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NATURAL AND MODIFIED CONJUGATE PRIORS IN EXPONENTIAL FAMILIES OF STOCHASTIC PROCESSES

BY

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Abstract. Modified conjugate families of prior distributions are investigated and their properties are examined in the context of applications to admissible and minimax estimation for the general exponential model for stochastic processes defined by (1). The conjugate priors are characterized as those which yield a linear admissible estimator under a weighted quadratic loss in a sequential statistical model. In Section 3, a new characterization of conjugate priors is presented which is relevant to the problem of finding minimax estimators in the statistical model that after a random time transformation cannot be reduced to a model for processes with stationary independent increments. Applications of the results obtained are presented in some special models, among others to a zero mean stationary Gaussian Markov process in the problem of estimating the variance parameter.

1. INTRODUCTION AND BACKGROUND

Let $X(t)$, $t \in \mathscr{T}$, be a stochastic process with either discrete or continuous time and with values in $(R^k, \mathscr{B}_{R^k})$. Suppose that the distribution of $X(t)$ belongs to a family of probability measures $\mathscr{P} = \{P_\vartheta, \vartheta \in \Theta\}$, where Θ is an open set in R^n. Let $P_{\vartheta,t}$ denote the restriction of P_ϑ to $\mathscr{F}_t = \sigma\{X(s): s \leqslant t\}$. Suppose that the family $\{P_{\vartheta,t}, \vartheta \in \Theta\}$ is dominated by a measure μ_t which is the restriction of a probability measure μ to $\mathscr{F}_t$. Moreover, assume that the density functions (likelihood functions) have the following exponential form:

$$(1) \qquad L(t, \vartheta) = dP_{\vartheta,t}/d\mu_t = \exp[\vartheta Z(t) - \Phi(\vartheta) S(t) - \Psi(\vartheta)],$$

where both $\Phi(\vartheta)$ and $\Psi(\vartheta)$, $\vartheta \in \Theta$, are real and strictly convex functions, and $(Z(t), S(t))$, $t \in \mathscr{T}$ ($Z(t)$ is n-dimensional, $S(t)$ is one-dimensional), is a stochastic process adapted to the filtration $\mathscr{F}_t$, $t \in \mathscr{T}$. Clearly, $(Z(t), S(t))$ is a sufficient statistic for ϑ relative to $\mathscr{F}_t$, $t \in \mathscr{T}$. The process $(Z(t), S(t))$, $t \in \mathscr{T}$, is assumed

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to satisfy the following conditions: $Z(t)$ is right continuous as a function of t, P_ϑ-a.s., and $S(t)$, $t \in \mathcal{T}$, are nonnegative random variables ($S(t)$ may be non--random as well) such that $S(t)$ is strictly increasing and continuous as a function of t and $S(t) \to \infty$ as $t \to \infty$, P_ϑ-a.s.

The family of (1) covers many counting, branching, diffusion-type etc. processes and the family of exponential-type processes which may start from a random state and/or time, comprising also some models for stationary Gaussian processes.

Let τ be a stopping time relative to $\mathcal{F}_t$, $t \in \mathcal{T}$, such that $P_\vartheta(\tau < \infty) = 1$ for each $\vartheta \in \Theta$. Assume that the process $X(t)$ is observed during the random time interval $[0, \tau]$ and denote by $P_{\vartheta,\tau}$ and μ_τ the restrictions of P_ϑ and μ to $\mathcal{F}_\tau$. Then, as well known, $P_{\vartheta,\tau}$ is absolutely continuous with respect to μ_τ and the following fundamental identity of sequential analysis holds:

(2) $$dP_{\vartheta,\tau}/d\mu_\tau = \exp[\vartheta Z(\tau) - \Phi(\vartheta) S(\tau) - \Psi(\vartheta)].$$

Consider estimation of the unknown vector-valued parameter $A\Phi(\vartheta) + B\Psi(\vartheta)$ in the exponential family (2), when the squared error or weighted squared error is taken as a loss. Amongst all the problems which one may state evaluating this parameter the following one seems to be of special interest: do there exist a vector z_0, a number n_0 and a finite stopping time τ for which the decision rule

$$d_0(Z(\tau), S(\tau)) = \frac{Z(\tau) + z_0}{n_0}$$

is an admissible or minimax estimator of a linear combination $A\Phi(\vartheta) + B\Psi(\vartheta)$? The standard way to solve both these problems (or at least the first of them) is to show that $d_0(Z(\tau), S(\tau))$ is a Bayes estimator of $A\Phi(\vartheta) + B\Psi(\vartheta)$ or a limit of Bayes estimators of this linear combination. For this it is necessary to find a suitable family Π of priors on Θ from which the desired prior, or a sequence of priors, may be chosen. A good candidate for such a Π is a family of priors which are conjugate to $\mathcal{P} = \{P_\vartheta, \vartheta \in \Theta\}$. No matter what is the structure of $\mathcal{P}$, the usual definition is that a family Π of priors is conjugate to the family of distributions $\mathcal{P}$ if it is closed under sampling from $P_\vartheta \in \mathcal{P}$. This means that, for each prior distribution $\pi \in \Pi$ of the parameter ϑ, the posterior distribution of this parameter also belongs to Π. This, obviously, greatly simplifies all the calculations and helps to solve problems of admissible and minimax estimation in $\mathcal{P}$. Unfortunately, the conjugate priors may not exist if $\mathcal{P} = \{P_\vartheta, \vartheta \in \Theta\}$ is an arbitrary set of probability distributions, but, as explained below, this is not the case for the family $\mathcal{P}$ defined by (1).

The first, general result concerning conjugate priors was obtained by Diaconis and Ylvisaker [1] who found the family conjugate to $\{P_\vartheta, \vartheta \in \Theta\}$, where, for each $\vartheta \in \Theta$, P_ϑ is the distribution of a random vector Z and

$$dP_\vartheta/d\mu = \exp[\vartheta Z - \Phi(\vartheta)]$$

for a fixed σ-finite measure μ on the Borel sets of R^n. They also characterized this family through the property of linear posterior expectation of the mean parameter $E_\vartheta Z$. This excellent result was generalized by Magiera and Wilczyński [3] who considered the family $\{P_\vartheta, \vartheta \in \Theta\}$ describing distributions of the k-dimensional process $X(t)$, $t \in \mathscr{T}$, whose likelihood function at time t belongs to the following n-parameter exponential family:

$$dP_{\vartheta,t}/d\mu_t = \exp[\vartheta Z(t) - \Phi(\vartheta) S(t)],$$

where processes $Z(t)$ and $S(t)$ are as in the model of (1). The latter result was, in turn, generalized by Magiera [2] who considered a slightly more general case where the likelihood function has the form of (1). In the next section we recall some properties of the conjugate family described in the last reference. Later on, we will use these properties to obtain the main result of the paper. This result concerns the structure of a modified conjugate family which is suitable in statistical inference for the process $X(t)$, $t \in \mathscr{T}$, when the error of estimation is measured by a weighted quadratic loss. In this model the quadratic loss function is divided by a quantity, which, roughly speaking, depends linearly on the elements of the matrix of the Fisher information for (1). The reason for such a modification will be explained in the next section.

2. PROPERTIES OF THE CONJUGATE FAMILY OF PRIORS

To describe properties of the priors which are conjugate to the family (1) we use the following notation.

Let $\mathscr{Y}$ be the interior of the convex hull of the set of all possible values of the process $(Z(t), S(t))$, $t \in \mathscr{T}$, defined in the previous section, and let the function $M\colon \Theta \times R_+ \to R$ be given by

$$M(\vartheta; \alpha) = \Phi(\vartheta)\alpha + \Psi(\vartheta).$$

The partial derivatives of the first and second order of $M(\vartheta; \alpha)$ with respect to the variables ϑ_i and ϑ_j will be denoted throughout by $M_i'(\vartheta; \alpha)$ and $M_{ij}''(\vartheta; \alpha)$, respectively, i.e., for $i, j = 1, \ldots, n$

$$M_i'(\vartheta; \alpha) = \partial M(\vartheta; \alpha)/\partial\vartheta_i \quad \text{and} \quad M_{ij}''(\vartheta; \alpha) = \partial^2 M(\vartheta; \alpha)/\partial\vartheta_i \partial\vartheta_j.$$

Moreover, by $\nabla M(\vartheta; \alpha)$ we denote the gradient of the function $M(\vartheta; \alpha)$ with respect to the variable ϑ, i.e.,

$$\nabla M(\vartheta; \alpha) = \big(M_1'(\vartheta; \alpha), M_2'(\vartheta; \alpha), \ldots, M_n'(\vartheta; \alpha)\big)^T.$$

The set Θ is assumed to be open in R^n, and, by Hölder's inequality, is convex in R^n. Therefore, for fixed coordinates $\vartheta_1, \ldots, \vartheta_{i-1}, \vartheta_{i+1}, \ldots, \vartheta_n$ the coordinate ϑ_i of any point $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ which lies in Θ belongs to an open interval whose ends will be denoted from now on by $\underline{\vartheta_i}$ and $\overline{\vartheta_i}$, i.e.,

$$\bigwedge_{1 \leqslant i \leqslant n} \underline{\vartheta_i} = \underline{\vartheta_i}(\vartheta_1, \ldots, \vartheta_n) < \vartheta_i < \overline{\vartheta_i} = \overline{\vartheta_i}(\vartheta_1, \ldots, \vartheta_n), \quad \vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \Theta.$$

The following results concerning characterization of the prior distributions on Θ which are conjugate to the family of (1) may be found in the papers by Magiera and Wilczyński [3] and Magiera [2].

THEOREM 1. *If* $\nu > 0$ *and* $(r, \alpha) = ((r_1, r_2, \ldots, r_n), \alpha) \in \mathscr{Y}$, *then*

(a) $$\int_{\Theta} \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty;$$

(b) $$\int_{\Theta} |M_i'(\vartheta; \alpha)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty;$$

(c) $$\int_{\Theta} M_i'(\vartheta; \alpha) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta = r_i \int_{\Theta} \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta;$$

(d) $$\bigwedge_{\vartheta=(\vartheta_1,\ldots,\vartheta_n)\in\Theta} \ \bigwedge_{1\leqslant i\leqslant n} \ \lim_{\vartheta\to\vartheta_i^0} \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] = 0$$

where $\vartheta_i^0 = \underline{\vartheta_i}$ *or* $\vartheta_i^0 = \overline{\vartheta_i}$;

(e) $$\sup_{\vartheta\in\Theta} \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] < \infty.$$

From the first part of this theorem it follows that, for each $\nu > 0$ and $(r, \alpha) \in \mathscr{Y}$, the measure $\pi(\vartheta; \nu, r, \alpha)$ on Θ whose density with respect to the Lebesgue measure has the form

$$\frac{d\pi(\vartheta; \nu, r, \alpha)}{d\vartheta} = C(\nu, r, \alpha) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right],$$

with the norming constant given by

(3) $$C^{-1}(\nu, r, \alpha) = \int_{\Theta} \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta,$$

defines the prior distribution of the parameter ϑ. Moreover, the family of priors defined in such a way has the following properties (cf. Magiera [2]).

THEOREM 2. *If* $\nu > 0$ *and* $(r, \alpha) = ((r_1, r_2, \ldots, r_n), \alpha) \in \mathscr{Y}$, *then*

(a) $\pi(\vartheta; \nu, r, \alpha)$ *is a conjugate prior to the family* $\{P_\vartheta, \vartheta \in \Theta\}$ *defined by* (1) *or* (2);

(b) *the posterior distribution of* ϑ, *given a random sample* $(Z^1(\tau), S^1(\tau)), \ldots, (Z^N(\tau), S^N(\tau))$ *from* $\{P_\vartheta, \vartheta \in \Theta\}$, *is*

$$\pi\left(\vartheta; N+\nu, (N\overline{Z(\tau)}+\nu r)/(N+\nu), (N\overline{S(\tau)}+\nu\alpha)/(N+\nu)\right),$$

where

$$\overline{Z(\tau)} = (1/N)\sum_{i=1}^{N} Z^i(\tau) \quad \textit{and} \quad \overline{S(\tau)} = (1/N)\sum_{i=1}^{N} S^i(\tau).$$

The properties of the conjugate family $\{\pi(\vartheta; \nu, r, \alpha): (r, \alpha) \in \mathscr{Y}\}$ we will use to solve the problem of admissible estimation considered below.

Let us consider, for a fixed number $s > 0$, the stopping time

$$\tau_s = \inf\{t:\ S(t) \geqslant s\}, \qquad s > 0. \tag{4}$$

By the assumptions concerning the process $S(t)$ this stopping time is finite for each $\vartheta \in \Theta$. Moreover, it follows from the exponential families theory that the regularity conditions are fulfilled. This allows to differentiate twice under the integral sign with respect to ϑ the identity

$$\int_{\Theta} \exp[\vartheta Z(\tau_s) - \Phi(\vartheta) S(\tau_s) - \Psi(\vartheta)]\, d\mu_{\tau_s} = 1,$$

and thus the following Wald identities hold for $1 \leqslant i, j \leqslant n$:

$$E_\vartheta Z_i(\tau_s) = M_i'(\vartheta; s), \tag{5}$$

$$E_\vartheta\big([Z_i(\tau_s) - M_i'(\vartheta; s)][Z_j(\tau_s) - M_j'(\vartheta; s)]\big) = M_{ij}''(\vartheta; s). \tag{6}$$

Let (r, α) be a point from $\mathscr{Y}$ and let

$$\tilde{\alpha} = \frac{\nu\alpha + s}{\nu + 1} \quad \text{for some } \alpha > 0 \text{ and } \nu > 0.$$

Moreover, let $(Z^1(\tau), S^1(\tau)), \ldots, (Z^N(\tau), S^N(\tau))$ be a random sample from (2), with $\tau = \tau_s$. Consider an admissible estimation of the vector $\nabla M(\vartheta; \tilde{\alpha})$ with respect to the loss function

$$L(d, \vartheta) = \big(d - \nabla M(\vartheta; \tilde{\alpha})\big)^T C \big(d - \nabla M(\vartheta; \tilde{\alpha})\big), \tag{7}$$

where C is an $n \times n$ nonnegative definite matrix. As a straightforward implication of Theorem 2 we obtain the following theorem:

THEOREM 3. *If* $\nu > 0$ *and* $(r, \alpha) = ((r_1, r_2, \ldots, r_n), \alpha) \in \mathscr{Y}$, *then*

$$d_0\big(Z(\tau_s), S(\tau_s)\big) = \frac{Z^1(\tau_s) + \ldots + Z^N(\tau_s) + \nu r}{N + \nu} \tag{8}$$

is an admissible estimator of $\nabla M(\vartheta; \tilde{\alpha})$ *when the loss is given by* (7).

Proof. From the previous theorem it follows that the decision rule defined by (8) is Bayes with respect to the prior $\pi(\vartheta; \nu, r, \alpha)$. Since the risk for $d_0\big(Z(\tau_s), S(\tau_s)\big)$ can be seen to be finite, admissibility of this decision rule holds true. ∎

Now suppose, as before, that one observes the process $X(t)$ during the random time interval $[0, \tau_s]$ and one wants to find a minimax estimator of $\nabla M(\vartheta; \tilde{\alpha})$ with respect to the loss function (7). In most cases this problem becomes trivial since, for each estimator d, the resulting expected loss is infinite, i.e.,

$$\sup_{\vartheta \in \Theta} E_\vartheta L\big(d(Z(\tau_s), S(\tau_s)), \vartheta\big) = \infty,$$

and thus any decision rule may be considered as minimax. To avoid this triviality one should modify the loss, e.g. by taking the weighted squared error

$$L(d, \vartheta) = (1/w(\vartheta))\big(d - \nabla M(\vartheta; \tilde{\alpha})\big)^T C\big(d - \nabla M(\vartheta; \tilde{\alpha})\big), \tag{9}$$

so that the corresponding risk of estimation was bounded. The simplest choice of the function $w(\vartheta)$ is $w(\vartheta) = R(d_0, \vartheta)$, where $R(d_0, \vartheta)$ is the risk function for $d_0(Z(\tau_s), S(\tau_s))$ and $N = 1$. This risk equals, by (5) and (6),

$$R(d_0, \vartheta) = \frac{1}{(\nu+1)^2}\Big[\sum_{i,j=1}^{n} c_{ij} M''_{ij}(\vartheta; s) + \nu^2 \sum_{i,j=1}^{n} c_{ij}\big(r_i - M'_i(\vartheta; \alpha)\big)\big(r_j - M'_j(\vartheta; \alpha)\big)\Big].$$

A slightly more general choice of the function $w(\vartheta)$ is

$$w(\vartheta) = \sum_{i,j=1}^{n} c_{ij} M''_{ij}(\vartheta; s) + \sum_{i,j=1}^{n} d_{ij}\big(r_i - M'_i(\vartheta; \alpha)\big)\big(r_j - M'_j(\vartheta; \alpha)\big), \tag{10}$$

where $D = [d_{ij}]$ is an $n \times n$ nonnegative definite matrix. Unfortunately, either choice of these weight functions leads to another difficulty. We are going to prove, using the Bayes methodology, that an admissible or minimax estimator of $\nabla M(\vartheta; \tilde{\alpha})$, under weighted loss, is linear in $Z(\tau_s)$. However, the Bayes estimator of this parameter for the conjugate prior $\pi(\vartheta; \nu, r, \alpha)$ is of the form

$$\frac{\int_\Theta (1/w(\vartheta))\, \nabla M(\vartheta; \tilde{\alpha}) \exp[\vartheta Z(\tau_s) - \Phi(\vartheta) S(\tau_s) - \Psi(\vartheta)] \exp[\nu(r\vartheta - M(\vartheta; \alpha))]\, d\vartheta}{\int_\Theta (1/w(\vartheta)) \exp[\vartheta Z(\tau_s) - \Phi(\vartheta) S(\tau_s) - \Psi(\vartheta)] \exp[\nu(r\vartheta - M(\vartheta; \alpha))]\, d\vartheta},$$

which, except for the trivial case $w(\vartheta) \equiv \text{const}$, is not a linear function of $Z(\tau_s)$. This assertion may be deduced from the following theorem being a modification of Theorem 3 of Diaconis and Ylvisaker [1].

THEOREM 1. *Suppose Θ is open in R^n and suppose that the support of μ_{τ_s} contains an open interval in R^n. If ϑ has the prior distribution π which does not concentrate at a single point, and if*

$$\frac{\int_\Theta (1/w(\vartheta))\, \nabla M(\vartheta; \tilde{\alpha}) \exp[\vartheta Z(\tau_s) - \Phi(\vartheta) S(\tau_s) - \Psi(\vartheta)]\, \pi(d\vartheta)}{\int_\Theta (1/w(\vartheta)) \exp[\vartheta Z(\tau_s) - \Phi(\vartheta) S(\tau_s) - \Psi(\vartheta)]\, \pi(d\vartheta)} = aZ(\tau_s) + b$$

for some constant a and a constant vector b, then $a \neq 0$ and π is absolutely continuous $(d\vartheta)$ with

$$(1/w(\vartheta)) \frac{d\pi(\vartheta)}{d\vartheta} = C \exp[a^{-1} b\vartheta - a^{-1}(1-a) M(\vartheta; \alpha)].$$

Thus, this theorem implies that if there is a prior π^* with respect to which a Bayes estimator of $\nabla M(\vartheta; \bar{\alpha})$ (under the weighted loss (9)) is a linear function of $Z(\tau_s)$, then $d\pi(\vartheta)/d\vartheta$ is proportional to $w(\vartheta)\pi(\vartheta; \nu, r, \alpha)$ for some $\nu > 0$ and $(r, \alpha) \in \mathscr{Y}$. Unfortunately, priors π^* of this form may be improper, since the inequality

$$\int_{\Theta} w(\vartheta)\pi(\vartheta; \nu, r, \alpha) < \infty$$

is not guaranteed. In the next section we give conditions under which these priors are proper and describe their properties, which can be used to obtain some minimax results.

3. PROPERTIES OF THE MODIFIED CONJUGATE FAMILY OF PRIORS

Let Π^* be the family of priors (possible improper) on Θ defined by

$$\Pi^* = \{\pi^*(\vartheta; \nu, r, \alpha): \pi^*(\vartheta; \nu, r, \alpha) \sim w(\vartheta)\pi(\vartheta; \nu, r, \alpha) \quad \text{with } \nu > 0 \text{ and } (r, \alpha) \in \mathscr{Y}\},$$

where the weight function $w(\vartheta)$ is given by (10). From the form of this weight function it follows that $\pi^*(\vartheta; \nu, r, \alpha)$ is proper if for each $1 \leqslant i, j \leqslant n$

$$\int_{\Theta} M''_{ij}(\vartheta; s)\exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty,$$

$$\int_{\Theta} M'_i(\vartheta; \alpha) M'_j(\vartheta; \alpha)\exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty.$$

To formulate conditions which guarantee finiteness of the integrals above we denote by A the following set:

$$A = \{(\nu, r, \alpha) \in R_+ \times \mathscr{Y}: \bigwedge_{\beta > 0} \sup_{\vartheta \in \Theta} \|\nabla M(\vartheta; \beta)\| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] < \infty\}$$

and assume that it is nonempty.

LEMMA 1. *Suppose that* $(\nu_0, r_0, \alpha_0) \in A$ *and that* $(\nu, r, \alpha) \in R_+ \times \mathscr{Y}$. *Then* $(\nu, r, \alpha) \in A$ *provided that at least one of the following four conditions is satisfied*:

(a)
$$\bigwedge_{\beta > 0} \sup_{\vartheta \in \Theta} \|\nabla M(\vartheta; \beta)\| < \infty;$$

(b)
$$(\nu, r, \alpha) = \left(\nu_0 + \nu_1, \frac{\nu_0 r_0 + \nu_1 r_1}{\nu_0 + \nu_1}, \frac{\nu_0 \alpha_0 + \nu_1 \alpha_1}{\nu_0 + \nu_1}\right) \quad \textit{and} \quad (\nu_1, r_1, \alpha_1) \in R_+ \times \mathscr{Y};$$

(c) $\quad \nu > \nu_0, \ \nu\alpha > \nu_0 \alpha_0 \quad$ *and* $\quad \liminf_{\|\theta\| \to \infty} \frac{M(\vartheta; 1)}{\|\vartheta\|} = \infty;$

(d) $\quad \Theta = (\underline{\vartheta}, \overline{\vartheta}) \subset (-\infty, \infty) \quad$ *and* $\quad \nu > \nu_0, \ \nu\alpha > \nu_0 \alpha_0.$

Proof. Let (ν, r, α) be a point from $R_+ \times \mathscr{Y}$. If the condition (a) is satisfied, then $(\nu, r, \alpha) \in A$ because (cf. Theorem 1)

$$\sup_{\vartheta \in \Theta} \exp\left[\nu \left(r\vartheta - M(\vartheta; \alpha)\right)\right] < \infty.$$

The same inequality implies that $(\nu, r, \alpha) \in A$ when the condition (b) is fulfilled, because $(\nu_0, r_0, \alpha_0) \in A$, $(\nu_1, r_1, \alpha_1) \in R_+ \times \mathscr{Y}$, and

$$\begin{aligned} &\sup_{\vartheta \in \Theta} \|\nabla M(\vartheta; \beta)\| \exp\left[\nu \left(r\vartheta - M(\vartheta; \alpha)\right)\right] \\ &\quad = \sup_{\vartheta \in \Theta} \|\nabla M(\vartheta; \beta)\| \exp\left[\nu_0 \left(r_0\vartheta - M(\vartheta; \alpha_0)\right)\right] \exp\left[\nu_1 \left(r_1 \vartheta - M(\vartheta; \alpha_1)\right)\right] \\ &\quad \leqslant \sup_{\vartheta \in \Theta} \|\nabla M(\vartheta; \beta)\| \exp\left[\nu_0 \left(r_0\vartheta - M(\vartheta; \alpha_0)\right)\right] \\ &\qquad\qquad \times \sup_{\vartheta \in \Theta} \exp\left[\nu_1 \left(r_1 \vartheta - M(\vartheta; \alpha_1)\right)\right] < \infty. \end{aligned}$$

Assume now that the condition (c) is satisfied. Then

$$\begin{aligned} &\|\nabla M(\vartheta; \beta)\| \exp\left[\nu \left(r\vartheta - M(\vartheta; \alpha)\right)\right] \\ &\quad = \|\nabla M(\vartheta; \beta)\| \exp\left[\nu_0 \left(r_0\vartheta - M(\vartheta; \alpha_0)\right)\right] \exp\left[\nu_1 \left(r_1 \vartheta - M(\vartheta; \alpha_1)\right)\right], \end{aligned}$$

where

$$(11) \qquad \nu_1 = \nu - \nu_0 > 0, \quad r_1 = \frac{\nu r - \nu_0 r_0}{\nu - \nu_0}, \quad \alpha_1 = \frac{\nu\alpha - \nu_0 \alpha_0}{\nu - \nu_0} > 0.$$

Therefore, to prove that $(\nu, r, \alpha) \in A$ it suffices to show that

$$\sup_{\vartheta \in \Theta} \exp\left[\nu_1 \left(r_1\vartheta - M(\vartheta; \alpha_1)\right)\right] < \infty.$$

For the purpose observe first that $r_1\vartheta - M(\vartheta; \alpha_1)$ is bounded from above when $\|\vartheta\|$ is bounded, because $M(\vartheta; \alpha_1)$ is a convex function of the variable ϑ and is bounded from below by an affine function, e.g.,

$$\bigwedge_{\vartheta \in \Theta} M(\vartheta; \alpha_1) \geqslant M(\vartheta_0, \alpha_1) + \nabla M(\vartheta_0, \alpha_1)(\vartheta - \vartheta_0) \quad \text{for a fixed } \vartheta_0 \in \Theta.$$

By the same argument, the convex functions $\Phi(\vartheta)$ and $\Psi(\vartheta)$ are bounded from below by two affine functions, and the assumption

$$\liminf_{\|\theta\| \to \infty} \frac{M(\vartheta; 1)}{\|\vartheta\|} = \infty$$

implies immediately that

$$\bigwedge_{\alpha>0} \liminf_{\|\theta\|\to\infty} \frac{M(\vartheta;\alpha)}{\|\vartheta\|} = \infty .$$

Therefore, if $\|\vartheta\| \to \infty$, we obtain

$$\limsup_{\|\vartheta\|\to\infty} [r_1\vartheta - M(\vartheta;\alpha_1)] = \limsup_{\|\vartheta\|\to\infty} \|\vartheta\| \left[\frac{r_1\vartheta}{\|\vartheta\|} - \frac{M(\vartheta;\alpha_1)}{\|\vartheta\|}\right] = -\infty,$$

which implies the desired result.

Now assume that the condition (d) holds. For each fixed $\beta > 0$ the function $M'(\vartheta;\beta)$ is increasing with respect to the variable $\vartheta \in \Theta = (\underline{\vartheta}, \overline{\vartheta}) \subset (-\infty, \infty)$. Therefore, for $(\nu, r, \alpha) \in R_+ \times \mathscr{Y}$, the function $|M'(\vartheta;\beta)| \exp[\nu(r\vartheta - M(\vartheta;\alpha))]$ may become unbounded only when

$$\vartheta \to \underline{\vartheta} \text{ and } \lim_{\vartheta\to\underline{\vartheta}} M'(\vartheta;\beta) = -\infty \quad \text{ or } \quad \vartheta \to \overline{\vartheta} \text{ and } \lim_{\vartheta\to\overline{\vartheta}} M'(\vartheta;\beta) = \infty .$$

We prove the boundedness of this function only in the first case, because the other one can be treated by analogy. So assume now that $\vartheta \to \underline{\vartheta}$ and that $\lim_{\vartheta\to\underline{\vartheta}} M'(\vartheta;\beta) = -\infty$. Since both functions $\Phi'(\vartheta)$ and $\Psi'(\vartheta)$ are increasing, and thus are bounded from above as $\vartheta \to \underline{\vartheta}$, this assumption implies that

$$\bigwedge_{\alpha>0} \lim_{\vartheta\to\underline{\vartheta}} M'(\vartheta;\alpha) = -\infty . \tag{12}$$

Using the same arguments as in the proof of (c) we deduce that to obtain the boundedness of the function $|M'(\vartheta;\beta)| \exp[\nu(r\vartheta - M(\vartheta;\alpha))]$ when $\vartheta \to \underline{\vartheta}$ it suffices to show that

$$\lim_{\vartheta\to\underline{\vartheta}} [r_1\vartheta - M(\vartheta;\alpha_1)] < \infty, \tag{13}$$

where the point (ν_1, r_1, α_1) is of the same form as in (11). So, suppose that

$$\lim_{\vartheta\to\underline{\vartheta}} [r_1\vartheta - M(\vartheta;\alpha_1)] = \infty .$$

By the same arguments as in the proof of (c) we deduce that it is impossible when $\underline{\vartheta} > -\infty$. Assume that $\underline{\vartheta} = -\infty$. Then, by the de l'Hôspital rule and (12),

$$0 > \lim_{\vartheta\to-\infty} \frac{r_1\vartheta - M(\vartheta;\alpha_1)}{\vartheta} = \lim_{\vartheta\to-\infty} [r_1 - M'(\vartheta;\alpha_1)] = \infty,$$

which is a contradiction. Therefore (13) holds, which completes the proof of the lemma. ∎

Let A_0 denote the interior of the set A. Then the following lemma holds:

LEMMA 2. *If* $(\nu, r, \alpha) \in A_0$, *then, for each* $\beta > 0$ *and each* $1 \leqslant i, j \leqslant n$,

(a) $$\int_{\Theta} |M'_i(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty;$$

(b) $$\int_{\Theta} |M'_i(\vartheta; \beta) M'_j(\vartheta; \alpha)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty;$$

(c) $$\left|\int_{\Theta} M''_{ij}(\vartheta; \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta\right| < \infty;$$

(d) $$\frac{1}{\nu}\int_{\Theta} M''_{ij}(\vartheta; \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta = -r_j \int_{\Theta} M'_i(\vartheta; \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta + \int_{\Theta} M'_j(\vartheta; \alpha) M'_i(\vartheta; \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta;$$

(e) $$\frac{1}{\nu}\int_{\Theta} M''_{ij}(\vartheta; \alpha) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta = \int_{\Theta} \left(r_i - M'_i(\vartheta; \alpha)\right)\left(r_j - M'_j(\vartheta; \alpha)\right) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta.$$

Proof. If $(\nu, r, \alpha) \in A_0$, then, for a sufficiently small $\varepsilon > 0$, $(\nu - \varepsilon, r, \alpha) \in A$ and, obviously, $(\varepsilon, r, \alpha) \in R_+ \times \mathscr{Y}$. Therefore

$$\int_{\Theta} |M'_i(\vartheta; \beta) M'_j(\vartheta; \alpha)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta = \int_{\Theta} |M'_i(\vartheta; \beta)| \exp\left[(\nu - \varepsilon)\left(r\vartheta - M(\vartheta; \alpha)\right)\right] |M'_j(\vartheta; \alpha)| \times \exp\left[\varepsilon\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta < \infty,$$

because the first factor under the second integral is bounded while the other one is integrable by Theorem 1. Thus the second part of the lemma and, by the same arguments, the first one are proved. We prove the third part of the lemma only for $i = 1$ because other cases can be treated by analogy. The boundedness of the factor $|M'_i(\vartheta; \beta)| \exp\left[(\nu - \varepsilon)\left(r\vartheta - M(\vartheta; \alpha)\right)\right]$ implies that, for fixed values of $\vartheta_2, \ldots, \vartheta_n$ and for $\vartheta_1^0 = \underline{\vartheta_1}$ or $\vartheta_1^0 = \overline{\vartheta}_1$,

$$\lim_{\vartheta \to \vartheta_1^0} |M'_1(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] = 0$$

(cf. Theorem 1 (d)). Integrating by parts we obtain, by integrability of the factors $|M'_1(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right]$ and $|M'_j(\vartheta; \alpha) M'_1(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right]$,

$$\int_{\Theta} M''_{1j}(\vartheta; \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] d\vartheta$$

$$= \int \ldots \int \left[M'_1(\vartheta;\, \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right]\right]_{\vartheta=\underline{\vartheta}_1}^{\vartheta=\bar{\vartheta}_1} d\vartheta_2 \ldots d\vartheta_n$$

$$-\int_{\Theta} M'_1(\vartheta;\, \beta)\, \nu\left(r_j - M'_j(\vartheta;\, \alpha)\right) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right] d\vartheta$$

$$= -\nu r_j \int_{\Theta} M'_1(\vartheta;\, \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right] d\vartheta$$

$$+\nu \int_{\Theta} M'_j(\vartheta;\, \alpha)\, M'_1(\vartheta;\, \beta) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right] d\vartheta.$$

This proves (c) and (d) of the lemma. The last part of the lemma can be deduced from the previous one and from the assertion (c) of Theorem 1. ∎

As a straightforward implication of Lemmas 1 and 2 we have the following theorem:

THEOREM 5. *Let the weight function* $w(\vartheta)$ *be of the form* (10). *If* $(\nu, r, \alpha) \in A_0$, *then the measure* $\pi^*(\vartheta; \nu, r, \alpha)$ *on* Θ *whose density with respect to the Lebesgue measure takes the form*

$$\frac{d\pi^*(\vartheta;\, \nu,\, r,\, \alpha)}{d\vartheta} = C^*(\nu, r, \alpha)\, w(\vartheta) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right],$$

with the norming constant given by

$$\left(C^*(\nu, r, \alpha)\right)^{-1} = \int_{\Theta} w(\vartheta) \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right] d\vartheta,$$

defines the prior distribution of the parameter ϑ.

In the case when the parameter ϑ is positive we present the following proposition useful to determine the set of parameters (ν, r, α) for which the modified priors are the proper ones. As it is well known, to derive the Bayes estimator of ϑ and the posterior expected loss, one has to consider the following conditions:

$$\int_{\Theta} \frac{d}{d\vartheta} \left\{\exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right]\right\} d\vartheta = 0, \tag{14}$$

$$\int_{\Theta} \frac{d}{d\vartheta} \left\{\left[r - M'(\vartheta;\, \alpha)\right] \exp\left[\nu\left(r\vartheta - M(\vartheta;\, \alpha)\right)\right]\right\} d\vartheta = 0. \tag{15}$$

It follows from Theorem 1 that the condition (14) is satisfied for all $\nu > 0$ and $(r, \alpha) \in \mathscr{Y}$. The following proposition determines conditions under which the equality (15) holds.

PROPOSITION 1. *Let* $\vartheta \in (0, \infty)$. *Suppose that* ϑ *and* $\Phi'(\vartheta)$ *can be expressed as*

$$\vartheta = \exp\left[\nu_0\left(r_0 \vartheta - M(\vartheta;\, \alpha_0)\right)\right] \tag{16}$$

and

$$\Phi'(\vartheta) = \exp\left[\nu_1 \left(r_1 \vartheta - M(\vartheta;\, \alpha_1)\right)\right], \tag{17}$$

respectively, for some choice of (ν_0, r_0, α_0) *and* (ν_1, r_1, α_1) *such that* $\nu_0 > 0$. *Then the equality* (15) *holds for* (ν, r, α) *satisfying the following conditions*:

$$\nu > 0, \quad (r+r_0,\, \alpha+\alpha_0) \in \mathscr{Y},$$

$$\nu - \nu_0 > 0, \quad \left(\frac{\nu}{\nu-\nu_0} r + r_0,\, \frac{\nu}{\nu-\nu_0}\alpha + \alpha_0\right) \in \mathscr{Y},$$

$$\nu + \nu_1 > 0, \quad \left(\frac{\nu(r+r_0)+\nu_1 r_1}{\nu+\nu_1},\, \frac{\nu(\alpha+\alpha_0)+\nu_1\alpha_1}{\nu+\nu_1}\right) \in \mathscr{Y}.$$

If $\Psi(\vartheta) \equiv 0$, *and if* ϑ *can be written as*

$$\vartheta = \exp\left[\varrho_0 \vartheta - \sigma_0 \Phi(\vartheta)\right] \tag{18}$$

for some choice of $\varrho_0, \sigma_0 > 0$, *then the condition* (15) *holds for* $(\varrho+\varrho_0,\, \sigma+\sigma_0) \in \mathscr{Y}$, *where* $(\varrho, \sigma) \in \mathscr{Y}$ *are the prior parameters.*

Proof. It follows from (16), after differentiation, that

$$M'(\vartheta;\, \alpha_0) = r_0 - \frac{1}{\nu_0 \vartheta}. \tag{19}$$

Consider the expression

$$I(\vartheta) = \left[r + r_0 - M'(\vartheta;\, \alpha+\alpha_0)\right] \exp\left\{\nu\left[(r+r_0)\vartheta - M(\vartheta;\, \alpha+\alpha_0)\right]\right\}$$

appearing in the integrand of (15) for $\nu > 0$ and $(r+r_0,\, \alpha+\alpha_0) \in \mathscr{Y}$. Taking into account (19) yields

$$M'(\vartheta;\, \alpha+\alpha_0) = r_0 - \frac{1}{\nu_0 \vartheta} + \alpha \Phi'(\vartheta). \tag{20}$$

Thus, in view of (16), (17) and (20),

$$\begin{aligned}
I(\vartheta) &= \left[r + \frac{1}{\nu_0 \vartheta} - \alpha\Phi'(\vartheta)\right] \exp\left\{\nu\left[(r+r_0)\vartheta - M(\vartheta;\, \alpha+\alpha_0)\right]\right\} \\
&= r \exp\left\{\nu\left[(r+r_0)\vartheta - M(\vartheta;\, \alpha+\alpha_0)\right]\right\} \\
&\quad + \frac{1}{\nu_0} \exp\left\{(\nu-\nu_0)\left[\left(\frac{\nu}{\nu-\nu_0} r + r_0\right)\vartheta - M\left(\vartheta;\, \frac{\nu}{\nu-\nu_0}\alpha + \alpha_0\right)\right]\right\} \\
&\quad - \alpha \exp\left\{(\nu+\nu_1)\left[\left(\frac{\nu(r+r_0)+\nu_1 r_1}{\nu+\nu_1}\right)\vartheta - M\left(\vartheta;\, \frac{\nu(\alpha+\alpha_0)+\nu_1\alpha_1}{\nu+\nu_1}\right)\right]\right\}.
\end{aligned}$$

Then, applying the condition (14) for each term of $I(\vartheta)$, we obtain

$$\lim_{\vartheta\to 0} I(\vartheta) = \lim_{\vartheta\to\infty} I(\vartheta),$$

which is equivalent to (15).

Remark that if $\Psi(\theta) \equiv 0$, then the dimension of the prior parameters (ν, r, α) is reduced to two parameters, say (ϱ, σ), where $\varrho = \nu r$ and $\sigma = \nu\alpha$. In this case, the relation between ϑ and $\Phi'(\vartheta)$ can be derived explicitly from (18). Namely,

$$\Phi'(\vartheta) = \frac{1}{\sigma_0}\left(\varrho_0 - \frac{1}{\vartheta}\right).$$

Thus

$$\begin{aligned}
&[(\varrho+\varrho_0)-(\sigma+\sigma_0)\Phi'(\vartheta)]\exp[(\varrho+\varrho_0)\vartheta-(\sigma+\sigma_0)\Phi(\vartheta)]\\
&\quad = (\varrho+\varrho_0)\exp[(\varrho+\varrho_0)\vartheta-(\sigma+\sigma_0)\Phi(\vartheta)]\\
&\qquad -\frac{\varrho_0(\sigma+\sigma_0)}{\sigma_0}\exp[(\varrho+\varrho_0)\vartheta-(\sigma+\sigma_0)\Phi(\vartheta)]+\frac{\sigma+\sigma_0}{\sigma_0}\exp[\varrho\vartheta-\sigma\Phi(\vartheta)],
\end{aligned}$$

and the result follows by applying the condition (14). ∎

4. APPLICATIONS

In this section we will apply Proposition 1 and Theorem 5 to find the modified conjugate prior $\pi^*(\vartheta; \nu, r, \alpha)$ for a one-parameter stationary Gaussian Markov process and for a two-parameter Markov chain. Next we solve one problem of an admissible and minimax estimation for the real process $X(t)$ whose likelihood function is of the form (1).

4.1. EXAMPLE 1. Let $X(t)$, $t \geqslant 0$, be a stochastic process satisfying the following stochastic differential equation:

$$dX(t) = -\vartheta X(t)\,dt + dW(t),$$

where $W(t)$, $t \geqslant 0$, denotes the standard Wiener process and $X(0) =_{\mathscr{D}} \mathcal{N}(0, 1/2\vartheta)$, $\vartheta \in \Theta = (0, \infty)$. The process $X(t)$, $t \geqslant 0$, is a stationary Gaussian Markov process with $E_\vartheta X(t) = 0$ and the covariance function $B(s, t) = (2\vartheta)^{-1}\exp(-\vartheta|t-s|)$. The likelihood function for this process is of the form (1) because

$$L(t, \vartheta) = \exp[\vartheta Z(t) - \Phi(\vartheta)S(t) - \Psi(\vartheta)],$$

where

$$Z(t) = \tfrac{1}{2}[t - X^2(t) - X^2(0)], \qquad S(t) = \int_0^t X^2(s)\,ds,$$

$$\Phi(\vartheta) = \vartheta^2/2 \quad \text{and} \quad \Psi(\vartheta) = -\tfrac{1}{2}\log\vartheta.$$

Theorem 1 implies that, for each $\nu > 0$ and $(r, \alpha) \in \mathscr{Y} = (-\infty, \infty) \times (0, \infty)$, the measure $\pi(\vartheta; \nu, r, \alpha)$ on Θ, whose density takes the form

$$\frac{d\pi(\vartheta; \nu, r, \alpha)}{d\vartheta} = C(\nu, r, \alpha)\,\vartheta^{\nu/2} \exp\left[\nu\left(r\vartheta - \alpha\frac{\vartheta^2}{2}\right)\right],$$

defines a proper conjugate prior. The norming constant $C(\nu, r, \alpha)$, given by (3), can be shown to satisfy the equation

$$[C(\nu, r, \alpha)]^{-1} = (\nu\alpha)^{-(\nu+2)/4}\,\Gamma\left(\frac{\nu+2}{2}\right)\exp\left(\frac{\nu r^2}{4\alpha}\right) D_{-(\nu+2)/2}\left(-\left(\frac{\nu}{\alpha}\right)^{1/2} r\right),$$

where $D_p(\kappa)$ denotes the parabolic cylinder function

$$D_p(\kappa) = \frac{\exp(-\kappa^2/4)}{\Gamma(-p)} \int_0^\infty \vartheta^{-p-1} \exp\left(-\kappa\vartheta - \frac{\vartheta^2}{2}\right) d\vartheta, \qquad p < 0$$

(see Magiera [2]).

In this model,

$$M(\vartheta; \alpha) = \alpha\frac{\vartheta^2}{2} - \frac{1}{2}\log\vartheta$$

and

$$|M'(\vartheta; \beta)| \exp[\nu(r\vartheta - M(\vartheta; \alpha))] = \left|\beta\vartheta - \frac{1}{2\vartheta}\right| \vartheta^{\nu/2} \exp\left[\nu\left(r\vartheta - \alpha\frac{\vartheta^2}{2}\right)\right].$$

Therefore

$$A = \{(\nu, r, \alpha) \in (0, \infty) \times (-\infty, \infty) \times (0, \infty)\colon\ \nu \geqslant 2\}$$

and, by Lemma 2, the priors $\pi^*(\vartheta; \nu, r, \alpha)$ are proper for any

$$(\nu, r, \alpha) \in A_0 = \{(\nu, r, \alpha) \in (0, \infty) \times (-\infty, \infty) \times (0, \infty)\colon\ \nu > 2\}.$$

The same result can be deduced from Proposition 1 because for the numbers $\nu_0 = \nu_1 = 2$, $r_0 = r_1 = 0$ and $\alpha_0 = \alpha_1 = 0$ the parameter ϑ and the function $\Phi'(\vartheta)$ can be expressed as

$$\vartheta = \exp\left[\nu_0\left(r_0\vartheta - \alpha_0\frac{\vartheta^2}{2} + \frac{1}{2}\log\vartheta\right)\right] = \exp[\nu_0(r_0\vartheta - M(\vartheta; \alpha_0))]$$

$$= \exp[\nu_1(r_1\vartheta - M(\vartheta; \alpha_1))] = \Phi'(\vartheta).$$

Now, assume that the weight function $w(\vartheta)$ takes the form

$$w(\vartheta) = M''(\vartheta;\beta)+A\big(M'(\vartheta;\alpha)\big)^2+B\,|M'(\vartheta;\delta)|+C \tag{21}$$

for some numbers A, B, C, β, γ and δ for which this weight is a nonnegative function of ϑ. Theorem 5 implies that, for each $v > 2$ and $(r, \alpha)\in\mathscr{Y} = (-\infty, \infty)\times(0, \infty)$, the measure $\pi^*(\vartheta; v, r, \alpha)$ on Θ, whose density is of the form

$$\frac{d\pi^*(\vartheta; v, r, \alpha)}{d\vartheta} = C^*(v, r, \alpha)\left[\left(\beta+\frac{1}{2\vartheta^2}\right)+A\left(\gamma\vartheta-\frac{1}{2\vartheta}\right)^2+B\left(\delta\vartheta-\frac{1}{2\vartheta}\right)+C\right] \times \vartheta^{v/2}\exp\left[v\left(r\vartheta-\alpha\frac{\vartheta^2}{2}\right)\right],$$

defines a proper prior on Θ.

EXAMPLE 2. Consider a two-state Markov chain with the matrix $(p_{ij})_{i,j=1}^{2}$ of the one-step transition probabilities and starting from state 1 with probability 1. Let $N_{ij}(t)$, $i, j = 1, 2$, denote the number of the one-step transitions from state i to state j in the time interval $[0, t]$. The likelihood function based on the observation of the process up to time t is

$$L(\tau, p) = \exp\{N_{11}(\tau)\log(1-p_{12})+N_{12}(\tau)\log p_{12} +N_{21}(\tau)\log p_{21}+N_{22}(\tau)\log(1-p_{21})\},$$

where $p = (p_{12}, p_{21})\in(0, 1)^2$. Suppose that the observation of the process is terminated at the following random time:

$$\tau_s^2 = \inf\{t\colon\ N_{22}(t) = s\},\qquad s = 1, 2, \ldots \tag{22}$$

Remark that $N_{22}(\tau_s^2) = s$ and that $N_{12}(\tau_s^2) = N_{21}(\tau_s^2)+1$. Thus the likelihood function at τ_s^2 is

$$L(\tau_s^2, \vartheta) = \exp\left\{\vartheta_1 N_{11}(\tau_s^2)+\vartheta_2 N_{21}(\tau_s^2)-s\log\frac{1-e^{\vartheta_1}}{1-e^{\vartheta_1}-e^{\vartheta_2}}+\log(1-e^{\vartheta_1})\right\}, \tag{23}$$

where $\vartheta_1 = \log(1-p_{12})$ and $\vartheta_2 = \log p_{12}p_{21}$.

In this case, $\mathscr{Y} = \{(r_1, r_2, \alpha)\colon\ r_1 > 0, r_2 > 0, \alpha > 0\}$ and

$$M(\vartheta;\alpha) = \alpha\log\frac{1-e^{\vartheta_1}}{1-e^{\vartheta_1}-e^{\vartheta_2}}-\log(1-e^{\vartheta_1}) = -\alpha\log(1-p_2)-\log p_1,$$

$$\frac{\partial M(\vartheta;\alpha)}{\partial\vartheta_1} = \frac{e^{\vartheta_1}}{1-e^{\vartheta_1}}\,\frac{1-e^{\vartheta_1}+(\alpha-1)e^{\vartheta_2}}{1-e^{\vartheta_1}-e^{\vartheta_2}} = \frac{1-p_1}{p_1(1-p_2)}[1+(\alpha-1)p_2],$$

$$\frac{\partial M(\vartheta;\alpha)}{\partial\vartheta_2} = \alpha\frac{e^{\vartheta_2}}{1-e^{\vartheta_1}-e^{\vartheta_2}} = \alpha\frac{p_2}{1-p_2},$$

where $p_1 = p_{12}$ and $p_2 = p_{21}$.

Thus, for each β and $(r_1, r_2, \alpha) \in \mathcal{Y}$,

$$|M_1'(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right]$$
$$= [1 + p_2(\beta - 1)]\, p_1^{\nu(r_2+1)-1} (1-p_1)^{\nu r_1 + 1} p_2^{\nu r_2} (1-p_2)^{\nu\alpha - 1}$$

and

$$|M_2'(\vartheta; \beta)| \exp\left[\nu\left(r\vartheta - M(\vartheta; \alpha)\right)\right] = \beta p_1^{\nu(r_2+1)} (1-p_1)^{\nu r_1} p_2^{\nu r_2 + 1} (1-p_2)^{\nu\alpha - 1},$$

which implies that

$$A = \{(\nu, r, \alpha) \in (0, \infty) \times \mathcal{Y}:\ \nu\alpha \geqslant 1,\ \nu(r_2+1) \geqslant 1\}.$$

Therefore, by Lemma 2, the priors $\pi^*(\vartheta; \nu, r, \alpha)$ are proper for any

$$(\nu, r, \alpha) \in A_0 = \{(\nu, r, \alpha) \in (0, \infty) \times \mathcal{Y}:\ \nu\alpha > 1,\ \nu(r_2+1) > 1\}.$$

4.2. Now, suppose that $X(t)$ is a real process whose likelihood function is of the form of (1), e.g., a process considered in Example 1, and assume, for simplicity, that Θ is an open and convex subset of the real line, i.e. put $n = 1$. Let (r, α) be a point from $\mathcal{Y}$ and let

$$\tilde{\alpha} = \frac{\nu\alpha + s}{\nu + 1} \quad \text{for some } \alpha > 0 \text{ and } \nu > 0.$$

Given an observation $\left(Z(\tau_s), S(\tau_s)\right)$ from (2), with $\tau = \tau_s$, we consider estimation of $M'(\vartheta; \tilde{\alpha}) = M_1'(\vartheta; \tilde{\alpha})$ with respect to the loss function

$$L(d, \vartheta) = \frac{1}{w(\vartheta)} \left(d - M'(\vartheta; \tilde{\alpha})\right)^2, \tag{24}$$

where the weight function is of the form (21).

THEOREM 6. *If* $(\nu, r, \alpha) \in A_0$, *then*

$$d_0\left(Z(\tau_s), S(\tau_s)\right) = \frac{Z(\tau_s) + \nu r}{1 + \nu}$$

is an admissible estimator of $M'(\vartheta; \tilde{\alpha}) = M'\left(\vartheta; (\nu\alpha + s)/(\nu + 1)\right)$ *when the loss is given by* (24). *Moreover, if the weight function is of the form*

$$w(\vartheta) = M''(\vartheta; s) + \nu^2 \left(r - M'(\vartheta; \alpha)\right)^2, \tag{25}$$

then this estimator is minimax.

Proof. From the result of the last section it follows that $d_0\left(Z(\tau_s), S(\tau_s)\right)$ is Bayes with respect to the proper prior $\pi^*(\vartheta; \nu, r, \alpha)$. It is easy to calculate that the risk for this decision rule is

$$R(d_0, \vartheta) = \frac{M''(\vartheta; s) + \nu^2 \left(r - M'(\vartheta; \alpha)\right)^2}{(\nu + 1)^2 w(\vartheta)}$$

and it is finite for all $\vartheta \in \Theta$. This implies admissibility of $d_0(Z(\tau_s), S(\tau_s))$. For the weight function (25) this admissible decision rule has a constant risk and thus is minimax. ■

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