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THE INFINITE DIVISIBILITY AND ORTHOGONAL POLYNOMIALS WITH A CONSTANT RECURSION FORMULA IN FREE PROBABILITY THEORY

BY

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Abstract. We calculate Voiculescu's R-transform of the compactly supported probability measure on $\boldsymbol{R}$ induced from the orthogonal polynomials with a constant recursion formula, and investigate its infinite divisibility with respect to the additive free convolution. In the case of infinite divisibility, we give the Lévy–Hinčin measure explicitly for the integral representation of the R-transform of the free analogue of the Lévy–Hinčin formula.

1. Introduction. In [15], Voiculescu began studying the operator algebra free products from the probabilistic point of view. His idea is to look at free products as an analogue of tensor products and to develop a corresponding highly noncommutative probabilistic framework, where freeness is given as the notion of independence. In [16] the operation of the additive free convolution has been introduced as the analogue of the usual convolution. In order to compute it, the R-transform (free cumulant) was also introduced, which linearizes the additive free convolution and its definition goes in terms of a certain family of formal Toeplitz operators. An alternative, combinatorial approach to the R-transform was found by Speicher in [13]. The most important advantage of this combinatorial approach is that it can be generalized in a straightforward way to multidimensional situations as in [12].

The infinite divisibility for the additive free convolution was also studied in [16]. The characterization of the $\boxplus$-infinitely divisible measure on $\boldsymbol{R}$ with a compact support was given, and it was explained in [17] that this characterization is an analogue of the classical Lévy–Hinčin theorem. Moreover, in [4] Bercovici and Voiculescu showed that the Lévy–Hinčin measure associated with a $\boxplus$-infinitely divisible measure can be calculated as a weak limit of measures related with the convolution semigroup.

Here we should note that Maassen gave in [10] a direct proof for the addition theorem for freely independent real-valued random variables which have unbounded supports in general, by using resolvents of self-adjoint operators without an assumption on the existence of moments above the second. The free infinite divisibility of a probability measure on $\boldsymbol{R}$ was also investigated in [10], and then the free analogue of the Lévy–Hinčin formula and the interpretation of the Lévy measure were given.

The probability measure on $\boldsymbol{R}$ is deeply related to the orthogonal polynomials as in [14]. In [7], Cohen and Trenholme calculated explicitly the measure for which a sequence of polynomials with a constant (Jacobi relation) recursion formula is orthogonal. If we normalize it so as to be a probability measure, many important distributions in the free probability theory can be realized as such measures, for instance, the semicircle law, the free Poisson distribution, the free binomial distribution. Many other investigations on the orthogonal polynomials in noncommutative probability theory can be found, for instance, in [1], [3], [6], and [11].

In this paper, we first find the Cauchy transform of the compactly supported probability measure on $\boldsymbol{R}$ induced from the orthogonal polynomials with a constant recursion formula and then calculate its R-transform. In the subsequent section, we investigate its infinite divisibility and give the Lévy–Hinčin measure for the integral representation of the R-transform of the free analogue of the Lévy–Hinčin formula in the case of infinite divisibility. Finally, we introduce the free negative binomial distribution as an interesting example of free infinitely divisible distribution.

2. The induced probability measure and the R-transform. Let $(\mathscr{A}, \varphi)$ be a noncommutative probability space, that is, $\mathscr{A}$ is a unital algebra over $\boldsymbol{C}$, together with a specified linear functional $\varphi\colon \mathscr{A} \to \boldsymbol{C}$ such that $\varphi(1) = 1$. An element $x \in \mathscr{A}$ will be viewed as a random variable, the distribution of which is the functional $\mu_x\colon \boldsymbol{C}[X] \to \boldsymbol{C}$ given by $\mu_x(1) = 1$, $\mu_x(X^n) = \varphi(x^n)$. If $\mathscr{A}$ is a C^*-algebra and φ is a state, which is called a *C^*-probability space*, then for a self-adjoint element x, the distribution μ_x can be canonically extended to the probability measure on $\boldsymbol{R}$ with a compact support.

In noncommutative probability theory, independence is usually based on tensor products, in particular independent random variables commute. In [15] free products were proposed as a replacement of tensor products for definition of new independence. This leads to a highly noncommutative independence called a *free* independence (for definition see, e.g., [17]).

If x and y are two free random variables, then the distribution μ_{x+y} can be shown to depend only on μ_x and μ_y. This allows one to define the operation by $\mu_x \boxplus \mu_y = \mu_{x+y}$. Considering self-adjoint elements x and y in a C^*-probability space, it is easily seen that if μ_x and μ_y are compactly supported probability measures on $\boldsymbol{R}$, then $\mu_x \boxplus \mu_y$ is also a compactly supported probability measure on $\boldsymbol{R}$. The operation $\boxplus$ for analytic functionals or compactly supported probability measures on $\boldsymbol{R}$ will be called the *free* (*additive*) *convolution*, abbreviated the $\boxplus$-*convolution*.

In this paper, we will restrict ourselves to compactly supported probability measures on $\boldsymbol{R}$, and we will denote by $\mathscr{P}$ the class of all such measures. Let us recall some basic facts on $\boxplus$-convolution in $\mathscr{P}$ (see, e.g., [16], [17]). For given $\mu \in \mathscr{P}$, one constructs first the Cauchy transform

$$G_\mu(\zeta) = \int_{\boldsymbol{R}} \frac{d\mu(t)}{\zeta - t}$$

which is called the *G-series* of μ and it is analytic at ∞ and $G_\mu(\zeta) - 1/\zeta$ is bounded for $|\zeta|$ large. Thus there exists a meromorphic function $K_\mu(z)$ with single pole at 0, such that $G_\mu(K_\mu(z)) = z$ in a neighborhood of 0. Then one can have the analytic function in a neighborhood of 0, $R_\mu(z) = K_\mu(z) - 1/z$, which we call the *R-transform* of μ. It was shown in [16] that $R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z)$ for $\mu_1, \mu_2 \in \mathscr{P}$, which means that the R-transform is the analogue of the logarithm of the Fourier transform, because it makes $\boxplus$-convolution linearize.

Next we will recall the measure induced from the sequence of polynomials generated by the following constant recursion formula:

$$(*) \qquad \begin{aligned} &P_0(X) = c, \quad P_1(X) = X - \alpha, \\ &P_{m+1}(X) = (X-a)P_m(X) - bP_{m-1}(X) \quad (m \geqslant 1), \end{aligned}$$

where α and a are real numbers, and b and c are positive numbers.

It is shown in [8] that there exists a unique compactly supported positive measure ν on $\boldsymbol{R}$, up to constant multiplication, such that $\int_{\boldsymbol{R}} P_k(t) P_m(t) d\nu(t) > 0$ if $k = m$, and is 0 otherwise. In [7], Cohen and Trenholme calculated the measure ν explicitly, for which the sequence of polynomials $\{P_m(X)\}$ is orthogonal, by applying the translation theorem of Christoffel in the case of Chebyshev polynomials. We call such a measure the *induced measure* by the $\{P_m\}$. The normalization for an induced measure, given by Cohen and Trenholme, however, is not one for the probability measure. Here we should note that there is an error in [7] that we have to either multiply by c on the continuous part or divide by c on the discrete part in their original result (Theorem 3 of [7]). By normalizing their measure, we can obtain the induced probability measure as in the next theorem.

THEOREM 2.1. *Assume that $\{P_m\}$ satisfies the constant recursion formula* (*). *Let*

$$f(t) = (1-c)(t-a)^2 + (c-2)(\alpha - a)(t-a) + (\alpha - a)^2 + bc^2.$$

Then the Lebesgue absolutely continuous part μ_C of the probability measure μ induced by the $\{P_m\}$ is given by

$$d\mu_C(t) = \frac{c\sqrt{4b-(t-a)^2}}{2\pi f(t)} \chi_{[a-2\sqrt{b}, a+2\sqrt{b}]} dt,$$

and the discrete part μ_D *is* 0 *except possibly in the following cases*:

CASE 1. $f(t)$ *has two real roots* $y_1 \neq y_2$. *Then*

$$d\mu_D = \lambda_1 \delta_{y_1} + \lambda_2 \delta_{y_2},$$

where

$$\lambda_i = \frac{1}{\sqrt{(\alpha-a)^2-4b(1-c)}}\left(\frac{bc}{|y_i-\alpha|}-\frac{|y_i-\alpha|}{c}\right)_+.$$

CASE 2. $c=1$ *and* $\alpha \neq a$ *so that* $f(t)$ *has one root* $y=\alpha+b/(\alpha-a)$. *Then*

$$d\mu_D = \left(1-\frac{b}{(\alpha-a)^2}\right)_+ \delta_y,$$

where dt denotes Lebesgue measure, δ_x *means Dirac unit mass at x, and* χ_I *is the indicator function for the interval I. Here we adopt the notation* $(r)_+ = (r+|r|)/2$.

We shall denote by $\mathscr{C}$ ($\subset \mathscr{P}$) the class of all such induced probability measures. Of course, it is clear that the class $\mathscr{C}$ is parametrized by α, $a \in \boldsymbol{R}$ and $b, c > 0$.

The next lemma can be obtained by direct calculations, which will help us with the calculation of $G_\mu(\zeta)$ for μ in $\mathscr{C}$.

LEMMA 2.2. *Let* $f(t)$ *be as in Theorem* 2.1. *Assume that* $f(t)$ *has two real roots* $y_1 < y_2$. *Then the following equalities hold*:

$$c\left(\frac{bc}{|y_i-\alpha|}-\frac{|y_i-\alpha|}{c}\right) = \operatorname{sign}(y_i-a)\{(c-2)y_i+(2\alpha-ac)\},$$

$$\{(c-2)y_i+(2\alpha-ac)\}^2 = c^2\{(y_i-a)^2-4b\},$$

$$y_2-y_1 = \frac{c}{|1-c|}\sqrt{(\alpha-a)^2-4b(1-c)},$$

where $\operatorname{sign}(x)=1$ *if* $x \geqslant 0$ *and* $\operatorname{sign}(x)=-1$ *if* $x<0$.

PROPOSITION 2.3. *The G-series of the probability measure* μ *in Theorem* 2.1 *is given by*

$$G_\mu(\zeta) = \frac{\{(c-2)\zeta+(2\alpha-ac)\}+c\sqrt{(\zeta-a)^2-4b}}{-2f(\zeta)},$$

where the branch of the analytic square root should be determined by the condition that $\operatorname{Im}\zeta>0 \Rightarrow \operatorname{Im}G(\zeta) \leqslant 0$.

Proof. Since $G_\mu(\zeta)$ is given as the Cauchy transform of μ, it is enough to check that $G_\mu(\zeta)$ yields the probability measure μ by using the Stieltjes inversion formula [2] because μ is compactly supported on $\boldsymbol{R}$. The Stieltjes inversion formula says that μ has point masses where $G_\mu(\zeta)$ has poles on $\boldsymbol{R}$ and the mass at each point equals the residue there, and μ is absolutely continuous with respect to Lebesgue measure where $G_\mu(\zeta)$ has a non-zero imaginary part on the real axis with the density

$$-\frac{1}{\pi}\lim_{\varepsilon\to+0}\operatorname{Im} G_\mu(t+i\varepsilon).$$

In our case, it is easily seen that μ is absolutely continuous on the interval $[a-2\sqrt{b},\, a+2\sqrt{b}]$ with the density

$$\frac{c\sqrt{4b-(t-a)^2}}{2\pi f(t)}.$$

Now we shall see the discrete part of the measure μ. That is, we should find the real roots of $f(\zeta)$ and compute the residue of $G_\mu(\zeta)$ there.

If $c>1$, it is easy to see that $f(\zeta)$ always has two real roots, say $y_1<y_2$, and the inclusion $[a-2\sqrt{b},\, a+2\sqrt{b}]\subset[y_1,\, y_2]$ holds by the concavity of f and the inequalities $f(a)>0$ and $|y_i-a|>2\sqrt{b}$ $(i=1,2)$. In this case, $G_\mu(\zeta)$ has the simple poles at y_1 and y_2. Taking care of the choice of the branches of the analytic square root in $G_\mu(\zeta)$, it follows with the help of Lemma 2.2 that

$$\begin{aligned}
\operatorname{Res}\big(G_\mu(\zeta);\, y_1\big) &= \frac{\{(c-2)\,y_1+(2\alpha-ac)\}-c\sqrt{(y_1-a)^2-4b}}{-2(1-c)(y_1-y_2)}\\
&= \frac{-c\big((bc)/|y_1-\alpha|-|y_1-\alpha|/c\big)-c\,\big|(bc)/|y_1-\alpha|-|y_1-\alpha|/c\big|}{-2c\sqrt{(\alpha-a)^2-4b(1-c)}}\\
&= \frac{1}{\sqrt{(\alpha-a)^2-4b(1-c)}}\left(\frac{bc}{|y_1-\alpha|}-\frac{|y_1-\alpha|}{c}\right)_+ = \lambda_1,\\
\operatorname{Res}\big(G_\mu(\zeta);\, y_2\big) &= \frac{\{(c-2)\,y_1+(2\alpha-ac)\}+c\sqrt{(y_1-a)^2-4b}}{-2(1-c)(y_2-y_1)}\\
&= \frac{1}{\sqrt{(\alpha-a)^2-4b(1-c)}}\left(\frac{bc}{|y_2-\alpha|}-\frac{|y_2-\alpha|}{c}\right)_+ = \lambda_2.
\end{aligned}$$

If $c=1$, then

$$G_\mu(\zeta)=\frac{\{-\zeta+(2\alpha-a)\}+\sqrt{(\zeta-a)^2-4b}}{-2\big(-(\alpha-a)\,\zeta+\alpha(\alpha-a)+b\big)}$$

and, in addition, if $\alpha\neq a$, $G_\mu(\zeta)$ has a simple pole at $y=\alpha+b/(\alpha-a)$. The residue at y is calculated as

$$\operatorname{Res}\big(G_\mu(\zeta);\, y\big)=\frac{\{-y+(2\alpha-a)\}+\sqrt{(y-a)^2-4b}}{2(\alpha-a)}$$

$$= \frac{((\alpha-a)-b/(\alpha-a))+|(\alpha-a)-b/(\alpha-a)|}{2(\alpha-a)}$$

$$= \left(1-\frac{b}{(\alpha-a)^2}\right)_+ .$$

If $\alpha = a$, then $f(\zeta) = b > 0$ is constant and it has no zero-point on $\boldsymbol{R}$.

In the case where $0 < c < 1$, if $(\alpha-a)^2 > 4b(1-c)$, then $f(\zeta)$ has two real roots, say $y_1 < y_2$, and $[a-2\sqrt{b}, a+2\sqrt{b}] \cap [y_1, y_2] = \emptyset$ follows from the convexity of f and the inequalities $f(a) > 0$ and $|y_i - a| > 2\sqrt{b}$ $(i = 1, 2)$. The computation of the residue at each simple pole is the same as in the case of $c > 1$, but the differences are in the choice of the branches, and in the signature of $(y_i - a)$. So we would like to omit the details. Moreover, if $(\alpha-a)^2 = 4b(1-c)$, then $f(\zeta)$ has one real root

$$y = \frac{2\alpha-\alpha c-ac}{2(1-c)}$$

of multiplicity two. However, it can be seen that

$$\lim_{\zeta\to y} \frac{\partial}{\partial\zeta}\{(\zeta-y)^2 G_\mu(\zeta)\} = -\frac{1}{2}\left((c-2)y \pm \frac{c(y-a)}{\sqrt{(y-a)^2-4b}}\right) = 0,$$

where $\pm$ depends on the choice of the branches. In our situation, this signature is given by $\operatorname{sign}(\alpha-a)$. Thus there is no point mass at y. If $(\alpha-a)^2 < 4b(1-c)$, then $f(\zeta)$ does not have real root any longer.

Summing up the above arguments, it follows that the discrete part of the probability measure μ might be given as one in Theorem 2.1. ∎

PROPOSITION 2.4. *The R-transform of the probability measure μ in Theorem* 2.1 *for $c = 1$ is given by*

$$R_\mu(z) = \frac{bz}{1-(a-\alpha)z} + \alpha,$$

and for $c \neq 1$ by

$$R_\mu(z) = \left(\frac{c}{1-c}\right)\left(\frac{-\{(a-\alpha)z-1\}+\sqrt{\{(a-\alpha)z-1\}^2-4b(1-c)z^2}}{2z}\right) + \alpha,$$

where the analytic square root is chosen as $\lim_{z\to 0} R_\mu(z) = \alpha$.

Proof. In order to obtain $R_\mu(\zeta)$, it will be required to invert the function $G_\mu(\zeta)$, that is, to solve the equation $z = G_\mu(\zeta)$ in ζ. By a direct calculation, we see that $z = G_\mu(\zeta)$ yields the following equation in ζ: $A\zeta^2 + B\zeta + C = 0$, where

$$A = (1-c)z^2, \quad B = \{(ac+c\alpha-2\alpha)z + (c-2)\}z,$$

$$C = (bc^2 - ac\alpha + \alpha^2)z^2 - (ac-2\alpha)z + 1.$$

Hence, if $c = 1$, then we have

$$\zeta = K_\mu(z) = \frac{(b-a\alpha+\alpha^2)z^2-(a-2\alpha)z+1}{\{1-(a-\alpha)z\}z},$$

and

$$R_\mu(z) = K_\mu(z)-\frac{1}{z} = \frac{(b-a\alpha+\alpha^2)z^2+\alpha z}{\{1-(a-\alpha)z\}z} = \frac{bz}{1-(a-\alpha)z}+\alpha.$$

In the case where $c \neq 1$, the equation is quadratic. Put $D = B^2-4AC$; then

$$D = c^2z^2(\{(a-\alpha)z-1\}^2-4b(1-c)z^2).$$

Thus we obtain

$$K_\mu(z) = \frac{-\{(ac+c\alpha-2\alpha)z+(c-2)\}+c\sqrt{\{(a-\alpha)z-1\}^2-4b(1-c)z^2}}{2(1-c)z},$$

and

$$R_\mu(z) = \frac{-\{(ac+c\alpha-2\alpha)z-c\}+c\sqrt{\{(a-\alpha)z-1\}^2-4b(1-c)z^2}}{2(1-c)z}$$

$$= \frac{c}{1-c}\left(\frac{-\{(a-\alpha)z-1\}+\sqrt{\{(a-\alpha)z-1\}^2-4b(1-c)z^2}}{2z}\right)+\alpha.$$

Here the analytic square root should be chosen as $\lim_{z\to 0} R_\mu(z) = \alpha$. ∎

Remark 2.5. It follows from the expression of the R-transform that, in the case where $c = 1$, the probability measure induced from the orthogonal polynomials is the (shifted) free Poisson with

$$R_\mu(z) = \left(\frac{b}{a-\alpha}\right)\left(\frac{1}{1-(a-\alpha)z}\right)+\left(\alpha-\frac{b}{a-\alpha}\right) \quad \text{if } a \neq \alpha,$$

and is the semicircle law with

$$R_\mu(z) = bz+a \quad \text{if } a = \alpha.$$

3. The infinite divisibility and the free negative binomial distributions. The infinitely divisible measures with respect to the free additive convolution in $\mathscr{P}$ were described in [16], and the characterization of its R-transform was given in [4] as the free analogue of the Lévy–Hinčin formula (see also [10]). We start with reviewing the results.

DEFINITION 3.1. A measure $\mu \in \mathscr{P}$ is said to be $\boxplus$-*infinitely divisible* if for every natural number n there exists a measure $\mu_n \in \mathscr{P}$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}.$$

The following results are from [4] and [17] (see also [5] and [10]): A measure $\mu \in \mathscr{P}$ is $\boxplus$-infinitely divisible if and only if its R-transform $R_\mu(z)$ has analytic continuation to a neighborhood of $(\boldsymbol{C}\backslash\boldsymbol{R}) \cup \{0\}$, such that $\operatorname{Im} R_\mu(z) \geqslant 0$ for $\operatorname{Im} z > 0$. Furthermore, let $\mu \in \mathscr{P}$ be $\boxplus$-infinitely divisible with R-transform $R_\mu(z)$. Then $R_\mu(z)$ can be represented in the integral as

$$R_\mu(z) = \int_{\boldsymbol{R}} \frac{z}{1-tz} d\nu(t) + \kappa$$

for all $z \in (\boldsymbol{C}\backslash\boldsymbol{R}) \cup \{0\}$, where the constant κ and the positive measure ν are obtained as follows. Let $\{\mu_\varepsilon \mid \varepsilon > 0\} \subset \mathscr{P}$ be the semigroup such that $R_{\mu_\varepsilon}(z) = \varepsilon R_\mu(z)$. Then the measure ν is the weak limit as $\varepsilon \to 0$ of the positive measure ν_ε defined by $\nu_\varepsilon(t) = (1/\varepsilon) t^2 d\mu_\varepsilon(t)$, and

$$\kappa = \lim_{\varepsilon \to 0} (1/\varepsilon) \int_{\boldsymbol{R}} t d\mu_\varepsilon(t).$$

It was explained in [17] why the above integral representation of R-transform is an analogue of the usual Lévy–Hinčin formula. We can also find it in Section 6 of [10]. The positive measure ν obtained by the above limiting procedure is called the *Lévy–Hinčin measure.* Using the above characterization, we obtain the following theorem:

THEOREM 3.2. *Assume that the sequence of polynomials $\{P_m\}$ is generated from the constant recursion formula* (*). *Let μ be the probability measure induced by the $\{P_m\}$. Then we have the following*:

(1) *If $c > 1$, then the probability measure μ is not $\boxplus$-infinitely divisible.*

(2) *If $0 < c \leqslant 1$, then the probability measure μ is $\boxplus$-infinitely divisible and the free analogue of the Lévy–Hinčin representation can be given as*

$$R_\mu(z) = \int_{\boldsymbol{R}} \frac{z}{1-tz} d\nu(t) + \alpha$$

with the Lévy–Hinčin measure ν such that

$$d\nu(t) = \begin{cases} b\delta_{a-\alpha} & \text{if } c = 1, \\ \dfrac{c}{2\pi(1-c)} \sqrt{4b(1-c) - \{t-(a-\alpha)\}^2}\, dt & \text{if } 0 < c < 1, \end{cases}$$

where dt denotes Lebesgue measure and δ_x means Dirac unit mass at x.

Proof. In the case where $c > 1$, the R-transform of μ is given by the lower case in Proposition 2.4 and it could not be analytic on $(\boldsymbol{C}\backslash\boldsymbol{R}) \cup \{0\}$ because the inside of the square root $(\{(a-\alpha)z-1\}^2-4b(1-c)z^2)$ has the roots

$$\frac{(a-\alpha)\pm 2\sqrt{b(c-1)}\,i}{(a-\alpha)^2+4b(c-1)},$$

which lie in $(\boldsymbol{C}\backslash\boldsymbol{R})\cup\{0\}$.

If $c=1$, then it is clear that the R-transform has the integral representation with the Lévy–Hinčin measure $d\nu(t)=b\delta_{a-\alpha}$ and with the constant α.

In the case where $0<c<1$, the R-transform of μ is also given by the lower case in Proposition 2.4 and it is not difficult to see that the R-transform can be continued analytically to a neighborhood of $(\boldsymbol{C}\backslash\boldsymbol{R})\cup\{0\}$, so that $\operatorname{Im} R_\mu(z)\geqslant 0$ for $\operatorname{Im} z>0$ because $R_\mu(z)$ has the singularities only on $\boldsymbol{R}\backslash\{0\}$.

For $\varepsilon>0$, let μ_ε be a compactly supported probability measure with $R_{\mu_\varepsilon}(z)=\varepsilon R_\mu(z)$. Then we can find the parameters α_ε, a_ε, b_ε, and c_ε satisfying the equation

$$\varepsilon R_\mu(z)=\frac{c_\varepsilon}{1-c_\varepsilon}\left(\frac{-\{(a_\varepsilon-\alpha_\varepsilon)z-1\}+\sqrt{\{(a_\varepsilon-\alpha_\varepsilon)z-1\}^2-4b_\varepsilon(1-c_\varepsilon)z^2}}{2z}\right)+\alpha_\varepsilon,$$

that is, $\mu_\varepsilon\in\mathscr{C}$ with the parameters α_ε, a_ε, b_ε, and c_ε. Indeed, we obtain

$$\alpha_\varepsilon=\varepsilon\alpha,\qquad a_\varepsilon=a-(1-\varepsilon)\alpha,\qquad b_\varepsilon=b\big(1-(1-\varepsilon)c\big),\qquad c_\varepsilon=\frac{\varepsilon c}{1-(1-\varepsilon)c}.$$

Note that $0<c_\varepsilon<1$ for all $\varepsilon>0$. Applying Theorem 2.1 we have the probability measure μ_ε explicitly and the Lévy–Hinčin measure ν can be obtained as the weak limit

$$\frac{t^2}{\varepsilon}\mu_\varepsilon\to\nu\qquad(\varepsilon\to 0),$$

which is positive and absolutely continuous with respect to Lebesgue measure with the density

$$\frac{c}{2\pi(1-c)}\sqrt{4b(1-c)-\{t-(a-\alpha)\}^2}.$$

It is clear that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\boldsymbol{R}} t\,d\mu_\varepsilon(t)=\lim_{\varepsilon\to 0}\frac{\alpha_\varepsilon}{\varepsilon}=\alpha. \ \blacksquare$$

Remark 3.3. Using the well-known formula

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{z-t}\,dt=z-\sqrt{z^2-1}\quad\text{ for } z\in\boldsymbol{C}\backslash[-1,1],$$

where we have to take the suitable branch of the square root, we can prove (2) of Theorem 3.2 without much difficulty in computing directly the integral $\int_{\boldsymbol{R}} [z/(1-tz)] d\nu(t)$ with given Lévy-Hinčin measure. The reason why we have adopted the above proof is that we would like to make the property of the probability measure μ_ε clearer. Actually, it can be seen from the above proof that for $0 < c < 1$ (respectively, $c = 1$) the induced probability measure μ is itself infinitely divisible in the subclass $\mathscr{C}$ for which $0 < c < 1$ (respectively, $c = 1$).

EXAMPLE 3.4 (The free binomial distribution $B_f(n, p)$). The free analogue of the binomial distribution $B_f(n, p)$ has been introduced as the n-fold $\boxplus$--convolution $((1-p)\delta_0 + p\delta_1)^{\boxplus n}$ of the Bernoulli distribution, the R-transform of which is

$$n\left(\frac{(z-1)+\sqrt{(z-1)^2+4pz}}{2z}\right),$$

where $0 < p < 1$ and $n \geqslant 2$ (see, for instance, [3], [4], [17]). It is easy to verify that the distribution $B_f(n, p)$ is in the class $\mathscr{C}$ with parameters

$$\alpha = np, \quad a = (n-1)p + (1-p), \quad b = (n-1)p(1-p), \quad c = \frac{n}{n-1}.$$

This is in the case of $c = n/(n-1) > 1$. Hence $B_f(n, p)$ is not $\boxplus$-infinitely divisible.

In the usual probability theory, it is well known that the characteristic function of the binomial distribution $B(n, p)$ can be given as $((1-p)+pe^{it})^n$. Another interesting feature of the discrete distribution in the usual probability theory is the negative binomial distribution (the Polyà distribution) $NB(\lambda, p)$, the probability function of which is

$$\binom{-\lambda}{x}(-p)^\lambda(1-p)^x \qquad (x = 0, 1, 2, \ldots), \text{ where } \lambda > 0, \ 0 < p < 1,$$

and we consider

$$\binom{-\lambda}{x} = (-1)^x \frac{\Gamma(\lambda + x)}{\Gamma(\lambda)\, x!}.$$

The characteristic function of the negative binomial distribution $NB(\lambda, p)$ can be given as

$$\frac{p^\lambda}{(1-(1-p)e^{it})^\lambda} = \left(\frac{1}{p} + \left(1 - \frac{1}{p}\right)e^{it}\right)^{-\lambda}$$

which could be derived from that of the binomial distribution by putting therein $(-\lambda)$ for n and $(1-1/p)$ for p (see, for instance, [9]).

From this point of view, it is natural to define the free analogue of the negative binomial distribution in terms of the R-transform by putting $(-\lambda)$ for n and $(1-1/p)$ for p in the R-transform of the free binomial distribution.

DEFINITION 3.5. The probability measure $\mu \in \mathcal{P}$ is said to be the *free negative binomial distribution* of parameters $\lambda > 0$ and $0 < p < 1$ if its R-transform can be written in the form

$$R_\mu(z) = -\lambda\left(\frac{(z-1)+\sqrt{(z-1)^2+4(1-1/p)z}}{2z}\right),$$

and we denote such a distribution by $NB_f(\lambda, p)$.

EXAMPLE 3.6. The free negative binomial distribution $NB_f(\lambda, p)$ is in the class $\mathcal{C}$ with parameters

$$\alpha = \lambda\left(\frac{1}{p}-1\right), \quad a = (\lambda+1)\left(\frac{1}{p}-1\right)+\frac{1}{p},$$
$$b = (\lambda+1)\left(\frac{1}{p}-1\right)\frac{1}{p}, \quad c = \frac{\lambda}{\lambda+1}.$$

This is in the case of $0 < c = \lambda/(\lambda+1) < 1$. Thus $NB_f(\lambda, p)$ is $\boxplus$-infinitely divisible, which is consistent with the commutative case that the negative binomial distribution is infinitely divisible with respect to the usual convolution.

It is easy to verify that the probability measure of the free negative binomial distribution can be given as

$$d\mu(t) = \frac{\lambda\sqrt{-(t-\gamma_+)(t-\gamma_-)}}{2\pi t(t+\lambda)}dt + \max\left\{0,\ 1-\lambda\left(\frac{1}{p}-1\right)\right\}\delta_0,$$

where

$$\gamma_\pm = \left(\sqrt{\frac{(\lambda+1)(1-p)}{p}} \pm \sqrt{\frac{1}{p}}\right)^2.$$

Finally, we mention here that it is natural to regard $NB_f(1, p)$ as the *free geometric distribution* of parameter $0 < p < 1$.

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