + T_\beta = E(\tau - s)$, that is

$$(2.31)\quad E\big(\tau(s, x) - s\big) = \pi_\alpha(s, x) E\big(\tau_\alpha(s, x) - s\big) + \pi_\beta(s, x) E\big(\tau_\beta(s, x) - s\big).$$

Remark 2.4. In Theorems 2.1, 2.2, and 2.3, we have limited ourselves to show that the functions M_n, π_α, π_β, T_α, T_β, there involved, satisfy certain PDDE's with suitable conditions. Concerning the uniqueness of the solutions of these PDDE problems, we notice that no general result can be achieved, without supposing any further condition on the operator L. Instead, if L_d is uniformly elliptic in Ω (i.e. $\sigma(x) > 0$ for all x), then by a maximum principle argument, analogous to that used in [13] and [9] to prove similar results for homogeneous SDE's, it follows that $M_1(s, x) = E[\tau(s, x) - s]$, for instance, can be determined as the smallest positive solution of the problem (2.15) with $n = 1$.

Remark 2.5. Let $f, g \in D$; then it is easily seen that

$$(2.32)\quad L_j(f \cdot g) = f L_j g + \int g\big(s, x + \gamma(u)\big)\big[f\big(s, x + \gamma(u)\big) - f(s, x)\big]\, \Pi(du).$$

Moreover,

$$(2.33)\quad L_d(f \cdot g) = f L_d g + g L_d f + \sigma^2 f'_x g'_x.$$

Then

$$(2.34)\quad L(f \cdot g)(s, x) = (L_d + L_j)(f \cdot g)(s, x) = f(s, x) Lg(s, x) + gLf$$
$$+ \int \big[f\big(s, x + \gamma(u)\big) - f(s, x)\big]\big[g\big(x + \gamma(u)\big) - g(x)\big]\, \Pi(du) + \sigma^2 f'_x(s, x) g'_x(s, x).$$

Now, setting $N \doteq \partial/\partial s + L$ and $\tau_\alpha = \tau + r$, and using (2.34), we can write the equation (2.26) (see also [1] and [2]) in the form

$$\begin{aligned} N\big(\pi_\alpha E(\tau_\alpha)\big) &= N\big(\pi_\alpha E(\tau + r)\big) = \pi_\alpha N\big(E(\tau)\big) + \pi_\alpha N\big(E(r)\big) \\ &\quad + E(\tau + r) N\pi_\alpha + \sigma^2 \pi'_\alpha E(\tau + r)' \\ &\quad + \int \big[E(\tau_\alpha)\big(s, x + \gamma(u)\big) - E(\tau_\alpha)(s, x)\big]\big[\pi_\alpha\big(s, x + \gamma(u)\big) \\ &\quad - \pi_\alpha(s, x)\big]\, \Pi(du) = -\pi_\alpha(s, x) \end{aligned}$$

and, since $N\big(E(\tau)\big) = -1$, $N\pi_\alpha = 0$, after some manipulations, we finally obtain

$$(2.35)\quad \frac{\partial E(\tau_\alpha)}{\partial s} + \frac{1}{2} E(\tau_\alpha)'' \sigma^2 + (b + \pi'_\alpha \sigma^2/\pi_\alpha) E(\tau_\alpha)'$$
$$+ \int \big[E(\tau_\alpha)\big(s, x + \gamma(u)\big) - E(\tau_\alpha)(s, x)\big] \frac{\pi_\alpha\big(s, x + \gamma(u)\big)}{\pi_\alpha(s, x)}\, \Pi(du) = -1,$$

where $'$ indicates derivation with respect to x. However, in the actual case, we are not able, as done in [2], to rewrite (2.35) as the equation for the mean of the unconditional exit time of a suitably modified jump-diffusion process. Indeed, in the case of simple-diffusions the integral in (2.35) disappears, so by a Girsanov transformation of the drift, it is possible to achieve the result for the new process $\tilde{X}(t)$ which is a solution of the SDE $d\tilde{X}(t) = (b+\pi_\alpha \sigma^2/\pi_\alpha)dt+\sigma d\tilde{B}_t$, $\tilde{B}_t$ being another suitable Brownian motion (see [2]). Now, unlike the diffusion case, due to the integral in (2.35), we are not able to use the argument of [2] to obtain an equation for the second order moment of the conditional exit time through one of the two boundaries $\alpha(t)$ or $\beta(t)$. However, the equation (2.35) may be more convenient for explicit calculations.

3. CONSTANT BOUNDARIES

In this section, we consider the special case of constant boundaries; the corresponding results can be obtained from the general case of Section 2, by setting to zero the derivatives with respect to the initial time s. Thus, the PDDE's become ordinary differential-difference equations (ODDE's). Let $\alpha(t) = \alpha = \text{const}$, $\beta(t) = \beta = \text{const}$ and let $X(t)$ be the solution of (2.1) starting from x such that $\alpha \leqslant x \leqslant \beta$. Now, we denote by D the class of functions $g(x)$ defined and continuous in R, twice differentiable in (α, β) for which the function $g(x+\gamma(x, u))-g(x)$ is Π-integrable for any x.

THEOREM 3.1 (Tuckwell [15]). *Let us suppose that the moments $M_n(x) = E(\tau^n_{\alpha\beta}(x))$ exist for all $n = 0, 1, \ldots$ Then we have:*

(i) *the probability $M_0(x)$ that $X(t)$ ever leaves the interval (α, β) satisfies the equation*

$$\begin{aligned} &-\Lambda M_0+b(x)\frac{dM_0}{dx}+\frac{1}{2}\sigma^2(x)\frac{d^2M_0}{dx^2}+\int M_0(x+\gamma(x, u))\,\Pi(du) = 0, \\ &M_0(x) = 1, \quad x \notin (\alpha, \beta); \end{aligned} \tag{3.1}$$

(ii) *if the solution of* (3.1) *is $M_0(x) = 1$ for all $x \in (\alpha, \beta)$, then the moments M_n satisfy*

(3.2)

$$\begin{aligned} &-\Lambda M_n(x)+b(x)\frac{dM_n}{dx}+\frac{1}{2}\sigma^2(x)\frac{d^2M_n}{dx^2}+\int M_n(x+\gamma(x, u))\,\Pi(du) = -nM_{n-1}(x), \\ &M_n(x) = 0, \quad x \notin (\alpha, \beta),\ n = 1, 2, \ldots \end{aligned}$$

Remark 3.1. If there exists a function $g \in D$, bounded in R and satisfying (3.2) with $n = 1$, then $M_0(x) = 1$ for all $x \in (\alpha, \beta)$, i.e. $\tau_{\alpha\beta}(x)$ is finite with probability one.

Indeed, by the generalized Itô's formula, if $\tau_t(x) = \min(t, \tau(x))$, we have for any $g \in D$:

$$E[g(X(\tau_t))]-g(x) = \int_0^{\tau_t} Lg(X(s))\,ds. \tag{3.3}$$

Since, by hypotheses, $Lg = -1$, we obtain

$$E(\tau_t) = g(x) - E\left[g\left(X(\tau_t)\right)\right].$$

Finally, by the Lebesgue theorem, passing to the limit as $t \to \infty$, we obtain

$$E\left(\tau(x)\right) = \lim_{t \to \infty} E(\tau_t) < \infty,$$

i.e. $\tau(x)$ is finite with probability one.

THEOREM 3.2. *$\pi_\alpha(x)$ satisfies the equation*

$$\begin{aligned} &Lu = 0, \quad x \in (\alpha, \beta); \\ &u(y) = 1, \ y \leqslant \alpha; \quad u(y) = 0, \ y \geqslant \beta. \end{aligned} \tag{3.4}$$

Analogously, $\pi_\beta(x)$ satisfies:

$$\begin{aligned} &Lv = 0, \quad x \in (\alpha, \beta); \\ &v(y) = 0, \ y \leqslant \alpha; \quad v(y) = 1, \ y \geqslant \beta. \end{aligned} \tag{3.5}$$

THEOREM 3.3. *If $T_\alpha(x) = \pi_\alpha(x) E\left(\tau_\alpha(x)\right)$, then $T_\alpha(x)$ satisfies the following problem*:

$$\begin{aligned} Lw &= -\pi_\alpha, \quad && x \in (\alpha, \beta), \\ w(x) &= 0, && x \notin (\alpha, \beta). \end{aligned} \tag{3.6}$$

Analogously, if $T_\beta(x) = \pi_\beta(x) E\left(\tau_\beta(x)\right)$, then $T_\beta(x)$ satisfies

$$\begin{aligned} Lw &= -\pi_\beta, \quad && x \in (\alpha, \beta), \\ w(x) &= 0, && x \notin (\alpha, \beta). \end{aligned} \tag{3.7}$$

Now, we will consider a simple example of jump diffusion allowing only upward jumps, and we will carry out explicit calculations.

EXAMPLE 3.1 (Brownian motion+upward Poisson jumps with constant amplitude). For fixed $\varepsilon > 0$, let us consider the SDE

$$dX(t) = dW_t + \varepsilon dN^+(t) \tag{3.8}$$

with the initial condition $X(0) = x$, and take $\alpha = 0$, $\beta = 2\varepsilon$, the Poisson process intensity, $\lambda = 1$.

(I) *For any x, $M_0(x) \equiv 1$.*

Indeed, $M_0(x)$ is the solution of the problem

$$\begin{aligned} \tfrac{1}{2} z''(x) + z(x+\varepsilon) - z(x) &= 0, \quad && x \in (0, 2\varepsilon), \\ z(x) &= 1, && x \notin (0, 2\varepsilon). \end{aligned} \tag{3.9}$$

We consider separately two cases:

(i) $\min_{(0,2\varepsilon)} z = z(0)$ or $\min_{(0,2\varepsilon)} z = z(2\varepsilon)$.

Then it is clear that $z(x) \equiv 1$ for all $x \in R$.

(ii) There exists $\bar{x} \in (0, 2\varepsilon)$ such that $\min_{(0,2\varepsilon)} z = z(\bar{x}) < 1$ with $z''(\bar{x}) > 0$.

If $\bar{x} \in (\varepsilon, 2\varepsilon)$, from (3.9) we have $z(\bar{x}) = 1 + \frac{1}{2} z''(\bar{x}) > 1$, which is a contradiction, since $z(x)$ is a probability. If $\bar{x} \in (0, \varepsilon)$, from (3.9) we have $z(\bar{x}+\varepsilon) - z(\bar{x}) = -\frac{1}{2} z''(\bar{x}) < 0$, which implies $z(\bar{x}+\varepsilon) < z(\bar{x})$, i.e. $\bar{x}$ is not the point at which $z(x)$ attains its global minimum; this is a contradiction.

We conclude from (i) and (ii) that $z(x) \equiv 1$ for all $x \in R$.

(II) *The solution of the equation* (3.2) *for* M_1 *is unique.*

First, we show that the problem

$$\begin{aligned} \tfrac{1}{2} z''(x) + z(x+\varepsilon) - z(x) &= 0, \quad x \in (0, 2\varepsilon), \\ z(x) &= 0, \quad x \notin (0, 2\varepsilon) \end{aligned} \tag{3.10}$$

has only the trivial solution $z(x) \equiv 0$. If not, $z(x)$ should have a positive maximum or a negative minimum. Let us suppose, for instance, that the first case occurs; first, we observe that if $z(\varepsilon) > 0$, then $z(x)$ has to be decreasing in $(\varepsilon, 2\varepsilon)$. Indeed, for $x \in (\varepsilon, 2\varepsilon)$, from (3.10) we have $z''(x) - 2z = 0$, which has a solution

$$z(x) = c\,(e^{-\sqrt{2}x} + e^{-4\varepsilon\sqrt{2}} e^{\sqrt{2}x})$$

and $z(\varepsilon) > 0$ implies $c > 0$; thus

$$z'(x) = -\sqrt{2}c\,(e^{-\sqrt{2}x} + e^{-4\varepsilon\sqrt{2}} e^{\sqrt{2}x}) < 0,$$

i.e. $z(x)$ is decreasing in $(\varepsilon, 2\varepsilon)$. Then, if $\bar{x}$ is the point at which $z(x)$ attains its global maximum, it must be $\bar{x} \leqslant \varepsilon$, and therefore

$$z(\bar{x}) \geqslant z(\varepsilon) > z(\bar{x}+\varepsilon). \tag{3.11}$$

From (3.10) we have $z(\bar{x}) - z(\bar{x}+\varepsilon) = \frac{1}{2} z''(\bar{x}) < 0$, i.e. $z(\bar{x}) < z(\bar{x}+\varepsilon)$, which contradicts (3.11).

If $z(\varepsilon) < 0$, a contradiction is obtained in an analogous manner (now, $z(x)$ is increasing in $(\varepsilon, 2\varepsilon)$). The case when $z(x)$ has a negative minimum can be treated similarly. We conclude that (3.10) has only the trivial solution $z(x) \equiv 0$.

From this fact it follows easily that the solution of the equation (3.2) for M_1 is unique. Indeed, if $z_1(x)$, $z_2(x)$ are two different solutions of (3.2), then $z(x) \doteq z_2(x) - z_1(x)$ satisfies (3.10), and therefore $z(x) \equiv 0$, i.e. $z_1(x) \equiv z_2(x)$.

(III) *The equations* (3.4) *and* (3.5) *for* π_α *and* π_β *have a unique solution.*

Also now, if $z_1(x)$, $z_2(x)$ are two different solutions, e.g. of (3.4), then $z(x) \doteq z_2(x) - z_1(x)$ satisfies (3.10), and therefore $z_1(x) \equiv z_2(x)$.

(IV) *The equations* (3.6) *and* (3.7) *for* $T_\alpha(x)$ *and* $T_\beta(x)$ *have a unique solution.*

Indeed, let us consider the problem

$$\begin{aligned} \tfrac{1}{2}u''+u(x+\varepsilon)-u(x) &= -\pi_0(x), \qquad x\in(0,\,2\varepsilon),\\ u(x) &= 0, \qquad\qquad\quad x\notin(0,\,2\varepsilon), \end{aligned} \tag{3.12}$$

where $\pi_0(x)$ is the (unique) solution of the problem

$$\begin{aligned} &\tfrac{1}{2}z''+z(x+\varepsilon)-z(x)=0, \qquad x\in(0,\,2\varepsilon);\\ &z(x)=1,\ x\leqslant 0; \qquad z(x)=0,\ x\geqslant 2\varepsilon. \end{aligned}$$

Let $u_1(x)$, $u_2(x)$ be two different solutions of (3.12); then $u \doteq u_2-u_1$ satisfies (3.10), and therefore $u_1\equiv u_2$.

After checking for the uniqueness of the solution of the equations involved, we can proceed to find their explicit formulas. To this end, for every equation, first we have to find a solution in the interval $(\varepsilon,\,2\varepsilon)$, up to some undetermined constant, then we must find a solution in $(0,\,\varepsilon)$; finally, the constant has to be determined requiring the solution to be C^2 in the whole interval $(0,\,2\varepsilon)$. By a straightforward, very long calculation, we obtain:

(V) *The solution of the problem for* $E(\tau)$:

$$\begin{aligned} \tfrac{1}{2}z''(x)+z(x+\varepsilon)-z(x) &= -1, \qquad x\in(0,\,2\varepsilon),\\ z(x) &= 0, \qquad\ \ x\notin(0,\,2\varepsilon), \end{aligned} \tag{3.13}$$

is given by

$$z(x)=\begin{cases} e^{-\sqrt{2}x}(A+ax)+e^{\sqrt{2}x}(B+bx)+2, & x\in(0,\,\varepsilon),\\[2mm] ce^{-\sqrt{2}x}-\dfrac{1+ce^{-2\varepsilon\sqrt{2}}}{e^{2\varepsilon\sqrt{2}}}e^{\sqrt{2}x}+1, & x\in(\varepsilon,\,2\varepsilon), \end{cases} \tag{3.14}$$

where

$$\begin{aligned} c &= \frac{e^{5\varepsilon\sqrt{2}}-8e^{4\varepsilon\sqrt{2}}+(2\sqrt{2}\varepsilon+3)\,e^{3\varepsilon\sqrt{2}}+4e^{2\varepsilon\sqrt{2}}}{4e^{4\varepsilon\sqrt{2}}+2(1-\varepsilon\sqrt{2})\,e^{3\varepsilon\sqrt{2}}-2(\varepsilon\sqrt{2}+1)\,e^{\varepsilon\sqrt{2}}-4},\\ a &= \frac{ce^{-\varepsilon\sqrt{2}}}{\sqrt{2}},\\ b &= \frac{1+ce^{-2\varepsilon\sqrt{2}}}{\sqrt{2}e^{\varepsilon\sqrt{2}}}, \end{aligned} \tag{3.15}$$

$$A=\frac{c(\sqrt{2}e^{2\varepsilon\sqrt{2}}-2\varepsilon e^{\varepsilon\sqrt{2}}-\sqrt{2})+e^{2\varepsilon\sqrt{2}}(2\sqrt{2}e^{2\varepsilon\sqrt{2}}-e^{\varepsilon\sqrt{2}}(\sqrt{2}+\varepsilon)-\sqrt{2})}{\sqrt{2}e^{2\varepsilon\sqrt{2}}(1-e^{2\varepsilon\sqrt{2}})},$$

$$B=-2-A.$$

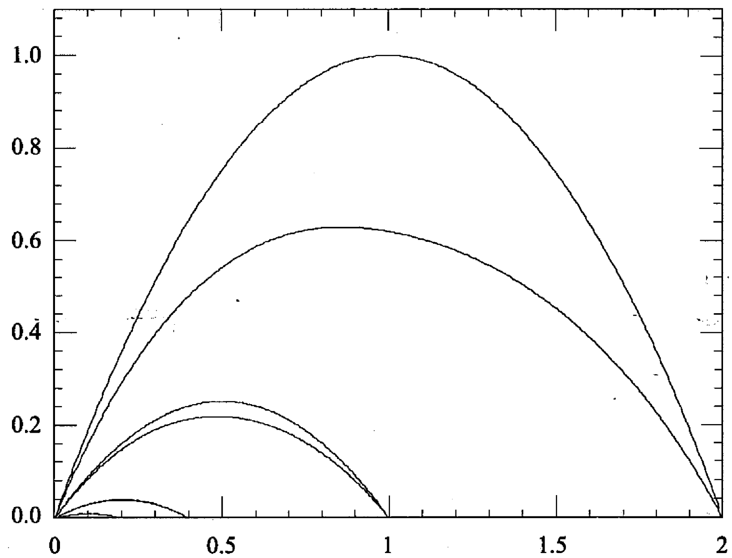

Fig. 1. For the process $X(t)$ which is the solution of $dX(t) = dW_t + \varepsilon dN_t^+$ (Example 3.1), $E(\tau_{0,2\varepsilon}(x))$ is compared with $E(\tilde{\tau}_{0,2\varepsilon}(x))$ (higher curve), that is the expected first-exit time of simple-diffusion, i.e. the Brownian motion, from the interval $(0, 2\varepsilon)$. The two expected exit times are reported as a function of the starting point $x \in (0, 2\varepsilon)$ for a set of values of $\varepsilon = 1, 0.5, 0.2, 0.1$ from the right to the left. Notice that the greater ε, the more accentuated the asymmetry of the graph of $E(\tau(x))$; for instance, when $\varepsilon = 1$, the maximum is attained at $\bar{x} \approx 0.83$. Further, the smaller ε, the more the two curves become close one to the other; for $\varepsilon = 0.1$, no difference can be detected.

As expected, the point at which $z(x) = E(\tau(x))$ attains its maximum does not coincide with $x = \varepsilon$, but it is shifted towards the left, i.e. $E(\tau(x))$ is not symmetric with respect to the middle point of the interval $(0, 2\varepsilon)$. This is because the process can exit on the left only trough continuous trajectories, while it can also exit on the right by a jump. The greater the jump amplitude ε, the more accentuated the asymmetry of the graph of $E(\tau(x))$; for instance, when $\varepsilon = 1$, the maximum is attained at $\bar{x} \approx 0.83$. In Figure 1, the graph of $E(\tau(x))$ is reported as a function of the starting point $x \in (0, 2\varepsilon)$, for a set of values of ε and it is compared with that corresponding to simple-diffusion, i.e. the Brownian motion. Of course, if $\tilde{\tau}(x)$ denotes the first-exit time of the Brownian motion from the interval $(0, 2\varepsilon)$, with the condition that it has started from x, we have $E(\tau(x)) \leqslant E(\tilde{\tau}(x))$.

(VI) *The solution of the problem for* $\pi_0(x)$:

$$\begin{aligned} &\tfrac{1}{2} z''(x) + z(x+\varepsilon) - z(x) = 0, \qquad x \in (0, 2\varepsilon); \\ &z(x) = 1, \ x \leqslant 0; \qquad z(x) = 0, \ x \geqslant 2\varepsilon, \end{aligned} \tag{3.16}$$

is given by

$$z(x) = \begin{cases} e^{-\sqrt{2}x}(A + ax) + e^{\sqrt{2}x}(B + bx), & x \in (0, 2\varepsilon), \\ c\,(e^{-\sqrt{2}x} - e^{-4\sqrt{2}\varepsilon + \sqrt{2}x}), & x \in (\varepsilon, 2\varepsilon), \end{cases} \tag{3.17}$$

where a is given by (3.15), $b = -ce^{-3\varepsilon\sqrt{2}}/\sqrt{2}$, $B = 1-A$, *and the values of A and c are obtained by requiring that* $z(x) \in C^2(0, 2\varepsilon)$.

(VII) *Analogous calculations allow to obtain the formula for* $E(\tau_0(x)) = T(x)/\pi_0(x)$, *where* $T(x)$ *is the solution of the problem*

$$\text{(3.18)} \qquad \begin{aligned} \tfrac{1}{2}T''(x)+T(x+\varepsilon)-T(x) &= -\pi_0(x), & x &\in (0, 2\varepsilon), \\ T(x) &= 0, & x &\notin (\varepsilon, 2\varepsilon). \end{aligned}$$

Remark 3.1. Actually, the calculations required to obtain the explicit solutions of problems with outer conditions, such as (3.13), (3.16), (3.18), are very long and tedious. By the way, we observe that if one had considered e.g. boundaries $\alpha = 0$ and $\beta = n\varepsilon$, n being an integer, the calculations would be still more complicated; indeed, the solution should be searched first in the interval $((n-1)\varepsilon, n\varepsilon)$, up to a set of undetermined constants, then it should be searched back in the preceding interval $((n-2)\varepsilon, (n-1)\varepsilon)$, and so on, recursively, until the first interval $(0, \varepsilon)$ is reached. Finally, all the constants should be found, by requiring the solution to be C^2 in the whole interval $(0, n\varepsilon)$.

Remark 3.2. If the jump amplitude ε is small enough, the solutions of the above problems are close to those (easier to obtain) of the corresponding problems with boundary conditions.

(i) Let us consider the problem with outer conditions

$$\text{(3.19)} \qquad \begin{aligned} &Lz(x) \doteq (1/2)\sigma^2(x)z''(x) \\ &\qquad + b(x)z'(x) + \int [z(x+\gamma(x, u)) - z(x)]\, \Pi(du) = -1, \qquad x \in (\alpha, \beta), \\ &z(x) = 0, \quad x \notin (\alpha, \beta), \end{aligned}$$

whose solution is $z(x) = E(\tau_{\alpha\beta}(x))$ and let $\bar{z}(x)$ be the solution of the corresponding problem with boundary conditions

$$\text{(3.20)} \qquad \begin{aligned} L\bar{z}(x) &= -1, \quad x \in (\alpha, \beta), \\ \bar{z}(\alpha) &= \bar{z}(\beta) = 0. \end{aligned}$$

By Itô's formula and taking expectation we obtain

$$\text{(3.21)} \qquad E(\tau(x)) = \bar{z}(x) - E(\bar{z}(X(\tau))).$$

Since $X(\tau) \in (\alpha-\varepsilon, \alpha] \cup [\beta, \beta+\varepsilon)$, for $M = \max\{-\bar{z}(\alpha-\varepsilon), -\bar{z}(\beta+\varepsilon)\} > 0$, from (3.21) we get (notice that $\bar{z}(X(\tau)) \leqslant 0$):

$$\text{(3.22)} \qquad \bar{z}(x) < E(\tau(x)) < \bar{z}(x) + M.$$

Thus, since $\bar{z} \in C^2$ and $\bar{z}(\alpha) = \bar{z}(\beta) = 0$, if ε is *small*, also M will be *small*, and (3.22) gives an approximation (easier to calculate) of the solution of (3.19).

(ii) The equation that gives $\pi_\alpha(x)$ is

$$(3.23)\qquad \begin{aligned} &Lz(x)=0, \quad x\in(\alpha,\beta);\\ &z(x)=1,\ x\leqslant\alpha; \quad z(x)=0,\ x\geqslant\beta, \end{aligned}$$

and the corresponding problem with boundary condition is

$$(3.24)\qquad \begin{aligned} &L\bar{z}=0, \quad x\in(\alpha,\beta),\\ &\bar{z}(\alpha)=1, \quad \bar{z}(\beta)=0. \end{aligned}$$

Let $\bar{\pi}_\alpha(x)$ be the solution of (3.24); since $X(\tau)\in(\alpha-\varepsilon,\alpha]\cup[\beta,\beta+\varepsilon)$, by the same argument as that used in the proof of Theorem 2.2 we obtain

$$(3.25)\qquad \pi_\alpha(x)\leqslant\bar{\pi}_\alpha(x)<\pi_\alpha(x)+[\bar{\pi}_\alpha(\alpha-\varepsilon)-1].$$

Since $\bar{\pi}_\alpha\in C^2$, if ε is *small*, the term in the brackets is also small. Thus (3.25) gives an approximation of the solution of (3.23).

(iii) Let $T(x)=E(\tau_\alpha\pi_\alpha)$ be the solution of the problem with outer conditions

$$(3.26)\qquad \begin{aligned} &LT=-\pi_\alpha, \quad x\in(\alpha,\beta),\\ &T(x)=0, \quad x\notin(\alpha,\beta), \end{aligned}$$

and let us consider the solution $\bar{T}(x)$ of the problem with boundary conditions

$$(3.27)\qquad \begin{aligned} &L\bar{T}=-\bar{\pi}_\alpha, \quad x\in(\alpha,\beta),\\ &\bar{T}(\alpha)=\bar{T}(\beta)=0. \end{aligned}$$

As in the case (i), we obtain

$$E\big(\bar{T}(X(\tau_\alpha))\big)=\bar{T}(x)-\bar{\pi}_\alpha(x)E(\tau_\alpha).$$

Since $\alpha-\varepsilon<X(\tau_\alpha)\leqslant\alpha$, and $\bar{T}(x)$ is increasing on the left of α, i.e. $\bar{T}(X(\tau_\alpha))>\bar{T}(\alpha-\varepsilon)$, we have (notice that $\bar{T}(X(\tau_\alpha))\leqslant 0$):

$$\bar{T}(x)<\bar{\pi}_\alpha(x)E(\tau_\alpha(x))<\bar{T}(x)-\bar{T}(\alpha-\varepsilon),$$

i.e.

$$(3.28)\qquad \frac{\bar{T}(x)}{\bar{\pi}_\alpha(x)}<E(\tau_\alpha(x))<\frac{1}{\bar{\pi}_\alpha(x)}[\bar{T}(x)-\bar{T}(\alpha-\varepsilon)].$$

Of course, the approximation above is meaningless for x close to β (at $x=\beta$, $\pi_\alpha(x)$ becomes zero).

Returning to Example 3.1, we consider now the corresponding problems with boundary conditions, discussed in Remark 3.2.

Actually, (3.20) becomes

$$(3.29)\qquad \begin{aligned} &\tfrac{1}{2}\bar{z}''(x)+\bar{z}(x+\varepsilon)-\bar{z}(x)=-1, \quad x\in(0,2\varepsilon),\\ &\bar{z}(0)=\bar{z}(2\varepsilon)=0, \end{aligned}$$

which has the solution

$$\bar{z}(x) = -\frac{2}{1-e^{2\varepsilon B}}e^{Bx} - \frac{1}{\varepsilon}x + \frac{2}{1-e^{2\varepsilon B}}, \tag{3.30}$$

where B is the negative solution of the equation

$$B^2/2 + e^{B\varepsilon} - 1 = 0. \tag{3.31}$$

The approximating problem (3.24) becomes

$$\begin{gathered} \tfrac{1}{2}\bar{z}''(x) + \bar{z}(x+\varepsilon) - \bar{z}(x) = 0, \quad x \in (0, 2\varepsilon), \\ \bar{z}(0) = 1, \quad \bar{z}(2\varepsilon) = 0, \end{gathered} \tag{3.32}$$

which has the solution

$$\bar{\pi}_0(x) = \frac{e^{2\varepsilon B} - e^{Bx}}{e^{2\varepsilon B} - 1}, \tag{3.33}$$

where B is given by (3.31).

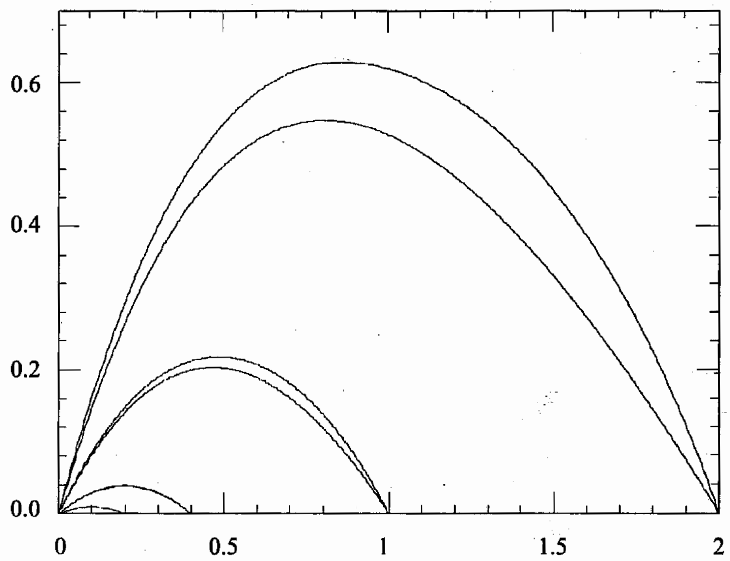

Fig. 2. For the same process as shown in Fig. 1, the solution $E(\tau_{0,2\varepsilon}(x))$ of the problem with outer conditions (3.19) is compared with the solution $E(\bar{\tau}_{0,2\varepsilon}(x))$ of the corresponding problem with boundary conditions (3.20) (lower curve). They are reported as a function of the starting point $x \in (0, 2\varepsilon)$ for a set of values of $\varepsilon = 1, 0.5, 0.2, 0.1$ from the right to the left. The smaller ε, the more the two curves become close one to the other; for $\varepsilon = 0.1$, no difference can be detected.

The approximating problem (3.27) is

$$\begin{aligned} &\tfrac{1}{2}\bar{T}''(x)+\bar{T}(x+\varepsilon)-\bar{T}(x) = -\bar{\pi}_0(x), \qquad x\in(0, 2\varepsilon), \\ &\bar{T}(0) = \bar{T}(2\varepsilon) = 0, \end{aligned} \tag{3.34}$$

which has the solution

$$\bar{T}(x) = e^{Bx}(ax+b)+cx+d, \tag{3.35}$$

where B is given by (3.31) and

$$\begin{aligned} &h = e^{2\varepsilon B}, \qquad a = [(h-1)(B+e^B)]^{-1}, \\ &c = -h/(h-1), \qquad b = -d = -2\varepsilon\frac{c+ae^{2B\varepsilon}}{e^{2B}-1}. \end{aligned} \tag{3.36}$$

In Figure 2, the solution of the problem with boundary conditions (3.20), relative to the SDE of Example 3.1, is compared with that of the corresponding problem with outer conditions (3.19) for a set of values of ε.

4. FIRST-PASSAGE-TIME DENSITY AND THE LAPLACE TRANSFORM

In this section, unlike the previous ones, we shall follow an approach based on the Laplace transform; indeed, our goal will be to express the conditional moments of first-passage time (FPT-moments) of a jump-diffusion process through each of two constant boundaries, in terms of the Laplace transform of the simple-diffusion FPT-density; moreover, in some cases we shall compare the jump-diffusion FPT-moments with the simple-diffusion ones. The unconditional case has been already considered in [12].

Let $X(t)$ be the jump-diffusion process which is the solution of the equation (see (2.5)):

$$dX(t) = b(X(t))\,dt+\sigma(X(t))\,dW_t+\varepsilon_1\,dN_1(t)+\varepsilon_2\,dN_2(t) \tag{4.1}$$

and let

$$g(t\,|\,x) = \frac{d}{dt}P[\tau_{\alpha\beta}(x) < t] \tag{4.2}$$

be the probability density function of the first-exit time (FPT-density) of the process $X(t)$ from (α, β), with the condition that $X(0) = x$. Moreover, let

$$g_\alpha(t\,|\,x) = \frac{d}{dt}P[\tau_\alpha(x) < t], \qquad g_\beta(t\,|\,x) = \frac{d}{dt}P[\tau_\beta(x) < t] \tag{4.3}$$

be the FPT-density of the conditional first-exit time through the end α and β, respectively. The n-th order moment of $\tau_{\alpha\beta}(x)$ is given by

$$E(\tau^n(x)) = \int_0^\infty t^n g(t\,|\,x)\,dt \tag{4.4}$$

and analogous definitions hold for $E(\tau_\alpha^n(x))$ and $E(\tau_\beta^n(x))$.

Now, let $\tilde{X}(t)$ be the simple-diffusion process obtained from the equation (4.1) disregarding the jumps, i.e. $\tilde{X}(t)$ is the solution of (4.1) with $\varepsilon_1 = \varepsilon_2 = 0$:

$$d\tilde{X}(t) = b(\tilde{X}(t))\,dt + \sigma(\tilde{X}(t))\,dW_t. \tag{4.5}$$

We denote, respectively, by $\tilde{g}(t \mid x)$, $\tilde{g}_\alpha(t \mid x)$, and $\tilde{g}_\beta(t \mid x)$ the probability density functions of the first-exit times $\tilde{\tau}(x)$, $\tilde{\tau}_\alpha(x)$, and $\tilde{\tau}_\beta(x)$, relative to the simple-diffusion process $\tilde{X}(t)$, i.e.

$$E(\tilde{\tau}^n(x)) = \int_0^\infty t^n \tilde{g}(t \mid x)\,dt \tag{4.6}$$

and analogous formulas hold for $E(\tilde{\tau}^n_\alpha(x))$, $E(\tilde{\tau}^n_\beta(x))$. Finally, we denote by $\hat{g}$ and $\hat{\tilde{g}}$ the Laplace transform of the densities g and $\tilde{g}$:

$$\hat{g}(\mu \mid x) = \int_0^\infty e^{-\mu t} g(t \mid x)\,dt, \qquad \hat{\tilde{g}}(\mu \mid x) = \int_0^\infty e^{-\mu t} \tilde{g}(t \mid x)\,dt, \tag{4.7}$$

and analogous definitions hold for $\hat{g}_A$ and $\hat{\tilde{g}}_A$, $A = \alpha, \beta$.

4.1. Constant amplitude Poissonian jumps. In this subsection, we deal with the case of constant amplitude Poissonian jumps. As already noted (see e.g. Example 3.1 of Section 3), a heavy computation is required to solve explicitly the differential-difference equations (3.2) satisfied by the moments $M_n(x) = E(\tau^n(x))$. Then it would be useful to find alternative formulas, involving the Laplace transform of FPT-densities.

Let us consider the special case in which $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon > 0$ in (4.1) and $\lambda_1 = \lambda$; therefore, we consider upward Poisson-distributed jumps with constant amplitude ε and intensity λ. The equations (3.2) for the moments $M_n(x)$ become

$$\begin{aligned} &-\lambda M_n(x) + b(x)\frac{dM_n}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2 M_n}{dx^2} + \lambda M_n(x+\varepsilon) = -nM_{n-1}(x), \\ &M_n(x) = 0, \qquad x \notin (\alpha, \beta),\ n = 1, 2, \ldots \end{aligned} \tag{4.8}$$

We further suppose that β is an absorbing barrier for the simple-diffusion process $\tilde{X}(t)$ associated with the equation (4.1). Let $\tilde{f}_\beta(y, t \mid x)$ be the transition probability density function of $\tilde{X}(t)$ constrained to be absorbed at the boundary β and denote by

$$\hat{\tilde{f}}_\beta(\mu, y \mid x) = \int_0^\infty e^{-\mu t} \tilde{f}_\beta(y, t \mid x)\,dt$$

its Laplace transform. Then, by adapting the results of [12] to the actual case, we are able to obtain:

THEOREM 4.1. *The probability $\pi_\beta(x)$ of ultimate absorption of the process $X(t)$ at the boundary β is the solution of the following integral equation:*

$$(4.9)\qquad \pi_\beta(x) = \lambda \int_{\alpha-\varepsilon}^{\beta} \hat{\tilde{f}}_\beta(\lambda, z-\varepsilon \mid x) I_{(\alpha+\varepsilon,\beta)} \pi_\beta(z)\, dz + \int_{\beta-\varepsilon}^{\beta+\varepsilon} \hat{\tilde{f}}_\beta(\lambda, y \mid x)\, dy + \hat{\tilde{g}}_\beta(\lambda \mid x),$$

where $I_{(a,b)}$ is the indicator function of the interval (a, b). ∎

THEOREM 4.2. *If $\pi_\beta(x) \equiv 1$ (i.e. the boundary α is repelling), then the first and second moments of the first-passage time of the process $X(t)$ through $x = \beta$ are solutions of the following integral equations:*

$$(4.10)\qquad E(\tau(x)) \equiv E(\tau_\beta(x)) = \lambda \int_{\alpha-\varepsilon}^{\beta} E(\tau_\beta(z))(z)\, \hat{\tilde{f}}_\beta(\lambda, z-\varepsilon \mid x) I_{(\alpha+\varepsilon,\beta)}(z)\, dz + \frac{1}{\lambda}(1-\hat{\tilde{g}}_\beta(\lambda \mid x)),$$

$$(4.11)\qquad E(\tau^2(x)) \equiv E(\tau_\beta^2(x)) = \lambda \int_{\alpha-\varepsilon}^{\beta} E(\tau_\beta^2(z))\, \hat{\tilde{f}}_\beta(\lambda, z-\varepsilon \mid x) I_{(\alpha+\varepsilon,\beta)}(z)\, dz - 2\frac{d\hat{\tilde{F}}_\beta(\lambda \mid x)}{d\lambda} - 2\lambda \int_{\alpha-\varepsilon}^{\beta} dz E(\tau_\beta(z)) I_{(\alpha+\varepsilon,\beta)}(z) \frac{d\hat{\tilde{f}}_\beta}{d\lambda}(\lambda, z-\varepsilon \mid x),$$

where $\hat{\tilde{F}}_\beta(\lambda \mid x)$ is the Laplace transform of the cumulative transition probability of the process $\tilde{X}(t)$ constrained to be absorbed at the boundary β, with respect to the parameter λ. ∎

The analogous results hold for the moments of $\tau_\alpha(x)$, in the case when α is absorbing and $\pi_\alpha(x) \equiv 1$.

4.2. Large Poissonian jumps. Now, we suppose that the amplitudes ε_1 and ε_2 of the jumps are not constant, but they are state-dependent, in order that, at any jump instant, the process exits from (α, β), irrespective of its state before the occurrence of the jump. For the sake of simplicity, we model the upward and downward jumps by means of a unique Poisson process with intensity λ, in such a way that, at each jump instant, an upward jump occurs with probability p, and a downward one occurs with probability $q = 1-p$. Then we have

PROPOSITION 4.1. *If τ, τ_α and τ_β are honest random variables, then the probability densities $g(t \mid x)$, $g_\alpha(t \mid x)$ and $g_\beta(t \mid x)$ are given by*

$$(4.12)\qquad g(t \mid x) = e^{-\lambda t} \tilde{g}(t \mid x) + \lambda e^{-\lambda t} \int_t^\infty \tilde{g}(s \mid x)\, ds,$$

$$(4.13)\qquad g_\alpha(t \mid x) = e^{-\lambda t}\left[\tilde{g}_\alpha(t \mid x) + \lambda \int_t^\infty \tilde{g}_\alpha(s \mid x)\, ds + \lambda(1-q)(\lambda t - 1)\right],$$

$$(4.14)\qquad g_\beta(t \mid x) = e^{-\lambda t}\left[\tilde{g}_\beta(t \mid x) + \lambda \int_t^\infty \tilde{g}_\beta(s \mid x)\, ds + \lambda(1-p)(\lambda t - 1)\right].$$

Proof. We shall prove only (4.14), since the other relations can be proved analogously; for $p = 1$ the equation (4.14) reduces to equation (5) of [12], i.e. the right member of (4.14) becomes that of (4.12).

The path of the process can be divided into two disjoint classes: one consists of the realizations which exit through β before the occurrence of a jump, the second class consists of all the others. Of course, the probability that, until the time t, the first jump has occurred upwardly (and the successive jumps have occurred upwardly or downwardly) is

$$p\lambda te^{-\lambda t}+1-e^{-\lambda t}-\lambda te^{-\lambda t} = 1-e^{-\lambda t}\big(\lambda t(1-p)+1\big).$$

Then

$$P\big(\tau_\beta(x) < t\big) = P\big(\tilde{\tau}_\beta(x) < t\big)e^{-\lambda t}+1-e^{-\lambda t}\big(\lambda t(1-p)+1\big). \tag{4.15}$$

Differentiating (4.15) with respect to t, we easily obtain (4.14). ∎

PROPOSITION 4.2. *The Laplace transform of g_β is given by*

$$\hat{g}_\beta(\mu|x) = \int_0^\infty e^{-\mu t} g_\beta(t|x)\,dt = \hat{\tilde{g}}_\beta(\lambda+\mu|x)\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}\left(1-\frac{q\mu}{\lambda+\mu}\right), \tag{4.16}$$

where $q = 1-p$ and $\hat{\tilde{g}}_\beta$ is the Laplace transform of the density of the first-exit time of the simple-diffusion through the end β. Analogous formulas hold for $\hat{g}$ and $\hat{g}_\alpha$.

The proposition can be obtained by a straightforward calculation, by using (4.14). ∎

Remark 4.1. The Laplace transform of the density of the first-exit time $\tau(x)$ from (α, β) is given by

$$\hat{g}(\mu|x) = \hat{\tilde{g}}(\lambda+\mu|x)\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}.$$

It can be obtained by the same argument as that used in [12] to get the Laplace transform of the density of the first-passage time τ_S of a process with upward jumps through a barrier S. Then

$$\hat{g}_\beta(\mu|x) = \hat{g}(\mu|x)-\frac{q\lambda\mu}{(\lambda+\mu)^2}. \tag{4.17}$$

(When $p = 1$, (4.17) reduces to the equation (7) of [12].)

THEOREM 4.1 (cf. [12]). *If $\tau(x)$ is an honest random variable, its moments of n-th order satisfy the following recursive equations ($n = 1, 2, \ldots$):*

$$E\big(\tau^n(x)\big) = \frac{n}{\lambda}\left\{E\big(\tau^{n-1}(x)\big)+(-1)^n\left[\frac{d^{n-1}\hat{\tilde{g}}(\lambda+\mu|x)}{d\mu^{n-1}}\right]_{\mu=0}\right\}. \tag{4.18}$$

Moreover,

$$E(\tau^n(x)) \leqslant E(\tilde{\tau}^n(x)), \tag{4.19}$$

i.e. the moments of the first-exit time of the jump-diffusion process from the interval (α, β) *are less than those of the corresponding simple-diffusion.*

Proof. The theorem can be obtained by the proof of the analogous Theorem 2.2 of [12], concerning the first-passage time τ_S. ∎

Now, we are going to investigate the moments of τ_α and τ_β.

THEOREM 4.2. *Under the hypotheses of Proposition 4.1, the moments of* $\tau_\beta(x)$ *satisfy the following recursive equation* $(n = 1, 2, \ldots)$:

(4.20)
$$E(\tau_\beta^n(x)) = \frac{n}{\lambda}\left\{E(\tau_\beta^{n-1}(x)) + (-1)^n \left[\frac{d^{n-1}\hat{g}_\beta(\lambda+\mu \mid x)}{d\mu^{n-1}}\right]_{\mu=0} + (-1)^n \frac{qn!}{\lambda^{n-1}}\right\}$$

and an analogous formula holds for $E(\tau_\alpha^n(x))$.

Proof. By differentiating (4.17) with respect to μ, we obtain

$$\begin{aligned} \frac{d^n \hat{g}_\beta(\mu \mid x)}{d\mu^n} &= \frac{d^n \hat{g}(\mu \mid x)}{d\mu^n} - q\lambda \frac{d^n}{d\mu^n}\left(\frac{\mu}{(\lambda+\mu)^2}\right) \\ &= \frac{d^n \hat{g}(\mu \mid x)}{d\mu^n} - q\lambda(-1)^n n!(n\lambda - \mu)/(\lambda+\mu)^{n+2}, \end{aligned} \tag{4.21}$$

where $\hat{g}$ is the Laplace transform of the density g of τ. Then, taking into account the equality

$$\left.\frac{d^n \hat{g}(\mu \mid x)}{d\mu^n}\right|_{\mu=0} = E(\tau^n(x))$$

obtained from (4.18), putting $\mu = 0$ in (4.21), we easily get (4.20). ∎

Now, we give an integral representation of the moments of first-exit times.

THEOREM 4.3. *The following formula holds for the n-th order moment of* $\tau_\beta(x)$:

$$\begin{aligned} E(\tau_\beta^n(x)) &= E(\tilde{\tau}_\beta^n(x)) \\ &+ \frac{n!}{\lambda^n}\int_0^\infty \left[1 - e^{-\lambda t}\sum_{i=0}^{n-1}\frac{(\lambda t)^i}{i!} - \frac{\lambda^n t^n}{n!}\right] g_\beta(t \mid x)\, dt + \frac{nn!}{\lambda^n}(1-p). \end{aligned} \tag{4.22}$$

Proof. From (4.14) we have

$$\begin{aligned} E(\tau_\beta^n(x)) &= \int_0^\infty t^n e^{-\lambda t}\tilde{g}_\beta(t \mid x)\, dt + \lambda \int_0^\infty \left[t^n e^{-\lambda t}\int_t^\infty \tilde{g}_\beta(s \mid x)\, ds\right] dt \\ &+ \lambda(1-p)\int_0^\infty t^n e^{-\lambda t}(\lambda t - 1)\, dt = J_1(x) + \lambda J_2(x) + \lambda(1-p) J_3(x). \end{aligned} \tag{4.23}$$

Integrating by parts we obtain for the second integral $J_2(x)$:

(4.24) $$J_2(x) = \Big[\int_t^\infty g_\beta(s\,|\,x)\,ds\cdot I_n(t)\Big]_0^\infty + \int_0^\infty I_n(t)\,g_\beta(t\,|\,x)\,dt,$$

where $I_n(t) = \int t^n e^{-\lambda t}\,dt$ can be found recursively by the relation

(4.25) $$I_n(t) = \frac{1}{\lambda}\big[-e^{-\lambda t}t^n + nI_{n-1}(t)\big] + \mathrm{const}, \qquad n = 1, 2, \ldots$$

Analogously, $J_3(x)$ can be found in terms of integrals of the form (4.25). Then it is straightforward to check that (4.22) holds. ∎

COROLLARY 4.1. *If p is large enough, then*

(4.26) $$E\big(\tau_\beta^n(x)\big) < E\big(\tilde{\tau}_\beta^n(x)\big).$$

If $p = 1$, (4.26) *becomes* (4.19).

Proof. Let us put

$$h(t) = 1 - e^{-\lambda t}\sum_{i=0}^{n-1}\frac{(\lambda t)^i}{i!} - \frac{\lambda^n t^n}{n!}.$$

As easily seen, $h(0) = 0$ and $h(t) < 0$, $t > 0$. Then the integral in (4.22) is negative, and thus (4.26) holds provided that $1-p$ is small enough. ∎

Remark 4.2. Notice that by using (4.20) one can obtain (4.26) for $n = 1$, while, for $n \geqslant 2$, only the weaker estimate can be found:

$$E\big(\tau_\beta^n(x)\big) < nE\big(\tilde{\tau}_\beta^n(x)\big).$$

The result of Corollary 4.1 is easily understood if one observes that, although the n-th order moment of the unconditional first-exit time of the jump-diffusion process from (α, β) is less than that of the simple diffusion (see (4.19)), in the case when the probability p of an upward jump is small, it might take a longer time to exit on the right. On the contrary, if $p \approx 1$, the process exits for the first time on the right more likely than in the simple-diffusion case.

We observe that (4.19) and (4.26) can also be proved by solving the equations obtained from (4.8) putting equal to infinity the amplitude ε of the jumps, and then by comparing those solutions with the moments of the simple diffusion. For instance, in the case when only upward large jumps are allowed, $E\big(\tau(x)\big)$ is the solution of the equation

(4.27) $$\begin{gathered} b(x)z'(x) + \tfrac{1}{2}\sigma^2(x)z''(x) - \lambda z(x) = -1, \\ z(x) = 0, \quad x \notin (\alpha, \beta), \end{gathered}$$

while $E(\tilde{\tau}(x))$ is the solution of the equation

$$
\begin{aligned}
&b(x)v'(x)+\tfrac{1}{2}\sigma^2(x)v''(x) = -1, \\
&v(\alpha) = v(\beta) = 0.
\end{aligned}
\tag{4.28}
$$

If $b(x)$ and $\sigma(x)$ are explicitly known, then by solving (4.27) and (4.28) it is possible to verify directly that $z(x) \leqslant v(x)$ for all $x \in (\alpha, \beta)$.

Acknowledgment. I am deeply grateful to my wife Laura for carrying out all the long and tedious calculations needed to obtain the explicit solutions of the differential-difference equations of Section 3.

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Received on 19.9.2000